Solution by Brandon Affenzeller and Jonathon Henson, Auburn University Montgomery, Montgomery, AL, USA.

If $x \geq -\frac{1}{2}$, then $|2x+1| = 2x+1$; if $x < -\frac{1}{2}$, then $|2x+1| = -(2x+1)$. If $x \geq -\frac{5}{2}$, then $|2x+5| = 2x+5$; if $x < -\frac{5}{2}$, then $|2x+5| = -(2x+5)$.

Therefore, the given functions can be written in the following form:

$$f(x) = \begin{cases} 5x & \text{if } x \geq -\frac{1}{2} \\ x - 2 & \text{if } x < -\frac{1}{2} \end{cases}$$

and

$$g(x) = \begin{cases} \frac{1}{5}x & \text{if } x \geq -\frac{5}{2} \\ x + 2 & \text{if } x < -\frac{5}{2} \end{cases}$$

If $x < -\frac{1}{2}$, then $x - 2 < -\frac{5}{2}$, and in that case we have the calculation $g(f(x)) = g(x-2) = (x-2)+2 = x$. If $x \geq -\frac{1}{2}$, then $5x \geq -\frac{5}{2}$, and in that case we have $g(f(x)) = g(5x) = \frac{1}{5}(5x) = x$.

Hence, $(g \circ f)(x) = x$ for all $x \in \mathbb{R}$.

Similarly, if $x < -\frac{5}{2}$, then $x + 2 < -\frac{5}{2}$, and in that case we have $f(g(x)) = f(\frac{1}{5}x) = (\frac{1}{5}x) - 2 = x$; and if $x \geq -\frac{5}{2}$, then $\frac{1}{5}x \geq -\frac{1}{2}$, and then we have $f(g(x)) = f(\frac{1}{5}x) = 5(\frac{1}{5}x) = x$. Hence, $f \circ g = g \circ f = i$, where $i$ is the identity function, that is, $i(x) = x$ for all $x \in \mathbb{R}$.

Since $f \circ g = g \circ f = i$, we conclude that $f = g^{-1}$ and $g = f^{-1}$.

Hence, $(f \circ f)^{-1} = (g^{-1} \circ g^{-1})^{-1} = g \circ g$.

Also solved by Samuel Gómez Moreno, Universidad de Jaén, Jaén, Spain; Miguel Marañón Grande, student, Universidad de La Rioja, Logroño, La Rioja, Spain; Missouri State University Problem Solving Group, Springfield, MO, USA; Ricard Peiró, IES "Abastos", Valencia, Spain; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Titu Zvonaru, Comenioși, Romania. There were 2 incorrect or incomplete solutions submitted.

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Problem of the Month

Ian VanderBurgh

After a few months of more investigative problems, let’s look at something algebraic this month.

Problem 1

Determine all pairs $(a, b)$ of real numbers that satisfy the system of equations

$$a + \log a = b,$$

$$b + \log b = a.$$

Anyone who has ever played around with equations involving both polynomial terms (like $a$) and logarithmic terms (like $\log a$) will know that things can get a tad tricky. Having a system of equations involving both of these is undoubtedly much worse than a system with just one of them.

Well, it’s actually not so bad. First, we should clarify to what base we
are taking logarithms. In fact, it doesn't matter at all, but if it makes you more comfortable, think of \( \log a \) as meaning \( \log_{10} a \).

Next, we should see if we can find any solutions at all. Often when a system of equations is symmetric (that is, we get the same system if we switch \( a \) and \( b \)), trying to find solutions with \( a = b \) is not a bad idea. Here, if \( a = b \), both equations become

\[
a + \log a = a
\]

which simplifies to \( \log a = 0 \) or \( a = 1 \). Therefore, \((a, b) = (1, 1)\) is a solution, which we can verify by substitution.

At this point, we can try as we might to find another solution, but... there are actually no more solutions! Let's look at two different approaches that show us why.

**Solution 1 to Problem 1.** One approach that we learn early on when solving systems of equations is to combine the equations somehow. Let's try adding the two equations, because this will allow us to do some cancellation:

\[
\begin{align*}
a + \log a + b + \log b &= a + b, \\
\log a + \log b &= 0, \\
\log(ab) &= 0, \\
ab &= 1,
\end{align*}
\]

hence, \( b = \frac{1}{a} \).

