SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.


A sequence $\{a_n\}_{n=0}^{\infty}$ of positive real numbers satisfies the recurrence relation $a_{n+3} = a_{n+1} + a_n$ for $n \geq 0$. Simplify

$$\sqrt{a_{n+5}^2 + a_{n+4}^2 + a_{n+3}^2 - a_{n+2}^2 + a_{n+1}^2 - a_n^2}.$$ 

Solution submitted independently by Arkady Alt, San Jose, CA, USA; Michel Bataille, Rouen, France; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Thanos Magkos, 3rd High School of Kozani, Kozani, Greece.

Clearly, all the terms of the given sequence are positive. Set

$$A = \sqrt{a_{n+5}^2 + a_{n+4}^2 + a_{n+3}^2 - a_{n+2}^2 + a_{n+1}^2 - a_n^2}.$$ 

By the recurrence relation, $a_{n+3} = a_{n+1} + a_n$, $a_{n+4} = a_{n+2} + a_{n+1}$, and $a_{n+5} = a_{n+3} + a_{n+2} = a_{n+1} + a_n + a_{n+1}$, so that

$$a_{n+5}^2 + a_{n+4}^2 + a_{n+3}^2 = (a_{n+2} + a_{n+1} + a_n)^2 + (a_{n+2} + a_{n+1})^2 + (a_{n+1} + a_n)^2 = 2a_{n+2}^2 + 3a_{n+1}^2 + 2a_n^2 + 4a_{n+2}a_{n+1} + 2a_{n+2}a_n + 4a_{n+1}a_n.$$ 

Thus,

$$A^2 = a_{n+2}^2 + 4a_{n+1}^2 + a_n^2 + 4a_{n+2}a_{n+1} + 2a_{n+2}a_n + 4a_{n+1}a_n = (a_{n+2} + 2a_{n+1} + a_n)^2.$$ 

Therefore,

$$A = a_{n+2} + 2a_{n+1} + a_n = (a_{n+2} + a_{n+1}) + (a_{n+1} + a_n) = a_{n+4} + a_{n+3} = a_{n+6}.$$ 

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIC, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; DIONNE BAILEY, ELsie CAMPBELL, CHARLES DIMINNIE, KARL HAVLAK and PAULA KOCA, Angelo State University, San Angelo, TX, USA; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; OLIVER GEUPEL, Brühl, NRW, Germany; RICHARD I. HESS.
3277. [2007 : 428, 430] Proposed by Ovidiu Furdui, student, Western Michigan University, Kalamazoo, MI, USA.

The Lucas numbers \( L_n \) satisfy the recurrence relation \( L_0 = 2, L_1 = 1, \) and \( L_{n+2} = L_{n+1} + L_n \) for \( n \geq 0. \) Let \( k \) be an even positive integer. Find

\[
\lim_{n \to \infty} \left( \{ \sqrt[k]{L_n} \} - \{ \sqrt[k]{L_{n-k}} + \sqrt[k]{L_{n-2k}} \} \right),
\]

where \( \{x\} \) is the fractional part of \( x \) (that is, \( \{x\} = x - \lfloor x \rfloor \), where \( \lfloor x \rfloor \) is the integer part of \( x \)).

Comment: All three submissions claimed that the limit is 0, but no satisfactory argument was provided that the limit exists for every positive even integer \( k \). Problem 3277, therefore, remains open.


Let \( P \) be a point in the plane of \( \triangle ABC \) such that \( PC = PB \) and \( PA = AB \). Let \( x \) be the measure of \( \angle PBC \). Prove that

\[
\sin(B - C) = 2 \sin C \cos(B + 2x),
\]

where \( \varepsilon = 1 \) if the line \( BC \) separates the points \( P \) and \( A \), and \( \varepsilon = -1 \) otherwise.

Solution by Michel Bataille, Rouen, France.

Let \( AB = c, BC = a, \) and \( CA = b \). First, suppose that \( BC \) separates \( P \) and \( A \) (see the figure on the left).
Then, \( \cos x = \frac{a/2}{PB} \) and \( \cos(x + B) = \cos(\angle PBA) = \frac{PB/2}{c} \), so that

\[
2 \cos x \cos(x+B) = \frac{a}{2c} = \frac{\sin A}{2\sin C}.
\]

This yields \( \cos(2x+B)+\cos B = \frac{\sin A}{2\sin C} \)

and, since \( \sin A = \sin(B + C) = \sin B \cos C + \sin C \cos B \), we obtain

\[
2 \sin C \cos(B + 2x) = \sin B \cos C - \sin C \cos B = \sin(B - C),
\]
as desired.

