Twin Problems on Non-Periodic Functions

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1 Introduction

Two very similar problems were proposed in the *American Mathematical Monthly* by P.P. Dalyay.

Problem 11111. ([3]) Let f and g be nonconstant, continuous periodic functions mapping $\mathbb{R}$ into $\mathbb{R}$. Is it possible that the function h on $\mathbb{R}$ given by $h(x) = f\big(xg(x)\big)$ is periodic?

Problem 11174. ([4]) Let f and g be nonconstant, continuous functions mapping $\mathbb{R}$ into $\mathbb{R}$ and satisfying the following conditions:

1. f is periodic.
2. There is a sequence $\langle x_n \rangle_{n\geq 1}$ such that $\lim\limits_{n\to\infty} x_n = \infty$ and $\lim\limits_{n\to\infty} \left|\dfrac{g(x_n)}{x_n}\right| = \infty$.
3. $f \circ g$ is not constant on $\mathbb{R}$.

Determine whether $h = f \circ g$ can be periodic.

The solutions to Problem 11111 and Problem 11174 appeared in [5] and [8]. In this note we are going to consider a new question which is similar to but more general than each of these problems. The proofs here are based on a particular case of the well-known Stolz-Cesàro Lemma and on the fact that a continuous periodic function on $\mathbb{R}$ is uniformly continuous. The use of the latter idea is not new as it was used in the published solutions of these problems. On the other hand, the use of the Stolz-Cesàro Lemma is a good example of where an old tool from analysis appears unexpectedly (see [1], [6], [7], and [10]). L'Hospital's rule, which is well known to calculus students, is its "differentiable" counterpart.

For each version of L'Hospital's rule, there is an analogous version of the Stolz-Cesàro Lemma. For example, one version of L'Hospital's Rule states that if f and g are two differentiable functions on (a, ∞) such that $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} g(x) = \infty$ and $\lim\limits_{x\to\infty} \dfrac{f'(x)}{g'(x)} = L$, then $\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)} = L$.

The Stolz-Cesàro analog of this version of L'Hospital's Rule is: For two sequences $\langle x_n \rangle_{n\geq 1}$ and $\langle y_n \rangle_{n\geq 1}$, if $\lim\limits_{n\to\infty} x_n = \infty$ and $\lim\limits_{n\to\infty} \dfrac{y_{n+1} - y_n}{x_{n+1} - x_n} = L$, then $\lim\limits_{n\to\infty} \dfrac{y_n}{x_n} = L$

Indeed, one may think of the derivative $y'(a)$ as a special quotient $y'(a) = \frac{y(a+\delta) - y(a)}{\delta}$, where δ takes its smallest possible *infinitesimal* value. In the discrete case we write $y_n = y(n)$ for $n \in \mathbb{N}$, and we let δ take its smallest value, namely $\delta = 1$, to obtain the discrete derivative $y'(n) = \frac{y(n+1) - y(n)}{1} = y_{n+1} - y_n$. Thus, $\lim_{n\to\infty} \frac{y_{n+1} - y_n}{x_{n+1} - x_n} = L$ indicates (in this interpretation) that a ratio of discrete derivatives tends to L.

The following problem may be solved by applying both the Stolz-Cesàro Lemma and L'Hospital's Rule. We include it for the interested reader.

Problem Let $x_0 \in (0, \pi)$ and define the sequence $\langle x_n \rangle_{n\geq 0}$ by the recursion $x_{n+1} = \sin x_n$, $n \geq 0$. Show that $\lim_{n\to\infty} x_n\sqrt{n} = \sqrt{3}$.

