SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Last year we received a batch of correct solutions from Steven Karp, student, University of Waterloo, Waterloo, ON, to problems 3289, 3292, 3294, 3296, 3297, 3298, and 3300, which did not make it into the December issue due to being misfiled. Our apologies for this oversight.

3301. [2008 : 44, 46] *Proposed by Ovidiu Furdui, University of Toledo, Toledo, OH, USA.*

Prove that

$$\sum_{n=1}^{\infty} \frac{\ln 2 - \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}\right)}{n} = \sum_{n=1}^{\infty} \frac{1 + \frac{1}{2} + \cdots + \frac{1}{n}}{(2n+1)(2n+2)}\,.$$

What is this common value?

Solution by Manuel Benito, Óscar Ciaurri, and Emilio Fernández, Logroño, Spain, expanded by the editor.

Let A and B denote the summations on the left side and the right side of the proposed equality, respectively. Also, let $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$. Then

$$\begin{aligned} \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} &= H_{2n} - H_n \\ &= H_{2n} - 2\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}\right) = \sum_{j=1}^{2n} \frac{(-1)^{j-1}}{j}\,. \end{aligned}$$

Since it is well known that $\sum\limits_{j=1}^{\infty} \dfrac{(-1)^{j-1}}{j} = \ln 2$, by changing the order of the double summation we have

$$\begin{aligned} A &= \sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{j=2n+1}^{\infty} \frac{(-1)^{j-1}}{j} \right) = \sum_{j=3}^{\infty} \frac{(-1)^{j-1}}{j} \left(\sum_{n=1}^{\lfloor \frac{j-1}{2} \rfloor} \frac{1}{n} \right) \\ &= \sum_{j=3}^{\infty} \frac{(-1)^{j-1}}{j} H_{\lfloor \frac{j-1}{2} \rfloor} = \sum_{k=1}^{\infty} H_k \left(\frac{1}{2k+1} - \frac{1}{2k+2} \right) \\ &= \sum_{k=1}^{\infty} \frac{H_k}{(2k+1)(2k+2)} = \sum_{n=1}^{\infty} \frac{1 + \frac{1}{2} + \cdots + \frac{1}{n}}{(2n+1)(2n+2)} = B\,. \end{aligned}$$

To find the common value of the two absolutely convergent series, let

$$f(x) \;=\; \sum_{j=3}^{\infty} \frac{(-1)^{j-1}}{j} H_{\lfloor \frac{j-1}{2} \rfloor} x^j \,,$$

where the power series for f converges for all $x \in (-1,1)$. Then

$$\begin{aligned} f'(x) \;&=\; \sum_{j=3}^{\infty} (-1)^{j-1} H_{\lfloor \frac{j-1}{2} \rfloor} x^{j-1} \;=\; \sum_{j=2}^{\infty} (-1)^{j} H_{\lfloor \frac{j}{2} \rfloor} x^{j} \\ &=\; \sum_{n=1}^{\infty} \left(H_n x^{2n} - H_n x^{2n+1}\right) \;=\; (1-x) \sum_{n=1}^{\infty} H_n x^{2n} \,. \end{aligned} \tag{1}$$

Now, it is well known that

$$\frac{1}{1-x} \ln\left(\frac{1}{1-x}\right) \;=\; \sum_{n=1}^{\infty} H_n x^n \,. \tag{2}$$

[*Ed*: Multiply $\dfrac{1}{1-x} = 1+x+x^2+\cdots$ with $-\ln(1-x) = x+\frac{1}{2}x^2+\frac{1}{3}x^3+\cdots$ and observe that the coefficient of x^n is $1+\frac{1}{2}+\cdots+\frac{1}{n}$.]

From (2) we have $(1-x) \sum_{n=1}^{\infty} H_n x^n = -\ln(1-x)$.

Thus, $(1-x^2) \sum_{n=1}^{\infty} H_n x^{2n} = -\ln(1-x^2)$, and it follows that

$$(1-x) \sum_{n=1}^{\infty} H_n x^{2n} \;=\; -\frac{1}{1+x} \ln(1-x^2) \,. \tag{3}$$

From (1) and (3) we obtain

$$f'(x) \;=\; -\frac{1}{1+x} \ln(1-x^2) \,.$$

Since $f(0) = 0$ and the last improper integral below is convergent, by applying Abel's Continuity Theorem for power series we have

$$A \;=\; \lim_{x \to 1^-} f(x) \;=\; \int_0^1 f'(x)\,dx \;=\; -\int_0^1 \frac{\ln\left(1-x^2\right)}{1+x}\,dx \,. \tag{4}$$

It remains to evaluate the last integral in (4).

With the change of variable $x = 2u-1$, we have

$$\int_0^1 \frac{\ln\left(1-x^2\right)}{1+x}\,dx \;=\; \int_{1/2}^1 \frac{\ln\big(4u(1-u)\big)}{u}\,du \;=\; I_1 + I_2 \,, \tag{5}$$

where

$$I_1 = \int_{1/2}^{1} \frac{\ln(4u)}{u}\,du = \frac{1}{2}\ln^2(4u)\Big|_{1/2}^{1} = \frac{1}{2}\left((\ln 4)^2 - (\ln 2)^2\right)$$
$$= \frac{1}{2}(\ln 4 + \ln 2)(\ln 4 - \ln 2) = \frac{1}{2}(\ln 8)(\ln 2) = \frac{3}{2}(\ln 2)^2 . \qquad (6)$$

On the other hand, using integration by parts and then making the change of variable $u = 1 - t$, we have

$$I_2 = \int_{1/2}^{1} \frac{\ln(1-u)}{u}\,du = (\ln u)\big(\ln(1-u)\big)\Big|_{1/2}^{1} + \int_{1/2}^{1} \frac{\ln u}{1-u}\,du$$
$$= -(\ln 2)^2 + \int_0^{1/2} \frac{\ln(1-t)}{t}\,dt = -(\ln 2)^2 + \int_0^1 \frac{\ln(1-t)}{t}\,dt - I_2 ,$$

from which we obtain

$$2I_2 = -(\ln 2)^2 + \int_0^1 \frac{\ln(1-t)}{t}\,dt . \qquad (7)$$

[Ed : Since the integrals in the computations above are improper, care must be taken ; e.g., the evaluation of $(\ln u)\big(\ln(1-u)\big)$ at $u = 1$ must be done by computing $\lim\limits_{u\to 1^-} (\ln u)\big(\ln(1-u)\big)$ using L'Hôspital's Rule.]