If \( P \) and \( A \) are on the same side of \( BC \), then we have \( \cos x = \frac{a/2}{PB} \) and

\( \angle PBA = x - B \) or \( \angle PBA = B - x \), depending on the location of point \( P \).

[Ed.: The cases are (a) \( P \) outside \( \triangle ABC \) and \( A \) inside \( \triangle PBC \) (figure on the left, \( P' \) replacing \( P \)), (b) \( P \) outside \( \triangle ABC \) and \( A \) outside \( \triangle PBC \) (figure on the right), and (c) \( P \) inside \( \triangle ABC \) (figure on the right, \( P' \) replacing \( P \)).] In any case, \( \frac{PB/2}{c} = \cos(x - B) \), and, in the same way as above, we obtain

\[
2 \cos x \cos(x-B) = \frac{a}{2c},
\]
which leads to \( 2 \sin C \cos(B - 2x) = \sin(B - C) \).

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ROY BARBARA, Lebanese University, Fanar, Lebanon; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; ANDREA MUNARO, student, University di Trento, Trento, Italy; JOEL SCHLOSBERG, Bayside, NY, USA; D.J. SNEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.


Let \( O, I, R, \) and \( r \) be the circumcentre, incentre, circumradius, and inradius of \( \triangle ABC \), and let \( a, b, \) and \( c \) be the lengths of the sides of \( \triangle ABC \) opposite the angles \( A, B, \) and \( C \), respectively. Let \( IO \) meet the lines \( AB \)
and \( AC \) at \( M \) and \( N \), respectively. Prove that the points \( B, C, \) \( N, \) and \( M \) are concyclic if and only if \( h_a = R + r \) (where \( h_a \) is the altitude to the side \( BC \)), and, in this case, we also have \( \frac{1}{MN} = \frac{1}{a} + \frac{1}{b+c} \).

A composite of similar solutions by Taichi Maekawa, Takatsuki City, Osaka, Japan and D.J. Sneenk, Zaltbommel, the Netherlands.

For an arbitrary triangle \( ABC \) let \( D \) be the foot of the altitude from \( A \) to \( BC \) and let \( P \) be the foot of the perpendicular from \( I \) to \( AD \); thus \( PD = r \). Moreover, it is always the case that \( AI \) bisects \( \angle DAO \); that is, \( \angle DAI = \angle IAO \). Because \( h_a = AP + PD = AP + r \), we therefore have

\[
h_a = R + r \iff AO = AP
\]

\[
\iff \angle API \cong \angle AOI
\]

\[
\iff \angle AOI = 90^\circ.
\]

The first part of the problem therefore reduces to proving that

\[
B, C, N, \text{ and } M \text{ lie on a circle } \iff IO \perp AO.
\]
This is not quite correct, however: the four cyclic points must be distinct to force a nontrivial condition. To that end we will assume the equivalent condition that no two angles of \( \triangle ABC \) are equal.

Under this assumption, we have that (because \( M \in AB \) and \( N \in AC \)) \( B, C, N, \) and \( M \) lie on a circle if and only if \( \angle ANM = \angle ABC, \) if and only if \( \angle ANM \) equals the angle between the chord \( AC \) and the tangent to the circumcircle at \( A \) (on the side that contains the arc \( AC \) opposite \( B \)), if and only if \( MN \) is parallel to that tangent, if and only if \( AO \perp MN. \)

Since \( MN \) is the same line as \( IO, \) the proof of the first part is complete.

For the claim concerning \( \frac{1}{MN}, \) we assume that \( B, C, N, \) and \( M \) lie on a circle, in which case \( \angle ANM = \angle B \) and \( \angle AMN = \angle C. \) Let \( E \) and \( F \) be the feet of the perpendiculars from \( I \) to \( AC \) and \( AB, \) respectively. Then in the right triangles \( INE \) and \( IMF \) we have

\[
NI = \frac{r}{\sin \beta} \quad \text{and} \quad IM = \frac{r}{\sin \gamma}.
\]

Hence, \( MN = MI + IN = \frac{r}{\sin \beta} + \frac{r}{\sin \gamma}, \) and the Law of Sines gives us

\[
MN = r \left( \frac{2R}{c} + \frac{2R}{b} \right) = 2rR \sin A \left( \frac{b + c}{bc \sin A} \right) = \frac{ar(b + c)}{2 \text{Area}(ABC)} = \frac{ar(b + c)}{r(a + b + c)}.
\]

Thus,

\[
\frac{1}{MN} = \frac{a + b + c}{a(b + c)} = \frac{1}{a} + \frac{1}{b + c},
\]

as claimed.