Substituting this into the first equation, we obtain \( a + \log a = \frac{1}{a} \) or \( \log a = \frac{1}{a} - a \).

At this stage we've still got an equation involving both logarithms and powers of \( a \). Here's a neat way to deal with it.

What happens if \( a > 1 \)? In this case, \( \log a > 0 \) and \( \frac{1}{a} - a < 0 \), which is not possible if the two sides are equal. Thus, \( a \) cannot be greater than 1.

What happens if \( 0 < a < 1 \)? In this case, \( \log a < 0 \). What about the right side? In fact, the right side here is positive, so again, there can be no solution.

Thus, \( a = 1 \) works, but neither \( a > 1 \) nor \( 0 < a < 1 \) can work, so there is a unique solution \( a = 1 \) (which gives \((a, b) = (1, 1)\) as above).

Our strategy was to find one solution and then to show that no other solutions are possible. Here's a second way to look at this.

**Solution 2 to Problem 1.** Let's leave the equations in their original form and examine the cases \( a > 1 \) and \( 0 < a < 1 \), knowing already that \( a = 1 \) yields a solution.

If \( a > 1 \), then \( \log a > 0 \), so \( b = a + \log a > a > 1 \). Now, since \( b > 1 \), then \( \log b > 0 \) which gives \( a = b + \log b > b \). However, this means that \( a > b > a \), which is impossible. Thus, \( a \) cannot be greater than 1.

If \( 0 < a < 1 \), then \( \log a < 0 \), so \( b = a + \log a < a < 1 \). Now, since
\( b < 1 \), then \( \log b < 0 \) which gives \( a = b + \log b < b \). However, this means that \( a < b < a \), which is also impossible.

Therefore, \( a = 1 \) and \( b = 1 \) is the only possible solution.

So there are two different approaches to this problem. If you feel ambitious, try solving the next problem from the 2008 Euclid Contest. (You may want to transform it first so that it looks more like Problem 1 above.)

**Problem 2.**

Determine all real solutions to the system of equations

\[
\begin{align*}
x &+ \log_{10} x = y - 1, \\
y &+ \log_{10}(y - 1) = z - 1, \\
z &+ \log_{10}(z - 2) = x + 2,
\end{align*}
\]

and prove that there are no more solutions.

**A Loose End.** At the end of last month’s column, I promised you a solution to the following problem:

**Problem 3.**

3 green stones, 4 yellow stones, and 5 red stones are placed in a bag. This time, two stones of different colours are selected at random, removed and replaced with two stones of the third colour. Show that it is impossible for all of the remaining stones to be the same colour, no matter how many times this process is repeated.

**Solution to Problem 3.** We’ll use the notation and terminology from last month. Let’s suppose that we have \( G \) green stones, \( Y \) yellow stones and \( R \) red stones at a given stage.

Let’s consider the remainder when the number of green stones, \( G \), is divided by 3. Let’s think about how a turn can change this remainder. Note that after a turn, \( G \) becomes either \( G - 1 \) (if a green stone was removed) or \( G + 2 \) (if two green stones are added).

Consider the flow chart \( 2 \rightarrow 1 \rightarrow 0 \rightarrow 2 \). After a turn, the remainder upon division by 3 has moved one position to the right. For example, \( G = 4 \) becomes either \( G = 3 \) or \( G = 6 \), so a remainder of 1 becomes a remainder of 0. Those familiar with modular arithmetic can feel free to use this idea formally.

The same thing is true for the yellow and red stones – the remainder when the number of stones after the turn is divided by 3 is one position to the right in the chart from where it was before the turn.

Initially, we have \( G = 3 \), \( Y = 4 \), and \( R = 5 \). Thus, the respective remainders are 0, 1, and 2. After the first turn, the (respective) remainders will be 2, 0, and 1. After the second turn, they will be 1, 2, and 0. After the third turn, they will be 0, 1, and 2, and so forth.

Can we get to a position where the stones are all the same colour? This would mean that two of the numbers \( G, R, \) and \( Y \) would equal 0 and so give a remainder of 0. But the three remainders are always different, so this is impossible.