Problem of the Month

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Approximation is one of the most important concepts in mathematics.

Problem (2006 Canadian Open Mathematics Challenge) Determine, with justification, the fraction $\frac{p}{q}$, where p and q are positive integers and $q < 100$, that is closest to, but not equal to, $\frac{3}{7}$.

While it is tempting to get out your calculator, it can initially only help so much. If you calculate $\frac{3}{7}$, you'll obtain $0.428571\ldots$. This doesn't help in any obvious way to answer the question.

A first approach after the calculator might be to go for the fraction with the largest possible denominator. This makes a lot of sense in some ways, as the fractions with the largest denominators will be closest together and so would seem to have the best chance of being closest to $\frac{3}{7}$. In our case, the largest possible denominator is $q = 99$. The given fraction, $\frac{3}{7}$, is between $\frac{42}{99} = 0.424242\ldots$ and $\frac{43}{99} = 0.434343\ldots$. After a quick look, we can tell that $\frac{3}{7}$ is closer to $\frac{42}{99}$. From the decimal approximations, $\frac{3}{7}$ and $\frac{42}{99}$ differ by about 0.004. Is this the closest of all possible fractions?

Another idea is to try to convert $\frac{3}{7}$ into the equivalent fraction with the largest possible denominator and then adjust from there. Multiplying numerator and denominator by 14, we obtain $\frac{42}{98}$. We could then add 1 or -1 to the numerator to obtain $\frac{41}{98}$ or $\frac{43}{98}$, which differ from $\frac{3}{7}$ by $\frac{1}{98}$. But this means that the difference is bigger than 0.01, which is worse than before, so this approach doesn't give a closer fraction.

Can we do better than $\frac{42}{99}$? It is possible that, even though fractions with smaller denominators are further apart, they can be between some of the other fractions that we've looked at, for example between $\frac{42}{99}$ and $\frac{3}{7}$ or between $\frac{43}{99}$ and $\frac{3}{7}$.

Solution We want to use the fraction $\frac{p}{q}$ to approximate $\frac{3}{7}$. Let's calculate their difference, which is what we want to minimize:

$$\left|\frac{p}{q} - \frac{3}{7}\right| = \left|\frac{7p - 3q}{7q}\right| = \frac{|7p - 3q|}{7q}.$$

What can we do to make this as small as possible? Two approaches would be to make the numerator of the difference as small as possible or to make the denominator of the difference as large as possible.

Let's focus initially on the numerator. The numerator cannot equal 0 because the fractions $\frac{p}{q}$ and $\frac{3}{7}$ are not equal. Thus, the smallest possible value for the numerator is 1, because p and q are integers. So let's try to find values of p and q for which the numerator equals 1. In this case, the difference equals $\frac{1}{7q}$ which is minimized when q is largest.

For the numerator to equal 1, we need $7p - 3q = \pm 1$. Since we also want to maximize q, we rewrite this as $7p = 3q \pm 1$ and work from the largest possible integer values of q to see when we also get an integer value for p.

If $q = 99$, the equation becomes $7p = 3(99) \pm 1 = 297 \pm 1$. Neither possibility is a multiple of 7.

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If $q = 96$, the equation becomes $7p = 3(96) \pm 1 = 288 \pm 1$. Since 287 is a multiple of 7, then taking $q = 96$ and $p = 41$ gives a difference with numerator 1.

So we have $\left|\frac{41}{96} - \frac{3}{7}\right| = \frac{1}{7 \cdot 96} = \frac{1}{672}$ and this is the smallest possible difference with the numerator equal to 1.

If the numerator equalled 2 or something larger, then the smallest possible difference occurs when the numerator is as small as possible and the denominator is as large as possible, so is $\frac{2}{7 \cdot 99} = \frac{2}{693}$. This is the smallest possible difference with numerator at least 2.

Combining the cases, the smallest possible difference is indeed $\frac{1}{672}$, and so the closest fraction to $\frac{3}{7}$ of all of the fractions under consideration is $\frac{p}{q} = \frac{41}{96}$.

The approximation of functions with polynomials is often seen in first-year university calculus courses. As part of these investigations, we learn how to estimate the amount of error when approximating, for example, $\sin x$ with $x - \frac{1}{6}x^3 + \frac{1}{120}x^5$. The techniques used to estimate this type of error are not dissimilar to what we have seen above, and are very useful in many types of calculations.