Also solved by RICARDO BARROSO CAMPOS, University of Seville, Seville, Spain; MICHEL BATAILLE, Rouen, France; OLIVER GEUPEL, Brühl, NRW, Germany; PETERY, WOO, Biola University, La Mirada, CA, USA; and the proposer.

Both Geu pel and the proposer avoid appealing to the Law of Sines to prove the final claim as follows: The triangles \( ABC \) and \( ANM \) are assumed to be similar. Since the angle bisector \( AI \) is common to both triangles, if \( W \) is the point where \( AI \) meets \( BC \) we have

\[
\frac{AI}{IW} = \frac{MN}{CB}.
\]

Since \( CB = a \) and it is known that \( \frac{AI}{IW} = \frac{b + c}{a + b + c} \) (see, for example, Nathan Altshiller Court, College Geometry, page 75, Theorem 121), it follows that

\[
MN = \frac{a(b + c)}{a + b + c},
\]

which is the reciprocal of the desired equality.

Let $O$ and $R$ be the circumcentre and circumradius, respectively, of $\triangle ABC$. Let $E$ and $F$ be points on $AB$ and $AC$, respectively, such that $O$ is the mid-point of segment $EF$. Let $A'$ be the point where the line $AO$ meets the circumcircle $\Gamma$ of $\triangle ABC$ a second time, and let $P$ be the point on the line $EF$ such that $A'P \perp EF$. Prove that the lines $EF, BC,$ and the tangent line to $\Gamma$ at $A'$ are concurrent, and that $\angle BPA' = \angle CPA'$.

A composite of solutions by Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina and by Andrea Munaro, student, University of Trento, Trento, Italy.

One sees that $E$ is uniquely defined as the point where $AB$ intersects the image of the line $AC$ under the halfturn about $O$, while $F$ is the intersection of $AC$ with the image of $AB$ under that halfturn. More relevant for us, however, is that because $A$ and $A'$ are interchanged by that halfturn it follows that $A'E||AF$ and $A'F||AE$. Because $A'P \perp PF$ (given) and $A'C \perp FC$ (because $A'A$ is a diameter of the circumcircle $\Gamma$ of $\triangle ABC$), the quadrilateral $A'PFC$ is cyclic, whence the directed angles satisfy $\angle A'PC = \angle A'FC$. The latter angle equals $\angle BAC$ (because their sides are parallel), so that

$$\angle A'PC = \angle A.$$  

Analogously, $A'BEP$ is cyclic and

$$\angle BPA' = \angle BEA' = \angle A.$$  

Consequently,  

$$\angle BPA' = \angle A'PC = \angle A,$$  

which is the second claim that we were to prove.

For the concurrency claim, we will prove that $EF, BC,$ and the tangent to $\Gamma$ at $A'$ all contain the image of $P$ under the inversion defined by $\Gamma$. We note that $\angle BOC = 2\angle A$, and we have just seen that $\angle BPC = 2\angle A$; we therefore conclude that the points $B, P, O,$ and $C$ lie on a circle. Inversion in $\Gamma$ fixes the line $EF$ (because that line passes through the centre $O$ of $\Gamma$), it takes line $BC$ to circle $BOC$ (which, we have seen, contains $P$), and it takes the tangent (to $\Gamma$) at $A'$ to the circle on diameter $OA'$ (which contains $P$ because $A'P \perp PO$). We therefore see that the images (under inversion) of the three lines contain $P$, so that these three lines themselves must concur at the inverse of $P$, as claimed. [Editor's comment. Malikić contributed the nice treatment of angles; it was Munaro's idea to invert the figure.]

