THE OLYMPIAD CORNER
No. 272
R.E. Woodrow

We begin this number with a selection of problems from the Olimpiada Matemática Española 2005, chosen from various sessions of the National and Local Stages of competition. My thanks go to Felix Recio, Canadian Team Leader to the IMO in Mexico, for collecting them.

OLIMPIADA MATEMÁTICA ESPAÑOLA 2005
National Stage
(selected questions)

1. In the plane, is it possible to colour the points with integer coordinates with three colours, in such a way that each colour is found infinitely many times on an infinite number of horizontal lines, and any three points of different colours are never collinear?

2. A triangle is said to be multiplicative if the product of the lengths of two of its sides equals the length of the third side.

Let $AB...Z$ be a regular polygon with $n$ sides, each of length 1. The $n - 3$ diagonals from the vertex $A$ divide the triangle $ZAB$ into $n - 2$ smaller triangles. Prove that all of these triangles are multiplicative.

3. Let $r, s, u,$ and $v$ be real numbers. Prove that

$$\min\{r - s^2, s - u^2, u - v^2, v - r^2\} \leq \frac{1}{4}.$$

4. In triangle $ABC$ the sides $BC$, $AC$, and $AB$ have lengths $a$, $b$, and $c$, respectively, and $a$ is the arithmetic mean of $b$ and $c$. Let $r$ and $R$ be the radius of the incircle and circumcircle of $ABC$, respectively. Prove that:

(a) $0^\circ \leq \angle BAC \leq 60^\circ$.

(b) The altitude from $A$ is three times the inradius $r$.

(c) The distance from the circumcentre of $ABC$ to the side $BC$ is $R - r$. 
Local Stage
(selected questions)

5. In triangle $ABC$ we have $\angle BAC = 45^\circ$ and $\angle ACB = 30^\circ$. Let $M$ be the mid-point of the side $BC$. Prove that $\angle AMB = 45^\circ$ and that $BC \cdot AC = 2 \cdot AM \cdot AB$.

6. Four black marks and five white marks are arbitrarily placed around a circle. If two consecutive marks are of the same colour, we put a new black mark between them and if the two marks are not the same colour, we put a new white mark between them. Then we remove all the previous marks.

Is it possible to obtain nine white marks by repeating this process?

7. Prove that the equation $x^2 + y^2 - z^2 - x - 3y - z - 4 = 0$ has infinitely many integer solutions.

8. On a $10 \times 10$ chessboard are placed 41 rooks. Prove that there are at least five rooks among them such that no two of the five attack each other.

Next we give the problems of the 54th Czech Mathematical Olympiad (2004/5), Category B, 10th Class. The D-, S-, and K-series questions correspond to the respective rounds "Domácí", "Školní", and "Krajské". Thanks again go to Felix Recio, Canadian Team Leader to the IMO in Mexico for obtaining them for us.

54th CZECH MATHEMATICAL OLYMPIAD 2004/5
Category B
10th Class

D1. Find all pairs $(a, b)$ of real numbers such that each of the equations

$$x^2 + ax + b = 0,$$
$$x^2 + (2a + 1)x + 2b + 1 = 0,$$

has two distinct real roots and the roots of the second equation are reciprocals of the roots of the first equation.

D2. Let $ABCD$ be a parallelogram. A line through $D$ meets the segment $AC$ in $G$, the side $BC$ in $F$, and the line $AB$ in $E$. The triangles $BEF$ and $CGF$ have the same area. Determine the ratio $|AG| : |GC|$.

D3. Let $k \geq 3$ be an integer. We have $k$ piles of stones with (respectively) $1, 2, \ldots, k$ stones in them. At each turn we choose three piles, merge them together, and add one stone (not already in a pile) to the resulting pile. Prove that if after some number of turns only one pile remains, then the number of stones in that pile is not divisible by 3.
D4. Let $ABC$ be a scalene triangle with orthocentre $H$ and circumcentre $O$. Prove that if $\angle ACB = 60^\circ$, then the bisector of $\angle ACB$ is the perpendicular bisector of $OH$.

D5. Find all real numbers $x$ such that
\[
\frac{x}{x + 4} = \frac{5[x] - 7}{7[x] - 5},
\]
where $[x]$ denotes the greatest integer not exceeding $x$.

D6. In a circle $\Gamma$ with radius $r$ are inscribed two mutually tangent circles, $\Gamma_1$ and $\Gamma_2$, each with radius $r/2$. Circle $\Gamma_3$ is tangent to $\Gamma_1$ and $\Gamma_2$ externally and to $\Gamma$ internally. Circle $\Gamma_4$ is tangent to $\Gamma_2$ and $\Gamma_3$ externally and to $\Gamma$ internally. Determine the radii of the circles $\Gamma_3$ and $\Gamma_4$.