Finally,

$$\int_0^1 \frac{\ln(1-t)}{t}\,dt = -\int_0^1 \frac{1}{t}\ln\left(\frac{1}{1-t}\right)dt$$
$$= -\int_0^1 \frac{1}{t}\left(\sum_{n=1}^{\infty}\frac{t^n}{n}\right)dt = -\sum_{n=1}^{\infty}\int_0^1 \frac{t^{n-1}}{n}\,dt$$
$$= -\sum_{n=1}^{\infty}\left(\frac{t^n}{n^2}\Big|_0^1\right) = -\sum_{n=1}^{\infty}\frac{1}{n^2} = -\frac{\pi^2}{6} . \qquad (8)$$

From (4) – (8), we conclude that $A = B = \dfrac{\pi^2}{12} - (\ln 2)^2$.

Also solved by PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; and the proposer. There were also three incomplete solutions, all of which only demonstrated that the two given summations are equal.

3302. [2008 : 44, 47] *Proposed by Mihály Bencze, Brasov, Romania.*

Let s, r, and R denote the semiperimeter, the inradius, and the circumradius of a triangle ABC, respectively. Show that

$$(s^2 + r^2 + 4Rr)(s^2 + r^2 + 2Rr) \ \geq\ 4Rr(5s^2 + r^2 + 4Rr) ,$$

and determine when equality holds.

Solution by Michel Bataille, Rouen, France.

First, using the two known formulas $s^2+r^2+4Rr = ab+bc+ca$ and $abc = 4Rrs$, where a, b, and c are the sides of the triangle, we deduce that

$$\begin{aligned}(a+b)(b+c)(c+a) &= (a+b+c)(ab+bc+ca) - abc \\ &= 2s\left(s^2+r^2+4Rr\right) - 4Rrs \\ &= 2s\left(s^2+r^2+2Rr\right) .\end{aligned}$$

For convenience, let $e_1 = a+b+c$, $e_2 = ab+bc+ca$, and $e_3 = abc$. It follows that the required inequality is successively equivalent to

$$\begin{aligned}(a+b)(b+c)(c+a)e_2 &\geq 8Rrs(5s^2+r^2+4Rr) , \\ (a+b)(b+c)(c+a)e_2 &\geq 2e_3\left(e_1^2+e_2\right) , \\ \left(ab^2+a^2b+bc^2+b^2c+ca^2+c^2a\right)e_2 &\geq 2e_3e_1^2 , \\ a^2b^3+a^3b^2+b^2c^3+b^3c^2+c^2a^3+c^3a^2 &\geq 2a^2b^2c+2a^2bc^2+2ab^2c^2 ,\end{aligned}$$

and finally to

$$a^2(b-c)^2(b+c)+b^2(c-a)^2(c+a)+c^2(a-b)^2(a+b) \geq 0 .$$

The last inequality is obviously true, which completes the proof. Equality holds if and only if $a = b = c$, that is, if and only if the triangle ABC is equilateral.

Also solved by ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, Lebanon; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; THANOS MAGKOS, 3rd High School of Kozani, Kozani, Greece; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; ANDREA MUNARO, student, University of Trento, Trento, Italy; PANOS E. TSAOUSSOGLOU, Athens, Greece; GEORGE TSAPAKIDIS, Agrinio, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

3303. [2008 : 44, 47] *Proposed by Mihály Bencze, Brasov, Romania.*

Let a, b, and c be positive real numbers. Show that

$$\prod_{\text{cyclic}}\left(2(a+b)^3\right) \geq \prod_{\text{cyclic}}\left((a+s_1)(bc+s_2)\right) ,$$

where $s_1 = a+b+c$ and $s_2 = ab+bc+ca$.

Composite of similar solutions by Manuel Benito, Óscar Ciaurri, Emilio Fernández, and Luz Roncal, Logroño, Spain; and Chip Curtis, Missouri Southern State University, Joplin, MO, USA.

By straightforward computations, we have

$$\begin{aligned} & 2(a+b)(b+c)(c+a) \;-\; (a+s_1)(bc+s_2) \\ = \;& 2(a+b)(b+c)(c+a) \;-\; (2a+b+c)(ab+2bc+ca) \\ = \;& 2(a^2b+ab^2+b^2c+bc^2+c^2a+ca^2+2abc) \\ & \;-\; (2a^2b+ab^2+2b^2c+2bc^2+c^2a+2ca^2+6abc) \\ = \;& ab^2+ac^2-2abc \;=\; a(b-c)^2 \;\geq\; 0\,. \end{aligned}$$

Hence,

$$2(a+b)(b+c)(c+a) \;\geq\; (a+s_1)(bc+s_2)\,.$$

Similarly, we have

$$\begin{aligned} 2(a+b)(b+c)(c+a) &\;\geq\; (b+s_1)(ca+s_2)\,; \\ 2(a+b)(b+c)(c+a) &\;\geq\; (c+s_1)(ab+s_2)\,. \end{aligned}$$

The result now follows by multiplying across the last three inequalities.

Also solved by ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; MICHEL BATAILLE, Rouen, France; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; OLIVER GEUPEL, Brühl, NRW, Germany; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; ANDREA MUNARO, student, University of Trento, Trento, Italy; NGUYEN MANH DUNG, High school student, HUS, Hanoi, Vietnam; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; DANIEL TSAI, student, Taipei American School, Taipei, Taiwan; PANOS E. TSAOUSSOGLOU, Athens, Greece; GEORGE TSAPAKIDIS, Agrinio, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

3304. [2008 : 45, 47] *Proposed by Mihály Bencze, Brasov, Romania.*

Let $a_1, a_2, \ldots, a_n$ be positive real numbers and identify a_{n+1} with a_1. Prove that

$$\sum_{k=1}^{n} a_k^3 \;\geq\; \sum_{k=1}^{n} a_k a_{k+1}^2\,.$$

Similar solutions by Michel Bataille, Rouen, France; Walther Janous, Ursulinengymnasium, Innsbruck, Austria; Steven Karp, student, University of Waterloo, Waterloo, ON; Xavier Ros, student, Universitat Politècnica de Catalunya, Barcelona, Spain; and Daniel Tsai, student, Taipei American School, Taipei, Taiwan.