Also solved by MICHEL BATAILLE, Rouen, France; PETER Y. Woo, Biola University, La Mirada, CA, USA; and the proposer. There was one incorrect submission.
Let $a_1, a_2, \ldots, a_n$ be positive real numbers. Prove that

$$\left( \sum_{k=1}^{n} a_{k}^{\frac{n+1}{k}} \right)^n \leq \prod_{k=1}^{n} \left( \sum_{j=1}^{n} a_{j}^{k} \right).$$

1. Solution by Michel Bataille, Rouen, France.

Let $A_k = \sum_{j=1}^{n} a_j^k$ and let $P = \prod_{k=1}^{n} A_k$ denote the right side of the given inequality. Then $P^2 = (A_1 A_n)(A_2 A_{n-1}) \cdots (A_n A_1)$. For each $k$ we have, by the Cauchy-Schwarz Inequality, that

$$A_k A_{n+1-k} = \left( \sum_{j=1}^{n} a_j^k \right) \left( \sum_{j=1}^{n} a_j^{n+1-k} \right) \geq \left( \sum_{j=1}^{n} a_j^{\frac{k}{2}} a_j^{\frac{n+1-k}{2}} \right)^2 = \left( \sum_{j=1}^{n} a_j^{\frac{n+1}{2}} \right)^2.$$ 

Hence, $P^2 \geq \left( \sum_{j=1}^{n} a_j^{\frac{n+1}{2}} \right)^{2n}$, from which the result follows.

II. Solution by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam.

Note first that $\frac{n(n+1)}{2} = 1 + 2 + \cdots + n$, and let $b_{k\ell} = \sqrt[n]{a_k^\ell}$ for each $1 \leq k, \ell \leq n$. For each $k$ we have $a_k^{\frac{n+1}{2}} = \left( a_k^{1+2+\cdots+n} \right)^{1/n} = b_{k1} b_{k2} \cdots b_{kn}$.

Therefore, by the generalized Hölder Inequality, we have

$$\sum_{k=1}^{n} a_k^{\frac{n+1}{2}} = \sum_{k=1}^{n} b_{k1} b_{k2} \cdots b_{kn}$$

$$\leq \left( \sum_{k=1}^{n} b_{k1}^n \right)^{1/n} \left( \sum_{k=1}^{n} b_{k2}^n \right)^{1/n} \cdots \left( \sum_{k=1}^{n} b_{k\ell}^n \right)^{1/n}$$

$$= \left( \sum_{k=1}^{n} a_k^n \right)^{1/n} \left( \sum_{k=1}^{n} a_k^2 \right)^{1/n} \cdots \left( \sum_{k=1}^{n} a_k^n \right)^{1/n}$$

$$= \prod_{k=1}^{n} \left( \sum_{j=1}^{n} a_j^k \right)^{1/n},$$

from which the given inequality follows.
Equality holds if and only if $a_1 = a_2 = \cdots = a_n$.

Also solved by MOHAMMED AASSILA, Strasbourg, France; ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIC, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brün, NRW, Germany; JOE HOWARD, Portales, NM, USA; SALEM MALIKIC, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina, DUNG NGUYEN MANH, High School of HUS, Hanoi, Vietnam; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; XAVIER ROS, student, Universitat Politècnica de Catalunya, Barcelona, Spain; PETER Y. WOO, Bierz University, La Mirada, CA, USA; BINGJIE WU, student, High School Affiliated to Fudan University, Shanghai, China; TITU ZVONARU, Comănești, Romania; and the proposers.

Wu proved a generalization: 
\[
\left( \sum_{i=1}^{n} a_{ij} \right)^m \leq \left( \sum_{i=1}^{n} \left( \frac{n}{m} \right) \right)^{m} \text{ whenever } a_{ij} \text{ is positive for } 1 \leq i \leq n \text{ and } 1 \leq j \leq m.
\]

The proposed inequality is the special case when $m = n$ and $a_{ij} = a_j^n$.


Of the $n!$ permutations $\sigma$ of $(1, 2, \ldots, n)$, for how many is $\sigma^3$ the identity permutation?

A composite of solutions by Michel Bataille, Rouen, France and Richard I. Hess, Rancho Palos Verdes, CA, USA.

The permutation $\sigma^3$ is the identity if and only if $\sigma$ is itself the identity or a product of disjoint 3-cycles. We will investigate the case $n = 8$ explicitly and then generalize.

<table>
<thead>
<tr>
<th>Cycle type</th>
<th>Total number of permutations</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1)(2)(3)\cdots(8)$</td>
<td>$1$</td>
</tr>
<tr>
<td>$(1, 2, 3)(4)(5)(6)(7)(8)$</td>
<td>$2(\binom{8}{3}) = 112$</td>
</tr>
<tr>
<td>$(1, 2, 3)(4, 5, 6)(7)(8)$</td>
<td>$\frac{1}{2} \left( 2(\binom{8}{3}) \right) = 1120$</td>
</tr>
</tbody>
</table>

Thus, for $n = 8$ the desired number is $1 + 112 + 1120 = 1233$.