Solution The required limit is equivalent to $\lim_{n\to\infty} \frac{1/x_n^2}{n} = \frac{1}{3}$. By the Stolz-Cesàro Lemma, it suffices to show that

$$\lim_{n\to\infty} \frac{\frac{1}{x_{n+1}^2} - \frac{1}{x_n^2}}{(n+1) - n} = \frac{1}{3},$$

or equivalently

$$\lim_{n\to\infty} \left(\frac{x_n^2 - \sin^2 x_n}{x_n^2 \sin^2 x_n} \right) = \frac{1}{3}.$$

Since $0 < \sin x < x$ whenever $x \in (0, \pi)$, the sequence $\langle x_n \rangle$ is decreasing and bounded below by 0 and so it converges to a limit $\ell \in [0, \pi)$. This limit ℓ satisfies $\sin \ell = \ell$, hence $\ell = 0$.

Thus, it suffices to show that $\lim_{x\to 0} \frac{x^2 - \sin^2 x}{x^2 \sin^2 x} = \frac{1}{3}$, which can be done by applying L'Hospital's Rule:

$$\begin{aligned}
\lim_{x\to 0} \frac{x^2 - \sin^2 x}{x^2 \sin^2 x} &= \lim_{x\to 0} \frac{(x - \sin x)(x + \sin x)}{x^2 \sin^2 x} \\
&= \lim_{x\to 0} \left(\frac{x - \sin x}{x^3} \right) \lim_{x\to 0} \left(1 + \frac{\sin x}{x} \right) \lim_{x\to 0} \left(\frac{x}{\sin x} \right)^2 \\
&= \left(\lim_{x\to 0} \frac{1 - \cos x}{3x^2} \right) \cdot (1+1) \cdot 1^2 \\
&= 2 \lim_{x\to 0} \frac{\sin x}{6x} = \frac{1}{3}.
\end{aligned}$$

We will use a variant of the Stolz-Cesàro Lemma to prove our main theorem, where we weaken the conditions in Problem 11174 but obtain the same conclusion.

Theorem 1 Let f and g be nonconstant, continuous functions from $\mathbb{R}$ into $\mathbb{R}$ that satisfy the following conditions:

(i) The function f is periodic.

(ii) There exist sequences $\langle x_n \rangle_{n\geq 1}$ and $\langle y_n \rangle_{n\geq 1}$ such that

$$\inf_n |x_n - y_n| \ > \ 0 \qquad \text{and} \qquad \lim_{n\to\infty} \left| \frac{g(x_n) - g(y_n)}{x_n - y_n} \right| \ = \ \infty \, .$$

Then the function $h = f \circ g$ is not periodic.

2 Some Facts from Real Analysis

The variant of the Stolz-Cesàro Lemma that we will use is stated next.

Lemma 1 Let $\langle a_n \rangle$ and $\langle b_n \rangle$ be two sequences such that $\langle b_n \rangle$ is increasing and $\lim\limits_{n\to\infty} b_n = \infty$. If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \infty$, then $\limsup\limits_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \infty$.

For completeness we include a proof of Lemma 1 along classical lines.

Assume to the contrary that $\gamma = \limsup\limits_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} < \infty$. Then there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$ we have

$$\frac{a_{n+1} - a_n}{b_{n+1} - b_n} \ \leq \ \gamma + 1 \, ,$$

or equivalently

$$a_{n+1} - a_n \ \leq \ (b_{n+1} - b_n)(\gamma + 1) \, , \tag{1}$$

for $n \geq n_0$. Adding up the inequalities in (1) for $n = k, \, k+1, \, \ldots, \, l$, where $n_0 \leq k < l$, we obtain

$$a_{l+1} - a_k \ \leq \ (b_{l+1} - b_k)(\gamma + 1) \, .$$

For sufficiently large l the term b_{l+1} is positive and we may divide the last inequality by b_{l+1} and then let $l \to \infty$. Using the hypothesis we then obtain $\infty \leq \gamma + 1$, a contradiction.

We now recall that a (real- or complex-valued) function with domain $\mathcal{D} \subset \mathbb{R}$ is *uniformly continuous* on $\mathcal{D}$ if for each $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $x, \, y \in \mathcal{D}$ and $|x - y| < \delta$. The following basic fact about continuous functions on a closed interval is all we need, though it can be generalized considerably (see [9], Theorem 4.19).