Reorder the numbers $a_1, a_2, \ldots, a_n$ from smallest to largest and rename them $x_1, x_2, \ldots, x_n$. Then $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n$ and if $y_i = x_i^2$

for each i then we also have $0 \le y_1 \le y_2 \le \cdots \le y_n$. The Rearrangement Inequality states that $\sum_{i=1}^{n} x_i y_i \le \sum_{i=1}^{n} x_i y_{\sigma(i)}$ for any permutation σ of $\{1, 2, \ldots, n\}$. Since $\sum_{k=1}^{n} a_k^3 = \sum_{i=1}^{n} x_i y_i$ and $\sum_{k=1}^{n} a_k a_{k+1}^2 = \sum_{i=1}^{n} x_i y_{\sigma(i)}$ for an appropriate permutation σ, the result follows.

Also solved by MOHAMMED AASSILA, Strasbourg, France; ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, Lebanon; MANUEL BENITO, ÓSCAR CIAURRI, EMILIO FERNANDEZ, and LUZ RONCAL, Logroño, Spain; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; OVIDIU FURDUI, University of Toledo, Toledo, OH, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOE HOWARD, Portales, NM, USA; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; DUNG NGUYEN MANH, High School of HUS, Hanoi, Vietnam; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; HENRY RICARDO, Medgar Evers College (CUNY), Brooklyn, NY, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; GEORGE TSAPAKIDIS, Agrinio, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

Ricardo comments that this problem appears as Problem 11.7 on p. 148 of Elementary Inequalities *by D.S. Mitrinović (P. Nordhoff, 1964), but that no solution is provided there.*

3305. [2008 : 45, 47] *Proposed by Stanley Rabinowitz, MathPro Press, Chelmsford, MA, USA.*

Prove that

$$\begin{aligned}\tan\frac{2\pi}{13} + 4\sin\frac{6\pi}{13} &= \tan\frac{4\pi}{13} + 4\sin\frac{\pi}{13}\\ &= \tan\frac{5\pi}{13} + 4\sin\frac{2\pi}{13} = \sqrt{13 + 2\sqrt{13}}\,.\end{aligned}$$

Solution by Manuel Benito, Óscar Ciaurri, Emilio Fernández, and Luz Roncal, Logroño, Spain, in memory of Jim Totten.

We will prove that

$$\begin{aligned}\tan\frac{2\pi}{13} + 4\sin\frac{6\pi}{13} &= \tan\frac{5\pi}{13} + 4\sin\frac{2\pi}{13}\\ &= \tan\frac{6\pi}{13} - 4\sin\frac{5\pi}{13} = \sqrt{13 + 2\sqrt{13}} \qquad (1)\end{aligned}$$

and

$$\begin{aligned}\tan\frac{4\pi}{13} + 4\sin\frac{\pi}{13} &= -\tan\frac{\pi}{13} + 4\sin\frac{3\pi}{13}\\ &= -\tan\frac{3\pi}{13} + 4\sin\frac{4\pi}{13} = \sqrt{13 - 2\sqrt{13}}\,. \qquad (2)\end{aligned}$$

We will make use of two elegant results due to K.F. Gauss and included in the *Sectio VII* of the *Disquisitiones Arithmeticæ* (DA).

Lemma (DA, art. 362, II). Let $n > 1$ be an odd number and $\omega = \dfrac{2k\pi}{n}$, where k is any of the numbers $1, 2, \ldots, n-1$. Then,

$$\tan\omega \;=\; 2\big[\sin(2\omega) - \sin(4\omega) + \sin(6\omega) + \cdots \mp \sin\big((n-1)\omega\big)\big]\,.$$

Theorem (DA, art. 356). Let $n > 1$ be an odd prime number, $\mathfrak{R}$ be the set of the (positive and less than n) quadratic residues modulo n, and $\mathfrak{N}$ be the set of the (positive and less than n) quadratic non-residues modulo n. Then,

$$\sum_{r\in\mathfrak{R}} \cos\frac{2\pi r}{n} - \sum_{m\in\mathfrak{N}} \cos\frac{2\pi m}{n} \;=\; \begin{cases} \sqrt{n} & if \quad n \equiv 1 \pmod 4\ , \\ 0 & if \quad n \equiv 3 \pmod 4\ , \end{cases}$$

and

$$\sum_{r\in\mathfrak{R}} \sin\frac{2\pi r}{n} - \sum_{m\in\mathfrak{N}} \sin\frac{2\pi m}{n} \;=\; \begin{cases} 0 & if \quad n \equiv 1 \pmod 4\ , \\ \sqrt{n} & if \quad n \equiv 3 \pmod 4\ . \end{cases}$$

For $n = 13$ with $\omega = \dfrac{2k\pi}{n}$ and $1 \le k \le 12$ the Lemma yields

$$\tan\omega = 2\big[\sin(2\omega) - \sin(4\omega) + \sin(6\omega) - \sin(8\omega) + \sin(10\omega) - \sin(12\omega)\big]\,.$$

We compute with different values of k in this identity as follows.
If $k = 1$ and $\omega = \dfrac{2\pi}{13}$, then the Lemma yields

$$\tan\frac{2\pi}{13} = 2\left(\sin\frac{4\pi}{13} - \sin\frac{5\pi}{13} + \sin\frac{\pi}{13} + \sin\frac{3\pi}{13} - \sin\frac{6\pi}{13} + \sin\frac{2\pi}{13}\right)\,. \quad (3)$$

If $k = 3$ and $\omega = \dfrac{6\pi}{13}$, then the Lemma yields

$$\tan\frac{6\pi}{13} = 2\left(\sin\frac{\pi}{13} + \sin\frac{2\pi}{13} + \sin\frac{3\pi}{13} + \sin\frac{4\pi}{13} + \sin\frac{5\pi}{13} + \sin\frac{6\pi}{13}\right)\,. \quad (4)$$

If $k = 4$ and $\omega = \dfrac{8\pi}{13}$, then $\tan\dfrac{5\pi}{13} = -\tan\dfrac{8\pi}{13}$ and the Lemma yields

$$\tan\frac{5\pi}{13} = 2\left(\sin\frac{3\pi}{13} + \sin\frac{6\pi}{13} + \sin\frac{4\pi}{13} + \sin\frac{\pi}{13} - \sin\frac{2\pi}{13} - \sin\frac{5\pi}{13}\right)\,. \quad (5)$$