In general, to find the number of permutations of $n$ elements into $k$ 3-cycles, we choose the first 3-cycle in $\binom{n}{3}$ ways, the next in $\binom{n-3}{3}$ ways, and so on until the $k^{th}$ 3-cycle, chosen in $\binom{n-3(k-1)}{3}$ ways. The product of these binomial coefficients is 
\[
\binom{n}{3} \binom{n-3}{3} \cdots \binom{n-3(k-1)}{3} = \frac{n!}{(n-3k)(3!)^k}.
\]

There are two possible cyclic permutations of each triple (such as $(1, 2, 3)$ and $(1, 3, 2)$ for the triple $\{1, 2, 3\}$), which means that we must multiply the above product by $2^k$; moreover, these $k$ 3-cycles can be chosen in $k!$ orders, so that we must divide the product by $k!$. We therefore have
Cycle type  | Total number of permutations
---|---
no 3-cycle | \[\frac{n!}{0! (n-0)! (3!)^0} = \frac{n!}{3^0! (n-0)!} = 1\]
1 3-cycle | \[\frac{n!}{1! (n-3)! (3!)^1} = \frac{n!}{3^1! (n-3)!}\]
2 3-cycles | \[\frac{n!}{2! (n-6)! (3!)^2} = \frac{n!}{3^2! (n-6)!}\]
\vdots
k 3-cycles | \[\frac{n!}{k! (n-3k)! (3!)^k} = \frac{n!}{3^k! (n-3k)!}\]

Of course, \(k\) can be any number from 0 to \(\lfloor \frac{n}{3} \rfloor\). It follows that the total number of permutations for which \(\sigma^3\) is the identity is

\[
\sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} \frac{n!}{3^k! (n-3k)!}
\]

The first eight values are 1, 1, 3, 9, 21, 81, 351, 1233.

Also solved by MOHAMMED AASSILA, Strasbourg, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Braith, NRW, Germany; and JOEL SCHOLOBERG, Bayside, NY, USA. The \(m\) were three incorrect submissions.

If we denote by \(P(n)\) the number of permutations for which \(\sigma^3\) is the identity, then alternative expressions obtained by our correspondent are

\[
P(n) = \sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} \frac{n!}{3^k! (n-3k)!} = 1 + \sum_{k=1}^{\lfloor \frac{n}{3} \rfloor} \frac{n(n-1) \cdots (n-3k+1)}{3^k!}
\]

Using an argument analogous to our featured solution, Geupel proved that for any positive integer \(n\) and any prime \(p\), the number \(P(n, p)\) of permutations \(\sigma\) of \(1, 2, \ldots, n\) such that \(\sigma^p\) is the identity is

\[
P(n, p) = \sum_{k=0}^{\lfloor \frac{n}{p} \rfloor} \frac{n!}{k! p^k}.
\]


Our sequence \(P(n)\) can be found in the on-line encyclopedia of integer sequences: http://www.research.att.com/~njas/sequences/ (entry 1, 1, 3, 9, 21, 81, 351). That web page provides several references, one of which gives (without proof) the formula of Chowla, Herstein, and Scott (from 1952) for the number \(P(n, m)\) of permutations for which \(\sigma^m\) is the identity, where \(m\) is any given integer: If \(d_0 = 1, d_1, d_2, \ldots, d_l = m\) are the divisors of \(m\), then \(P(n, m)/(n!)\) is the coefficient of \(x^n\) in the Taylor expansion of

\[
e^x e^{\frac{1}{2} x^{d_1}} e^{\frac{1}{3} x^{d_2}} \cdots e^{\frac{1}{l} x^{d_l}}.
\]

In particular, for our problem \(m = 3\) and \(P(n, 3) = P(n)\) equals \(n!\) times the coefficient of \(x^n\) in the Taylor expansion of \(e^x e^{\frac{1}{2} x^3}\).