S1. We have 54 piles of stones with (respectively) 1, 2, \ldots, 54 stones in them. At each turn we choose an arbitrary pile, say consisting of $k$ stones, and remove it along with $k$ stones from each pile which has at least $k$ stones. For example, if we choose the pile with 52 stones at the first turn, then after the turn there will be 53 piles remaining with 1, 2, 3, \ldots, 51, 1, 2 stones in them, respectively. Suppose that after some number of turns only one pile remains. How many stones can be in that pile?

S2. Let $ABC$ be a right triangle with $a = |BC|$, $b = |AC|$, and $c = |BC|$ and such that $a < b < c$. Let $Q$ be the mid-point of $BC$ and let $S$ be the mid-point of $AB$. The line $CA$ meets the perpendicular bisector of $AB$ at $R$. Prove that $|RQ| = |RS|$ if and only if $a^2 : b^2 : c^2 = 1 : 2 : 3$.

S3. Find all real numbers $x$ such that
\[
\left\lfloor \frac{x}{1 - x} \right\rfloor = \frac{[x]}{1 - [x]},
\]
where $[x]$ denotes the greatest integer not exceeding $x$.

K1. Circle $\Gamma_1$ with radius 1 is externally tangent to circle $\Gamma_2$ with radius 2. Each of the circles $\Gamma_1$ and $\Gamma_2$ is internally tangent to circle $\Gamma_3$ with radius 3. Determine the radius of the circle $\Gamma$, which is tangent externally to the circles $\Gamma_1$ and $\Gamma_2$ and internally to the circle $\Gamma_3$.

K2. On a public website participants vote for the world’s best hockey player of the last decade. The percentage of votes a player receives is rounded off to the nearest percent and displayed on the website. After Jožko votes for Miroslav Šatan, the hockey player’s score of 7% remains unchanged. What is the minimum number of people (including Jožko) who voted? (Each participant votes exactly once and for a single player only.)
K3. Let $ABC$ be an acute triangle. Let $K$ and $L$ be the feet of the altitudes from $A$ and $B$, respectively. Let $M$ be the mid-point of $AB$ and let $H$ be the orthocentre of triangle $ABC$. Prove that the bisector of $\angle KML$ bisects the line segment $HC$.

K4. Find all triples of real numbers $x$, $y$, $z$ such that

$$\lfloor x \rfloor - y = 2\lfloor y \rfloor - z = 3\lfloor z \rfloor - x = \frac{2004}{2005},$$

where $\lfloor x \rfloor$ denotes the greatest integer not exceeding $x$.

Next we give the problems of the three Rounds of the 23rd Iranian Mathematical Olympiad, 2005–2006. Our thanks go to Robert Morewood, Canadian Team Leader to the 47th IMO in Slovenia in 2006, for collecting them for our use.

23rd IRANIAN MATHEMATICAL OLYMPIAD
2005–2006
First Round

1. Let $n$ be a positive integer and $p$ a prime number such that $n \mid p - 1$ and $p \mid n^3 - 1$. Show that $4p - 3$ is a perfect square.

2. In triangle $ABC$ we have $\angle A = 60^\circ$. Let $D$ be any point on $BC$. Let $O_1$ be the circumcentre of $ABD$ and $O_2$ be the circumcentre of $ACD$. Let $M$ be the intersection of $BO_1$ and $CO_2$ and $N$ be the circumcentre of $DO_1O_2$. Prove that $MN$ passes through a fixed point.

3. Given are $10^5$ points in Euclidean space. If we consider the set of distances between pairs of these points, show that this set has at least 79 elements.

4. Consider a $2 \times n$ rectangular grid with $2n$ cells, some of which have coins in them. In each step we choose a cell with more than one coin, then we remove two coins from that cell and put one coin either in the cell immediately above it or in the cell immediately to the right of it. In the beginning there are at least $2^n$ coins on the grid. Show that by a series of steps, we can always arrange that there will be at least one coin in the right-most upper cell.
5. The segment $BC$ is the diameter of a circle and $XY$ is a chord perpendicular to $BC$. The points $P$ and $M$ are chosen on $XY$ and $CY$, respectively, such that $CY \parallel PB$ and $CX \parallel MP$. Let $K$ be the intersection of the lines $CX$ and $PB$. Prove that $PB \perp MK$.

6. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$, such that for all $x, y \in \mathbb{R}^+$, we have

$$(x + y)f(f(x)y) = x^2f(f(x) + f(y)),$$

where $\mathbb{R}^+$ denotes the set of positive real numbers.