By comparing the equations (3), (4), and (5) we see that the first three expressions in equation (1) are equal.
If $k = 2$ and $\omega = \dfrac{4\pi}{13}$, then the Lemma yields

$$\tan\frac{4\pi}{13} = 2\left(\sin\frac{5\pi}{13} + \sin\frac{3\pi}{13} - \sin\frac{2\pi}{13} - \sin\frac{6\pi}{13} - \sin\frac{\pi}{13} + \sin\frac{4\pi}{13}\right)\,. \quad (6)$$

If $k = 5$ and $\omega = \dfrac{10\pi}{13}$, then $\tan\dfrac{3\pi}{13} = -\tan\dfrac{10\pi}{13}$ and the Lemma yields

$$\tan\frac{3\pi}{13} = 2\left(\sin\frac{6\pi}{13} - \sin\frac{\pi}{13} - \sin\frac{5\pi}{13} + \sin\frac{2\pi}{13} + \sin\frac{4\pi}{13} - \sin\frac{3\pi}{13}\right)\,. \quad (7)$$

If $k = 6$ and $\omega = \dfrac{12\pi}{13}$, then $\tan\dfrac{\pi}{13} = -\tan\dfrac{12\pi}{13}$ and the Lemma yields

$$\tan\frac{\pi}{13} = 2\left(\sin\frac{2\pi}{13} - \sin\frac{4\pi}{13} + \sin\frac{6\pi}{13} - \sin\frac{5\pi}{13} + \sin\frac{3\pi}{13} - \sin\frac{\pi}{13}\right). \quad (8)$$

By comparing the equations (6), (7), and (8) we see that the first three expressions in equation (2) are equal.

Now we take $A = \tan\dfrac{2\pi}{13} + 4\sin\dfrac{6\pi}{13}$ and $B = \tan\dfrac{4\pi}{13} + 4\sin\dfrac{\pi}{13}$. Clearly A and B are positive numbers. From (3) and (6) it follows that

$$A + B = 4\left(\sin\frac{\pi}{13} + \sin\frac{3\pi}{13} + \sin\frac{4\pi}{13}\right)$$

and

$$A - B = 4\left(\sin\frac{2\pi}{13} - \sin\frac{5\pi}{13} + \sin\frac{6\pi}{13}\right).$$

Then,

$$A^2 - B^2 = 16\left(\sin\frac{\pi}{13} + \sin\frac{3\pi}{13} + \sin\frac{4\pi}{13}\right)\left(\sin\frac{2\pi}{13} - \sin\frac{5\pi}{13} + \sin\frac{6\pi}{13}\right).$$

Applying the identity $2\sin a\sin b = \cos(a-b) - \cos(a+b)$, we have

$$A^2 - B^2 = 8\left(\cos\frac{\pi}{13} + \cos\frac{2\pi}{13} + \cos\frac{3\pi}{13} - \cos\frac{4\pi}{13} - \cos\frac{5\pi}{13} + \cos\frac{6\pi}{13}\right).$$

However, for $n = 13 \equiv 1 \pmod 4$, the sets $\mathfrak{R}$ and $\mathfrak{N}$ in the Theorem are $\mathfrak{R} = \{1,4,9,3,12,10\}$ and $\mathfrak{N} = \{2,8,6,11,5,7\}$; thus, by the Theorem

$$2\left(\cos\frac{2\pi}{13} + \cos\frac{6\pi}{13} - \cos\frac{5\pi}{13} + \cos\frac{\pi}{13} + \cos\frac{3\pi}{13} - \cos\frac{4\pi}{13}\right) = \sqrt{13},$$

and therefore

$$A^2 - B^2 = 4\sqrt{13}.$$

Similarly, using the identity $2\sin^2 a = 1 - \cos(2a)$ we deduce that

$$\begin{aligned} AB &= 4\left(\sin\frac{\pi}{13} + \sin\frac{2\pi}{13} + \sin\frac{3\pi}{13} + \sin\frac{4\pi}{13} - \sin\frac{5\pi}{13} + \sin\frac{6\pi}{13}\right) \\ &\quad \times\left(\sin\frac{\pi}{13} - \sin\frac{2\pi}{13} + \sin\frac{3\pi}{13} + \sin\frac{4\pi}{13} + \sin\frac{5\pi}{13} - \sin\frac{6\pi}{13}\right) \\ &= 6\left(\cos\frac{\pi}{13} + \cos\frac{2\pi}{13} + \cos\frac{3\pi}{13} - \cos\frac{4\pi}{13} - \cos\frac{5\pi}{13} + \cos\frac{6\pi}{13}\right) \\ &= \frac{6\sqrt{13}}{2} = 3\sqrt{13}. \end{aligned}$$

For positive real numbers A and B with $A > B$, the solutions of the equations $A^2 - B^2 = 4\sqrt{13}$ and $AB = 3\sqrt{13}$ are

$$A = \sqrt{13 + 2\sqrt{13}} \quad \text{and} \quad B = \sqrt{13 - 2\sqrt{13}}.$$

This completes the proof of the identities (1) and (2). The following similar identities can be deduced when $n = 11$:

$$\begin{aligned}
\tan\frac{\pi}{11} + 4\sin\frac{3\pi}{11} &= -\tan\frac{2\pi}{11} + 4\sin\frac{5\pi}{11} \\
&= \tan\frac{3\pi}{11} + 4\sin\frac{2\pi}{11} = \tan\frac{4\pi}{11} + 4\sin\frac{\pi}{11} \\
&= \tan\frac{5\pi}{11} - 4\sin\frac{4\pi}{11} \\
&= \sqrt{11}\,.
\end{aligned}$$

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece (2 solutions); JOHN HAWKINS and DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; and PETER Y. WOO, Biola University, La Mirada, CA, USA.