Let $x$, $y$, and $z$ be positive real numbers which satisfy $x^2 + y^2 = z^2$. Construct a line segment $AC$ with length $z$. Let $B$ be any point such that $BC = x$ and $90^\circ < \angle ABC < 180^\circ$. Let $M$ be a point on $AC$ such that $\angle MAB = \angle MBC$. Let $D$ be the point on line $BM$ on the opposite side of $AC$ from $B$ such that $AD = y$. Show that $\angle ADM = \angle DCM$.

Solution by Oliver Geupel, Brühl, NRW, Germany; and Michael Parmenter, Memorial University of Newfoundland, St. John’s, NL.

Since the triangles $CAB$ and $CBM$ are similar, we have successively

$$\frac{MC}{BC} = \frac{CB}{CA},$$

$$\frac{MC}{x} = \frac{x}{z},$$

$$MC = \frac{x^2}{z}.$$

Then we have

$$AM = z - \frac{x^2}{z} = \frac{z^2 - x^2}{z} = \frac{y^2}{z}.$$}

Thus, $\frac{y}{z} = AM$, or equivalently $\frac{AD}{AC} = \frac{AM}{AD}$, which implies that triangles $AMD$ and $ADC$ are similar. Therefore,

$$\angle ADM = \angle ACD = \angle DCM,$$

as claimed.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; SEFKET ARSLANAGIC, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; PAUL DEIERMANN, Southeast Missouri State University, Cape Girardeau, MO, USA; SALEM MALIKIC, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposers.

3285. [2007 : 429, 432] Proposed by Gregory Akulov, student, University of Regina, Regina, SK.

Solve the following for $x$:

$$x \left( \sqrt{3 - 2x} + \sqrt{5(1 - x^2)} \right) = \sqrt{\frac{2}{3}}.$$
**Solution by Michel Bataille, Rouen, France.**

The equation can be rewritten as

\[ x \sqrt{6} \left( \sqrt{3 - 2x + \sqrt{5(1 - x^2)}} \right) = 2 - 3x, \]

so that the given equation is equivalent to \( 0 < x < \frac{2}{3} \) and

\[ 6x^2 \left( 3 - 2x + \sqrt{5(1 - x^2)} \right) = (2 - 3x)^2. \]

Since \((3 - 2x)^2 - (2 - 3x)^2 = 5(1 - x^2)\), we successively rewrite the above equation as

\[ 6x^2 = \frac{(2 - 3x)^2 \left( 3 - 2x - \sqrt{5(1 - x^2)} \right)}{(3 - 2x)^2 - 5(1 - x^2)}, \]

\[ 6x^2 + 2x - 3 = -\sqrt{5(1 - x^2)}. \]

Setting \( \alpha = \arccos \left( \frac{2}{3} \right) \) and \( x = \cos \theta \) where \( \theta \in (\alpha, \frac{\pi}{2}) \), the last equation is equivalent to each of

\[ -\sqrt{5} \sin \theta = 6 \cos^2 \theta + 2 \cos \theta - 3, \]

\[ -\frac{\sqrt{5}}{3} \sin \theta - \frac{2}{3} \cos \theta = 2 \cos^2 \theta - 1 = \cos(2\theta), \]

\[ \cos(\theta - \alpha) = \cos(\pi - 2\theta). \]

Hence, \( \theta = \frac{\pi + \alpha}{3} \) and \( x = \cos \left( \frac{\pi + \arccos \left( \frac{2}{3} \right)}{3} \right) \).

*Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŞEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; DIONNE BAILEY, ELSIE CAMPBELL, CHARLES DIMINNIE, KARL HAVLAK and PAULA KOSTA, Angelo State University, San Angelo, TX, USA; ROY BARBARA, Lebanese University, Fanar, Lebanon; PAUL DEIERMANN, Southeast Missouri State University, Cape Girardeau, MO, USA; KEE-WAI LAU, Hong Kong, China; JOEL SCHLOSBERG, Bayside, NY, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. There were two incomplete and two incorrect solutions submitted.*

**3286. [2007 : 430,432] Proposed by Neven Jurić, Zagreb, Croatia.**

Is it possible to find a function \( f : [0,1] \to \mathbb{R} \) such that

\[ f(x) = 1 + x \int_0^1 f(t) \, dt + x^2 \int_0^1 [f(t)]^2 \, dt? \]
All submitted solutions were similar to those of Michel Bataille, Rouen, France and Richard I. Hess, Rancho Palos Verdes, CA, USA.