Theorem 2 Every continuous, real-valued function whose domain is a closed interval $\mathcal{D} = [a, b]$ is uniformly continuous on $\mathcal{D}$.

As an easy consequence of this theorem, we have

Corollary 1. Every continuous, periodic function f on $\mathbb{R}$ is uniformly continuous.

This can be seen by applying Theorem 2 to the restriction of f to the closed interval $[0, 2T]$, where $T > 0$ is a period of f. That is, given $\epsilon > 0$, by Theorem 2 there is a δ such $|f(x) - f(y)| < \epsilon$ whenever x, $y \in [0, 2T]$ and $|x - y| < \delta$. Thus, if x, $y \in \mathbb{R}$ and $|x - y| < \delta$, then there exist integers n and m such that both $x_1 = x - nT$ and $y_1 = y - mT$ are in $[0, 2T]$ and $|x_1 - y_1| < \delta$, so that $|f(x) - f(y)| = |f(x_1) - f(y_1)| < \epsilon$.

The idea of our proofs is to show that the function h is not uniformly continuous. It then follows from Corollary 1 that h is not periodic.

Let us see how Problem 11111 can be solved using Theorem 1. Since g is not constant there exist a and b such that $g(a) - g(b) \neq 0$. Let $T > 0$ be a period of g, and define the sequences $\langle x_n \rangle_{n \geq 1}$ and $\langle y_n \rangle_{n \geq 1}$ by $x_n = a + nT$ and $y_n = b + nT$. Then $|x_n - y_n| = |a - b| > 0$ and

$$\begin{aligned} &\lim_{n\to\infty} \frac{|x_n g(x_n) - y_n g(y_n)|}{|x_n - y_n|} \\ &\quad = \; |a-b|^{-1} \lim_{n\to\infty} \big|ag(a) - bg(b) + nT\big(g(a) - g(b)\big)\big| \; = \; \infty\,, \end{aligned}$$

which says that f and the function g_1 on $\mathbb{R}$ given by $g_1(x) = xg(x)$ both satisfy the conditions (i) and (ii) in Theorem 1, hence, $h = f \circ g_1$ is not periodic. We have solved Problem 11111.

To show that Problem 11174 can be solved using Theorem 1, we need the weaker version of the Stolz-Cesàro Lemma given in Lemma 1.

Let us assume that f, g, and $\langle x_n \rangle_{n \geq 1}$ satisfy the three conditions in Problem 11174. There is a subsequence $\langle x_{n_k} \rangle_{k \geq 1}$ of $\langle x_n \rangle_{n \geq 1}$, such that $x_{n_{k+1}} - x_{n_k} \geq 1$ for all k, and for which either $\lim\limits_{k\to\infty} \dfrac{g(x_{n_k})}{x_{n_k}} = \infty$ or $\lim\limits_{k\to\infty} \dfrac{g(x_{n_k})}{x_{n_k}} = -\infty$. Without loss of generality we may suppose the former, because the latter case follows from the former case by replacing g with $-g$ and f with $f_1(x) = f(-x)$, $x \in \mathbb{R}$. By Lemma 1 we have

$$\limsup_{k\to\infty} \frac{g(x_{n_{k+1}}) - g(x_{n_k})}{x_{n_{k+1}} - x_{n_k}} \; = \; \infty\,,$$

which proves the existence of the two sequences in the hypothesis (ii) of Theorem 1. Hence, Theorem 1 can be applied to f and g and we deduce that $h = f \circ g$ is not periodic. We have solved Problem 11174.