All solvers noted that $\tan\frac{4\pi}{13} + 4\sin\frac{\pi}{13} \neq \sqrt{13 + 2\sqrt{13}}$, *as did George Apostolopoulos, Messolonghi, Greece; and Luyan Zhong-Qiao, Columbia International College, Hamilton, ON. Wagon used Mathematica to check that the first and third identities are correct and the second is incorrect. The proposer offered a partially correct solution.*

Woo wondered if similar results hold for $\frac{\pi}{5}$, $\frac{\pi}{7}$, $\frac{\pi}{11}$ *or* $\frac{\pi}{17}$. *For the case of* $\frac{\pi}{11}$ *Benito et al. answered (above) in the affirmative. The interested reader may want to investigate the other cases. Woo also challenges the readers to find geometric proofs for the equalities in (1) and (2).*

3306. [2008 : 45, 47] *Proposed by Stanley Rabinowitz, MathPro Press, Chelmsford, MA, USA.*

Find a real number t and polynomials $f(x)$, $g(x)$, and $h(x)$ with integer coefficients, such that

$$f(t) = \sqrt{2}\,, \qquad g(t) = \sqrt{3}\,, \qquad \text{and} \qquad h(t) = \sqrt{7}\,.$$

Solution by Roy Barbara, Lebanese University, Fanar, Lebanon.

Set $\theta = \sqrt{2} + \sqrt{3} + \sqrt{7}$ and $\psi = \sqrt{2}\sqrt{3}\sqrt{7}$. Computing θ^3, θ^5, and θ^7, we obtain

$$\begin{aligned}
\sqrt{2} + \sqrt{3} + \sqrt{7} &= \theta\,, \\
16\sqrt{2} + 15\sqrt{3} + 11\sqrt{7} + 3\psi &= \frac{1}{2}\theta^3\,, \\
281\sqrt{2} + 241\sqrt{3} + 161\sqrt{7} + 60\psi &= \frac{1}{4}\theta^5\,, \\
4796\sqrt{2} + 3975\sqrt{3} + 2611\sqrt{7} + 1043\psi &= \frac{1}{8}\theta^7\,.
\end{aligned}$$

This is a linear system for $\sqrt{2}$, $\sqrt{3}$, $\sqrt{7}$, and ψ. Solving for $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{7}$

yields

$$\begin{aligned}
\sqrt{2} &= \frac{59}{20}\theta - \frac{1}{2}\theta^3 + \frac{1}{80}\theta^5 ,\\
\sqrt{3} &= \frac{313}{80}\theta - \frac{297}{160}\theta^3 + \frac{67}{320}\theta^5 - \frac{3}{640}\theta^7 ,\\
\sqrt{7} &= -\frac{469}{80}\theta + \frac{377}{160}\theta^3 - \frac{71}{320}\theta^5 + \frac{3}{640}\theta^7 .
\end{aligned}$$

Finally, setting $t = \frac{\theta}{80} = \frac{\sqrt{2}+\sqrt{3}+\sqrt{7}}{80}$ we obtain $\sqrt{2} = f(t)$, $\sqrt{3} = g(t)$, and $\sqrt{7} = h(t)$, where the polynomials $f(x)$, $g(x)$, and $h(x)$ have integer coefficients :

$$\begin{aligned}
f(x) &= 236x - \frac{1}{2}(80)^3x^3 + (80)^4x^5 ,\\
g(x) &= 313x - \frac{1}{2}(80)^2\cdot 297x^3 + \frac{1}{4}(80)^4\cdot 67x^5 - \frac{3}{8}(80)^6x^7 ,\\
h(x) &= -469x + \frac{1}{2}(80)^2\cdot 377x^3 - \frac{1}{4}(80)^4\cdot 71x^5 + \frac{3}{8}(80)^6x^7 .
\end{aligned}$$

Also solved by MOHAMMED AASSILA, Strasbourg, France; BRIAN D. BEASLEY, Presbyterian College, Clinton, SC, USA; MANUEL BENITO, ÓSCAR CIAURRI, EMILIO FERNANDEZ, and LUZ RONCAL, Logroño, Spain; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposer.

3307. [2008 : 45, 47] *Proposed by D.E. Prithwijit, University College Cork, Republic of Ireland.*

Eliminate θ from the system

$$\begin{aligned}
\lambda\cos(2\theta) &= \cos(\theta+\alpha) ,\\
\lambda\sin(2\theta) &= 2\sin(\theta+\alpha) .
\end{aligned}$$

Similar solutions by Manuel Benito, Óscar Ciaurri, Emilio Fernández, and Luz Roncal, Logroño, Spain; Joe Howard, Portales, NM, USA; and George Tsapakidis, Agrinio, Greece.

The given system can be rewritten as a linear system in $\cos\alpha$ and $\sin\alpha$:

$$\begin{aligned}
\cos\theta\cos\alpha - \sin\theta\sin\alpha &= \lambda(\cos^2\theta - \sin^2\theta) ,\\
\sin\theta\cos\alpha + \cos\theta\sin\alpha &= \lambda\sin\theta\cos\theta .
\end{aligned}$$

Its solution is then

$$\cos\alpha = \lambda\cos^3\theta , \quad \text{and} \quad \sin\alpha = \lambda\sin^3\theta ;$$

whence,

$$(\cos\alpha)^{2/3} + (\sin\alpha)^{2/3} \;=\; \lambda^{2/3}\,. \qquad (1)$$

Comments from the Spanish team. Note that the original system has a solution if and only if α and λ satisfy (1). In particular, letting $x = \cos\alpha$ and $y = \sin\alpha$, we see that for $1 \le |\lambda| \le 2$ the solutions can be represented by the intersection points of the unit circle $x^2 + y^2 = 1$ with the astroid $x^{2/3} + y^{2/3} = \lambda^{2/3}$. Thus for each λ with absolute value between 1 and 2, (1) will be satisfied for eight values of α; for $\lambda \in \{\pm 1, \pm 2\}$, it will be satisfied by four values of α. There can be no real solutions for other values of λ.

Also solved by ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; JOSÉ LUIS DÍAZ-BARRERO, Universitat Politècnica de Catalunya, Barcelona, Spain; OLIVER GEUPEL, Brühl, NRW, Germany; JOHN HAWKINS and DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; ANDREA MUNARO, student, University of Trento, Trento, Italy; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; XAVIER ROS, student, Universitat Politècnica de Catalunya, Barcelona, Spain; BOB SERKEY, Leonia, NJ, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer. There was one incorrect submission.