Let \( a = \int_0^1 f(t) \, dt \) and \( b = \int_0^1 [f(x)]^2 \, dt \). Thus, \( f(t) = 1 + at + bt^2 \) and we have

\[
a = \int_0^1 (1 + at + bt^2) \, dt = \left( t + \frac{at^2}{2} + \frac{bt^3}{3} \right) \bigg|_0^1 = 1 + \frac{a}{2} + \frac{b}{3},
\]

hence \( a = 2 + \frac{2b}{3} \). Also

\[
b = \int_0^1 \left( 1 + 2at + (2b + a^2) t^2 + 2abt + b^2t^4 \right) \, dt
\]

\[
= \left( t + \frac{at^2}{2} + \frac{(2b + a^2) t^3}{3} + \frac{abt^4}{2} + \frac{b^2t^5}{5} \right) \bigg|_0^1
\]

\[
= 1 + a + \frac{2b + a^2}{3} + \frac{ab}{2} + \frac{b^2}{5}
\]

\[
= 1 + 2 + \frac{2b}{3} + \frac{13}{9} \left( 2b + 4 + \frac{8b}{3} + \frac{4b^2}{9} \right) + b + \frac{b^2}{3} + \frac{b^2}{5}.
\]

Hence,

\[
\frac{92b^2}{135} + \frac{20b}{9} + \frac{13}{3} = 0,
\]

and the two roots of the equation \( 92b^2 + 300b + 585 = 0 \) are

\[
b = -\frac{75 \pm 3i\sqrt{870}}{46}.
\]

Thus, such a function \( f \) does not exist. However, the complex valued function

\[
f(z) = 1 + \left( \frac{21 \pm i\sqrt{870}}{23} \right) z + \left( -\frac{75 \pm 3i\sqrt{870}}{46} \right) z^2
\]

do satisfy the equation of the problem.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ROY BARBARA, Lebanese University, Fanar, Lebanon; DIONNE BAILEY, ELISIE CAMPBELL, CHARLES DIMINNIE, KARL HAFLAK and PAULA KOCA, Angelo State University, San Angelo, TX, USA; PAUL BRACKEN, University of Texas, Edinburg, TX, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; JOSE LUIS DIAZ-BARRERO, Universitat Politècnica de Catalunya, Barcelona, Spain; OLIVER GEUPEL, Biel, NRW, Germany; KEE-WAI LAU, Hong Kong, China; THANOS MAGROS, 3rd High School of Karani, Karani, Greece; XAVIER ROS, student, Universitat Politècnica de Catalunya, Barcelona, Spain; ROBERT P. SEALY, Mount Allison University, Sackville, NB; DIGBY SMITH, Mount Royal College, Calgary, AB; DANIEL TSAI, student, Taipei American School, Taipei, Taiwan; GEORGE TSAPAKIDIS, Agrinio, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

Let \( x, y, \) and \( z \) be positive real numbers satisfying
\[
xy + yz + zx + xyz = 4.
\]

Prove that
\[
\begin{align*}
(a) \quad (x+2)(y+2) + (y+2)(z+2) + (z+2)(x+2) &= (x+2)(y+2)(z+2); \\
(b) \quad \text{there is a triangle whose sides have lengths } (x+2)(y+2), \ (y+2)(z+2), \ \text{and } (z+2)(x+2).
\end{align*}
\]

Solution by Joe Howard, Portales, NM, USA.

Let \( a = x + 2, b = y + 2, \) and \( c = z + 2. \) A simple calculation shows that
\[
4 = xy + yz + zx + xyz = (a - 2)(b - 2) + (b - 2)(c - 2) + (c - 2)(a - 2) + (a - 2)(b - 2)(c - 2)
\]
\[
= abc - (ab + bc + ca) + 4.
\]

Hence, \( abc = ab + bc + ca, \) and the equation in part (a) holds.

Clearly \( a, b, \) and \( c \) are positive. By part (a)
\[
ab + bc - ca = ab + bc + ca - 2ca = abc - 2ca = ac(b - 2) = acy > 0.
\]

Similarly, \( bc + ca - ab > 0 \) and \( ab + ca - bc > 0, \) and the result in part (b) follows.