3 Proof of Theorem 1

Let f and g satisfy the hypotheses of Theorem 1. Since g is continuous and satisfies condition (ii), the interval $I_n = g\big([x_n, y_n]\big)$ (or $I_n = g\big([y_n, x_n]\big)$)

has length greater than the period T of f for sufficiently large n. Hence, f and $h = f \circ g$ have the same range. Since f is not constant, h is not constant. Therefore, there exist α and β such that $f(g(\alpha)) \neq f(g(\beta))$, and we let $\epsilon_0 = |f(g(\alpha)) - f(g(\beta))| > 0$. As we said in the introduction, the key idea is to prove that h is not uniformly continuous. In fact, we will show that the definition of uniform continuity is not satisfied for this ϵ_0.

We fix $n \in \mathbb{N}$ large enough so that $|g(x_n) - g(y_n)| > 2T$, and we denote by $\sharp(g(\alpha))$ the number of integers k for which $g(\alpha) + kT$ is in I_n. It is then easy to see that

$$\sharp(g(\alpha)) \;>\; \frac{|g(x_n) - g(y_n)|}{T} - 1 \;>\; 1\,.$$

Similarly, we denote by $\sharp(g(\beta))$ the number of integers k for which $g(\beta)+kT$ is in I_n. Similarly, we have $\sharp(g(\beta)) > 1$.

It is clear that the values $g(\alpha) + kT$, $k \in \mathbb{Z}$, interlace with those of $g(\beta) + kT$, $k \in \mathbb{Z}$. Using again the fact that g is continuous and by repeated application of the Intermediate Value Theorem, we can find two finite sequences $\langle u_k \rangle$ and $\langle v_k \rangle$ in the interval $[x_n, y_n]$ (or $[y_n, x_n]$) both increasing and interlacing and such that $g(u_k) = g(\alpha) + l_k T$ and $g(b_k) = g(\beta) + s_k T$, with l_k, $s_k \in \mathbb{Z}$. The number of intervals of the form $[u_k, v_k)$ (or $[v_k, u_k)$) is at least

$$M \;=\; \min\left\{2\Big(\sharp((g(\alpha)) - 1\Big), 2\Big(\sharp((g(\beta)) - 1\Big)\right\} \;\geq\; 2\,.$$

These intervals form a partition of a subinterval of $J_n = [x_n, y_n]$ (or of $J_n = [y_n, x_n]$) of length $|x_n - y_n|$. Then one of these intervals has length at most $\frac{|x_n - y_n|}{M}$. We denote such an interval by $[\zeta_n, \eta_n]$ and notice that

$$|\zeta_n - \eta_n| \;\leq\; \frac{|x_n - y_n|}{M} \;<\; \frac{|x_n - y_n|}{2\left(\dfrac{|g(x_n) - g(y_n)|}{T}\right) - 4}$$

$$= \frac{1}{\dfrac{2}{T}\left(\dfrac{|g(x_n) - g(y_n)|}{|x_n - y_n|}\right) - \dfrac{4}{|x_n - y_n|}} \;\longrightarrow\; 0 \quad \text{as} \quad n \to \infty\,, \tag{2}$$

and $|f(g(\zeta_n)) - f(g(\eta_n))| = \epsilon_0$. Given $\delta > 0$, we may choose n so large that $|\zeta_n - \eta_n| < \delta$. This can be done because of (2). For such an n we still have $|h(\zeta_n) - h(\eta_n)| \geq \epsilon_0$, which proves that h is not uniformly continuous.

4 Conclusion

We would like to leave the reader with a natural question: Can Theorem 1 be generalized to almost periodic functions? There are various concepts of almost periodicity, but here we will only give Bohr's definition:

A continuous function $F : \mathbb{R} \to \mathbb{R}$ is said to be *almost periodic* if for all $\epsilon > 0$, there is an $L > 0$ such that every interval of length L contains an ϵ-period, that is, a number T such that $|F(x+T) - F(x)| < \epsilon$ for all $x \in \mathbb{R}$.

What is interesting and related to the question above is the fact that every almost periodic function is uniformly continuous (see [2]).

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