Our readers produced solutions in a variety of formats. Here are a few of the nicest. Instead of (1), Alt, Bataille, and the proposer independently obtained the equivalent equation

$$\sin^2(2\alpha) \;=\; \frac{4\left(\lambda^2-1\right)^3}{27\lambda^2}\,.$$

Geupel found that in terms of a real parameter t, the solutions of the given system satisfy

$$\theta \;=\; \arctan t + m\pi\,, \quad \alpha \;=\; \arctan(t^3) + n\pi\,, \quad \textit{and} \quad \lambda \;=\; (-1)^{m+n}\sqrt{\frac{\left(1+t^2\right)^3}{1+t^6}}\,,$$

for integers m and n. In addition, there were several implicit solutions where the solver simply presented an equation for θ in terms of α or λ; for example, $\theta = \arctan\sqrt[3]{\tan\alpha} + k\pi$ came from Arslanagić and from Ros.

3308. [2008 : 45, 48] *Proposed by D.E. Prithwijit, University College Cork, Republic of Ireland.*

Given $\triangle ABC$, let AD be the altitude to BC. If $AB : AC = 1 : \sqrt{3}$, prove that $AD \le \frac{\sqrt{3}}{2}\,BC$. When does equality hold?

I. *Solution by Joe Howard, Portales, NM, USA.*

It suffices to take $AB = 1$ and $AC = \sqrt{3}$; consequently, $AD = \sin B$. Writing $a = BC$, we therefore must show that

$$\sin B \;\le\; \frac{\sqrt{3}}{2}a\,.$$

We start with the inequality

$$\begin{aligned} 1 \;\geq\; \sin(60^\circ + A) \;&=\; \sin 60^\circ \cos A \;+\; \sin A \cos 60^\circ \\ &=\; \frac{\sqrt{3}}{2}\cos A \;+\; \frac{1}{2}\sin A\,, \end{aligned}$$

which is equivalent to

$$2 - \sqrt{3}\cos A \;\geq\; \sin A\,. \tag{1}$$

By the cosine law, $a^2 = 4 - 2\sqrt{3}\cos A$, so the inequality (1) is equivalent to

$$a^2 \;\geq\; 2\sin A\,. \tag{2}$$

By the sine law, $\dfrac{\sin A}{a} = \dfrac{\sin B}{\sqrt{3}}$, or

$$a \;=\; \frac{\sqrt{3}\sin A}{\sin B}\,,$$

whence (2) is equivalent to

$$a\frac{\sqrt{3}\sin A}{\sin B} \;\geq\; 2\sin A\,,$$

which reduces immediately to what we were to show. Equality occurs when $60^\circ + A = 90^\circ$, in which case $A = 30^\circ$ and $a^2 = 4 - 2\sqrt{3}\cos 30^\circ = 1$. Thus, equality holds if and only if $\triangle ABC$ is isosceles with $A = C = 30^\circ$.

II. *Solution by Michel Bataille, Rouen, France.*

We are given that vertex A belongs to the locus of those points P for which $\dfrac{PB}{PC} = \dfrac{1}{\sqrt{3}}$. We recognize this locus to be a circle Γ called the *circle of Apollonius*; it intersects symmetrically the line joining B to C in points that divide the segment BC internally and externally in the ratio $1 : \sqrt{3}$. As an easy consequence, the centre K of Γ satisfies $\overrightarrow{BK} = -\frac{1}{2}\overrightarrow{BC}$, and its radius is $\dfrac{\sqrt{3}BC}{2}$. Let NN' be the diameter of Γ that is perpendicular to BC. Then N and N' are the points of Γ that are farthest from the line BC, hence

$$AD \;=\; d(A, BC) \;\leq\; d(N, BC) \;=\; \frac{\sqrt{3}BC}{2}\,.$$

Equality holds if and only if A is situated at N or N', in which case $\triangle ABC$ is isosceles with $BA = BC$ and $\angle ABC = 120^\circ$.

Also solved by MOHAMMED AASSILA, Strasbourg, France; ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, Lebanon; MANUEL BENITO, ÓSCAR CIAURRI, EMILIO FERNANDEZ, and LUZ RONCAL, Logroño, Spain; CHIP CURTIS, Missouri Southern State University, Joplin, *MO, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; IAN JUNE L. GARCES, Ateneo de Manila University, Quezon City, The Philippines; FRANCISCO JAVIER GARCÍA CAPITÁN, IES Álvarez Cubero, Priego de Córdoba, Spain; OLIVER GEUPEL, Brühl, NRW, Germany; STEVEN KARP, student, University of Waterloo, Waterloo, ON; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; KEE-WAI LAU, Hong Kong, China; THANOS MAGKOS, 3rd High School of Kozani, Kozani, Greece; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; ANDREA MUNARO, student, University of Trento, Trento, Italy; JOEL SCHLOSBERG, Bayside, NY, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; GEORGE TSAPAKIDIS, Agrinio, Greece; DANIEL TSAI, student, Taipei American School, Taipei, Taiwan; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.*

More generally, Konečný proved that $AD \leq \dfrac{q}{q^2-1}BC$ *if* $AC : AB = q$ *with* $q > 1$. *His argument was much like that of solution II above.*

3309. [2008 : 45, 48] *Proposed by Virgil Nicula, Bucharest, Romania.*

Let α, β, and γ be fixed non-zero real numbers. Show that the system

$$\begin{aligned} \alpha x + \beta y + \gamma z &= 1\,, \\ xy + yz + zx &= 1\,, \end{aligned}$$

has a unique solution for (x, y, z) if and only if

$$\alpha^2 + \beta^2 + \gamma^2 + 1 = 2(\alpha\beta + \beta\gamma + \gamma\alpha)\,,$$

and, in this case find that unique solution.

Solution by George Tsapakidis, Agrinio, Greece, modified by the editor.