Also solved by Arkady Alt, San Jose, CA, USA; George Apostolopoulos, Messolonghi, Greece; Sefret Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; Dionne Bailey, Elsie Campbell, and Charles R. Diminnie, Angelo State University, San Angelo, TX, USA; Roy Barbara, Lebanese University, Fanar, Lebanon; Michel Bataille, Rouen, France; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Jose Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain; Oliver Geipel, Brühl, NRW, Germany; Karl Havlak, Angelo State University, San Angelo, TX, USA; Richard J. Hess, Rancho Palos Verdes, CA, USA; Thanos Magkos, 3rd High School of Kozani, Kozani, Greece; Saleem Malik, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; Dung Nguyen Manh, High School of HUS, Hanoi, Vietnam; Andrea Munic, student, University of Trento, Trento, Italy; Michael Parmenter, Memorial University of Newfoundland, St. John's, NL; Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam; Xavier Ros, student, Universitat Politècnica de Catalunya, Barcelona, Spain; Joel Schlossberg, Bayside, NY, USA; Bob Serkey, Leonia, NJ, USA; D.J. Smeenk, Zaltbommel, the Netherlands; Panos E. Tsapoussoglou, Athens, Greece; George Tsaparidis, Agrinio, Greece; Peter Y. Woo, Biola University, La Mirada, CA, USA; Bingjie Wu, student, High School Affiliated to Fudan University, Shanghai, China; Titu Zvonaru, Comănești, Romania; and the proposer.
3288. [2007 : 430, 432] Proposed by Ovidiu Furdui, student, Western Michigan University, Kalamazoo, MI, USA.

Let $n$ be a positive integer. Evaluate the sum:

$$
\sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i-1}{i} \frac{2^{n-2i-1}}{n-2i},
$$

where $\lfloor x \rfloor$ is the integer part of $x$.

**Solution by Michel Bataille, Rouen, France.**

Let $S_n$ denote the sum to be evaluated. Then

$$
S_n = \frac{1}{2} \int_0^2 \left( \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i-1}{i} x^{n-2i-1} \right) dx.
$$

(1)

From a known result (for example, see the solution to *Crux* problem 3217 in [2008: 112-3]) we have

$$
\sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i-1}{i} x^i = F_n(x)
$$

for non-negative $x$, where

$$
F_n(x) = \frac{1}{\sqrt{1+4x}} \left( \left( \frac{1+\sqrt{1+4x}}{2} \right)^n - \left( \frac{1-\sqrt{1+4x}}{2} \right)^n \right).
$$

(2)

Calculating $x^{n-1}F_n \left( \frac{1}{x^2} \right)$ for $x > 0$ using (1) and then (2) yields

$$
\sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i-1}{i} x^{n-2i-1}
$$

$$
= \frac{1}{2^n \sqrt{x^2+4}} \left( \left( x + \sqrt{x^2+4} \right)^n - \left( x - \sqrt{x^2+4} \right)^n \right),
$$

(3)

a formula that still holds if $x = 0$ [Ed.: if $x = 0$, $n$ is odd, and $i = \frac{n-1}{2}$, then the last term of the sum in (3) is $0^0$, which we interpret as 1]. It then follows from (1) and (3) that

$$
S_n = \frac{1}{2^{n+1}} \int_0^2 \frac{x^{n} + (\sqrt{x^2+4})^n}{\sqrt{x^2+4}} \frac{(x - \sqrt{x^2+4})^n}{\sqrt{x^2+4}} dx.
$$

Substituting $x = 2 \sinh t = e^t - e^{-t}$ and $dx = 2 \cosh t \, dt$ in the above integral yields

$$
S_n = \int_0^{\ln(1+\sqrt{2})} \frac{e^{nt} + (-1)^{n-1} e^{-nt}}{2} \, dt.
$$
Thus, if \( n \) is odd,
\[
S_n = \int_0^{\ln(1+\sqrt{2})} \cosh(nt) \, dt = \frac{1}{n} \sinh \left( \ln \left( 1 + \sqrt{2} \right)^n \right) = \frac{1}{2n} \left( (\sqrt{2} + 1)^n - (\sqrt{2} - 1)^n \right),
\]
and if \( n \) is even,
\[
S_n = \int_0^{\ln(1+\sqrt{2})} \sinh(nt) \, dt = \frac{1}{n} \cosh \left( \ln \left( 1 + \sqrt{2} \right)^n \right) - \frac{1}{n} \cosh(0) = \frac{1}{2n} \left( (\sqrt{2} + 1)^n + (\sqrt{2} - 1)^n \right) - \frac{1}{n}.
\]

Also solved by the proposer, whose proof used Fibonacci polynomials and similar arguments as given in the featured solution above.

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