Substituting $\alpha x = 1 - \beta y - \gamma z$ into $\alpha(xy + yz + zx) = \alpha$ we obtain $(y+z)(1-\beta y-\gamma z)+\alpha yz = \alpha$, which upon simplifying yields the following quadratic equation in y

$$\beta y^2 - \big[1 + (\alpha - \beta - \gamma)z\big]y + \gamma z^2 - z + \alpha = 0\,. \tag{1}$$

Equation (1) has a unique solution in y if and only if

$$[1 + (\alpha - \beta - \gamma)z]^2 - 4\beta(\gamma z^2 - z + \alpha) = 0\,,$$

which can be written as the following quadratic equation in z

$$\big[(\alpha - \beta - \gamma)^2 - 4\beta\gamma\big]\, z^2 \;+\; 2(\alpha + \beta - \gamma)z + 1 - 4\alpha\beta = 0\,. \tag{2}$$

Equation (2) has a unique solution in z if and only if

$$(\alpha + \beta - \gamma)^2 - \big[(\alpha - \beta - \gamma)^2 - 4\beta\gamma\big]\,(1 - 4\alpha\beta) = 0\,,$$

which, by straightforward computations, reduces to

$$\alpha^2 + \beta^2 + \gamma^2 + 1 = 2(\alpha\beta + \beta\gamma + \gamma\alpha)\,. \tag{3}$$

Now, suppose (3) holds. Then we have

$$(\alpha-\beta-\gamma)^2-4\beta\gamma \;=\; \alpha^2+\beta^2+\gamma^2-2\alpha\beta-2\beta\gamma-2\gamma\alpha \;=\; -1\,.$$

Thus, (2) reduces to $-z^2+2(\alpha+\beta-\gamma)z+1-4\alpha\beta=0$, the unique solution of which is given by $z=\alpha+\beta-\gamma$.

Using this and (3), we find that

$$\begin{aligned} y &= \frac{1+(\alpha-\beta-\gamma)(\alpha+\beta-\gamma)}{2\beta} = \frac{1+(\alpha-\gamma)^2-\beta^2}{2\beta} \\ &= \frac{1+\alpha^2-\beta^2+\gamma^2-2\alpha\gamma}{2\beta} = \frac{-2\beta^2+2\alpha\beta+2\beta\gamma}{2\beta} = \gamma+\alpha-\beta\,. \end{aligned}$$

Finally, using (3) we have

$$\begin{aligned} \alpha x &= 1-\beta y-\gamma z = 1-\beta(\gamma+\alpha-\beta)-\gamma(\alpha+\beta-\gamma) \\ &= -\alpha^2+\alpha\beta+\gamma\alpha\,, \end{aligned}$$

from which it follows that $x=\beta+\gamma-\alpha$.

Therefore, the unique solution to the given system is

$$(x,y,z) \;=\; (\beta+\gamma-\alpha,\; \gamma+\alpha-\beta,\; \alpha+\beta-\gamma)\,.$$

Also solved by ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; MANUEL BENITO, ÓSCAR CIAURRI, EMILIO FERNANDEZ, and LUZ RONCAL, Logroño, Spain; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; STEVEN KARP, student, University of Waterloo, Waterloo, ON; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; DANIEL TSAI, student, Taipei American School, Taipei, Taiwan; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer. There was also one partially incorrect solution submitted.

3310. [2008 : 46, 48] *Proposed by Virgil Nicula, Bucharest, Romania.*

Let a, b, and c denote, as usual, the lengths of the sides BC, CA, and AB, respectively, in $\triangle ABC$. Let s be the semiperimeter of $\triangle ABC$, r the inradius, h_a the altitude to side BC, and r_a, r_b, and r_c the exradii to A, B, and C, respectively.

(a) Show that for $x>0$, we have $h_a=\dfrac{2s(s-a)x}{x^2+s(s-a)}$ if and only if $x=r_b$ or $x=r_c$.

(b) Show that for $x>0$, we have $h_a=\dfrac{2(s-b)(s-c)x}{|x^2-(s-b)(s-c)|}$ if and only if $x=r$ or $x=r_a$.

Similar solutions by Michel Bataille, Rouen, France; Oliver Geupel, Brühl, NRW, Germany; Titu Zvonaru, Comănești, Romania; and the proposer.

It is well known that the area F of $\triangle ABC$ can be variously expressed as $\frac{1}{2}ah_a$, rs, $r_a(s-a)$, $r_b(s-b)$, $r_c(s-c)$, or $\sqrt{s(s-a)(s-b)(s-c)}$. We have

$$\begin{aligned} r_b + r_c &= \frac{F}{s-b} + \frac{F}{s-c} = \frac{F(2s-b-c)}{(s-b)(s-c)} \\ &= \frac{a\cdot F\cdot(s-a)}{s(s-a)(s-b)(s-c)} = \frac{as(s-a)}{F} \\ &= \frac{2as(s-a)}{ah_a} = \frac{2s(s-a)}{h_a} \qquad (1) \end{aligned}$$

and

$$r_b r_c = \frac{F^2}{(s-b)(s-c)} = \frac{s(s-a)(s-b)(s-c)}{(s-b)(s-c)} = s(s-a)\,. \qquad (2)$$

It follows from (1) and (2) that $x = r_b$ and $x = r_c$ are the solutions of

$$h_a x^2 - 2s(s-a)x + h_a s(s-a) = 0\,,$$

hence these are the solutions of the equation in (a).

We also have

$$\begin{aligned} r_a - r &= \frac{F}{s-a} - \frac{F}{s} = \frac{aF}{s(s-a)} \\ &= \frac{a\cdot F\cdot(s-b)(s-c)}{F^2} = \frac{2(s-b)(s-c)}{h_a} \qquad (3) \end{aligned}$$

and

$$r_a r = \frac{F^2}{s(s-a)} = (s-b)(s-c)\,. \qquad (4)$$

By (3) and (4) it follows that the equation

$$h_a x^2 - 2(s-b)(s-c)x - (s-b)(s-a)h_a = 0$$

has the solutions $x = r_a$ and $x = -r$ and that the equation

$$h_a x^2 + 2(s-b)(s-c)x - (s-b)(s-a)h_a = 0$$

has the solutions $x = -r_a$ and $x = r$. Thus, the only positive solutions of the preceding two equations are r and r_a, and it follows that these are the only positive solutions to the equation in (b).

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; MANUEL BENITO, ÓSCAR CIAURRI, EMILIO FERNANDEZ, and LUZ RONCAL, Logroño, Spain; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; GEORGE TSAPAKIDIS, Agrinio, Greece; and PETER Y. WOO, Biola University, La Mirada, CA, USA.

3311. [2008 : 46, 48] *Proposed by Michel Bataille, Rouen, France.*

Let n be an integer with $n \geq 2$. Suppose that for $k = 0, 1, \ldots, n-2$ we have

$$\binom{n-2}{k} \equiv (-1)^k(k+1) \pmod{n} .$$

Show that n is a prime.

Solution by Oliver Geupel, Brühl, NRW, Germany.

It is sufficient to prove that if n is a composite integer with $n \geq 2$, then there exists an integer k with $0 \leq k \leq n-2$ such that

$$\binom{n-2}{k} \not\equiv (-1)^k(k+1) \pmod{n} . \qquad (1)$$

Toward that end, let p be the least prime factor of n. If (1) holds for some $k < p$, then we are done. Otherwise the given congruence holds in particular for $k = p-2$, and we have

$$\begin{aligned} (p-1)\binom{n-2}{p} &= \binom{n-2}{p-2} \cdot \frac{n-p}{p} \cdot (n-p-1) \\ &\equiv (-1)^{p-2}(p-1) \cdot \left(\frac{n}{p} - 1\right) \cdot (-p-1) \\ &\equiv (p-1) \cdot (-1)^p \left(p + 1 - \frac{n}{p}\right) \pmod{n} . \end{aligned}$$

Because $p-1$ is coprime to the modulus n, we can divide both sides by it and conclude that (1) holds with $k = p$:

$$\binom{n-2}{p} \equiv (-1)^p \left(p + 1 - \frac{n}{p}\right) \not\equiv (-1)^p(p+1) \pmod{n} .$$

Also solved by MANUEL BENITO, ÓSCAR CIAURRI, EMILIO FERNANDEZ, and LUZ RONCAL, Logroño, Spain; JOHN HAWKINS and DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; STEVEN KARP, student, University of Waterloo, Waterloo, ON; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; JOEL SCHLOSBERG, Bayside, NY, USA; and the proposer.

The converse is a known result : If p is a prime, then $\binom{p-2}{k} \equiv (-1)^k(k+1) \pmod{p}$ for $k = 0, 1, \ldots, p-2$. *Karp provided a simple proof (by induction on k); Bataille provided a reference : E. Lucas,* Théorie des nombres, *A. Blanchard (1961), p. 420.*

3312. [2008 : 46, 48] *Proposed by Michel Bataille, Rouen, France.*

Let n be a positive integer congruent to 1 modulo 6. Show that $3/n$ can be expressed as

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k}$$

for some distinct positive integers $a_1, a_2, \ldots, a_k$, and find the minimal value of k.

Solution by Steven Karp, student, University of Waterloo, Waterloo, ON.

We solve the problem for all positive integers n. If $n \equiv 0$ or $3 \pmod 6$, then $\frac{3}{n} = \frac{1}{n/3}$. If $n \equiv -1 \pmod 6$, then

$$\frac{3}{n} = \frac{1}{\frac{n+1}{3}} + \frac{1}{\frac{n(n+1)}{3}}.$$

If $n \equiv \pm 2 \pmod 6$, then

$$\frac{3}{n} = \frac{1}{\frac{n}{2}} + \frac{1}{n}.$$

If $n \equiv 1 \pmod 6$ and n has a factor c such that $c \equiv -1 \pmod 6$, then

$$\frac{3}{n} = \frac{1}{\frac{n+c}{3}} + \frac{1}{\frac{n(n+c)}{3c}}.$$

We see that k is minimal in all of these cases, since $\frac{3}{n} = \frac{1}{a_1}$ for some positive integer a_1 only if $n \equiv 0 \pmod 3$. Now, if $n > 1$, $n \equiv 1 \pmod 6$ and all divisors of n are congruent to 1 modulo 6, then

$$\frac{3}{n} = \frac{1}{\frac{n+1}{2}} + \frac{1}{n} + \frac{1}{\frac{n(n+1)}{2}},$$

and we claim that $k = 3$ is minimal in this case. To prove this suppose for the sake of contradiction that

$$\frac{3}{n} = \frac{1}{a} + \frac{1}{b}$$

for distinct positive integers a and b. Then

$$n = \frac{b(3a-n)}{a},$$

and we have $a|b(3a-n)$. Therefore, $a = pq$ where p and q are positive integers such that $p|b$ and $q|(3a-n)$. Then $\frac{b}{p}|n$, whence $\frac{b}{p} \equiv 1 \pmod 6$. Since $q|a$ and $q|(3a-n)$, we also have $q|n$, so $q \equiv 1 \pmod 6$. We now have

$$1 \equiv qn = \left(\frac{b}{p}\right)(3a-n) \equiv 3a - n \equiv -1 \text{ or } 2 \pmod 6,$$

a contradiction.

Finally, for $n = 1$ we have

$$3 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{15} + \frac{1}{230} + \frac{1}{57960}.$$

A computer search shows that $k = 13$ is minimal in this case.

Also solved by ROY BARBARA, Lebanese University, Fanar, Lebanon; MANUEL BENITO, ÓSCAR CIAURRI, EMILIO FERNANDEZ, and LUZ RONCAL, Logroño, Spain; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; JOHN HAWKINS and DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; JOEL SCHLOSBERG, Bayside, NY, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

Benito et al. refer to the note by Thomas R. Hagedorn, A proof of a conjecture on Egyptian fractions, Amer. Math. Monthly, **107** *(2000) 62-63, where it is proved that for each odd integer* $n \geq 3$ *not divisible by* 3 *there exist distinct odd, positive integers* a, b, *and* c *such that*

$$\frac{3}{n} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}\,.$$

The result had been conjectured by R. Hardin and N. Sloane. When $n = 6p+1$ *Hagedorn gives the decomposition*

$$\frac{3}{n} = \frac{3}{6p+1} = \frac{1}{2p+1} + \frac{1}{(2p+1)(4p+1)} + \frac{1}{(4p+1)(6p+1)}\,.$$

Janous refers to a paper by Andrzej Schinzel, Sur quelques propriétés des nombres $3/n$ *et* $4/n$, *où n est un nombre impair,* Mathesis *65 (1956) 219-222, for a treatment of our problem. Since the improper fraction* $3 = 3/1$ *arises, he asks if any* ***CRUX*** *readers know the minimum number* $\ell(n)$ *of distinct Egyptian fractions needed to represent the positive integer* n.

A Happy New Year to all ***CRUX with MAYHEM*** readers. The Jim Totten special issue was slated to be completed in May of this year, but due to delays we are now going to release the special issue in September 2009 instead.

This year we plan on improving our database of names and affiliations of you, the readers. If your name does not look quite right, for example, if the accents are not quite right or your family name is incorrect, etc., then please let us know and we will update our files. Many international readers subscribe to ***CRUX with MAYHEM*** and we want to get these (fascinating !) details right.

Václav Linek.