**Second Round**

1. Let $P(x) \in \mathbb{Q}[x]$ be an irreducible polynomial whose degree is an odd number. Let $P(x) \mid (Q(x)^2 + Q(x)R(x) + R(x)^2)$, where $Q(x)$ and $R(x)$ are polynomials with rational coefficients. Prove that

$$P(x)^2 \mid (Q(x)^2 + Q(x)R(x) + R(x)^2).$$

2. Let $H$ and $O$ be the orthocentre and the circumcentre of triangle $ABC$, respectively. Let $\omega$ be the circumcircle of $ABC$ and let $AO$ intersect $\omega$ at $A_1$. Let $A_1H$ intersect $\omega$ at $A'$ and let $AH$ intersect $\omega$ at $A''$. We define the points $B', B'', C'$, and $C''$ similarly. Prove that $A'A'', B'B''$, and $C'C''$ are concurrent at a point on the Euler line of the triangle $ABC$.

3. Let $a$, $b$, and $c$ be non-negative real numbers. If

$$\frac{1}{a^2 + 1} + \frac{1}{b^2 + 1} + \frac{1}{c^2 + 1} = 2,$$

then show that $ab + bc + ca \leq \frac{3}{2}$.

4. Let $k$ be an integer. The sequence $\{a_n\}_{n=0}^{\infty}$ is defined by $a_0 = 0$, $a_1 = 1$, and for $n \geq 2$ by the recursion $a_n = 2ka_{n-1} - (k^2 + 1)a_{n-2}$. If $p$ is a prime number of the form $4m + 3$, prove that

(a) $a_{n+p^2-1} \equiv a_n \pmod{p}$,

(b) $a_{n+p^3-p} \equiv a_n \pmod{p^2}$.

5. The sets $A_1$, $A_2$, $\ldots$, $A_{35}$ are such that the intersection of any three of them is a singleton and $|A_k| = 27$ for $1 \leq i \leq 35$. Show that the intersection of $A_1$, $A_2$, $\ldots$, $A_{35}$ is non-empty.

6. Triangle $ABC$ is given. The point $L$ is on $BC$ and $M$, $N$ are on the extensions of $AB$, $AC$ (respectively) such that $B$ is between $M$ and $A$, $C$ is between $N$ and $A$, $2 \angle AMC = \angle ALC$, and $2 \angle ANB = \angle ALB$. If $O$ is the circumcentre of $AMN$, show that $OL$ is perpendicular to $BC$. 

Third Round

1. Let \(ABC\) be a triangle whose circumradius equals the radius of the excircle which is tangent to the side \(BC\). Let this excircle touch the side \(BC\) and the lines \(AC\) and \(AB\) at \(M\), \(N\), and \(L\), respectively. Show that the circumcentre of triangle \(ABC\) is the orthocentre of triangle \(MNL\).

2. Let \(x_1, x_2, \ldots, x_n\) be real numbers. Prove that
\[
\sum_{i=1}^{n} \sum_{j=1}^{n} |x_i + x_j| \geq n \sum_{i=1}^{n} |x_i|.
\]

3. Let \(G\) be a tournament with each edge coloured red or blue. Show that there exists a vertex \(v\) of \(G\) with the property that, for every other vertex \(u\), there is a mono-coloured directed path from \(v\) to \(u\).

4. Given are \(n\) points in the plane, no three on a line. If a subset \(E\) of these points is such that the members of \(E\) are the vertices of a convex polygon containing no other points of \(E\) in its interior, then \(E\) is a polite set. Let \(c_k\) be the number of polite sets with \(k\) points. Show that \(\sum_{i=3}^{n} (-1)^i c_i\) depends only on \(n\) and not on the configuration of the \(n\) points.

5. Let \(n > 1\) be an integer, and let the entries of \((a_1, a_2, \ldots, a_n)\) be pairwise distinct positive integers which are coprime in pairs. Find all such \(n\)-tuples for which \((a_1 + a_2 + \cdots + a_n) | (a_1^i + a_2^i + \cdots + a_n^i)\) for \(1 \leq i \leq n\).

6. Suppose we have a simple polygon (that is, it does not intersect itself, but it need not be convex). Show that this polygon has a diagonal which is contained completely inside it and divides the perimeter into two parts such that each part has at least one third of the vertices of the polygon.

As a final set of problems for this number we give the 9th and 10th Grade of the Romanian Mathematical Olympiad, Final Round, April 15th, 2006. Thanks go to Robert Morewood, Canadian Team Leader to the 47th IMO in Slovenia in 2006, for collecting them.

**ROMANIAN MATHEMATICAL OLYMPIAD**

April 15, 2006
Final Round – 9th Grade

1. (Dan Schwarz) Find the maximum value of \((x^3 + 1)(y^3 + 1)\) if \(x\) and \(y\) are real numbers such that \(x + y = 1\).
2. (Manuela Prajea) Let triangles $ABC$ and $DBC$ be such that $AB = BC$, $DB = DC$, and $\angle ABD = 90^\circ$. Let $M$ be the mid-point of $BC$. Points $E$, $F$, and $P$ are interior to the segments $AB$, $MC$, and $AF$, respectively, and $\angle BDE = \angle ADP = \angle CDF$. Prove that $P$ is the mid-point of $EF$ and $DP \perp EF$.

3. (Virgil Nicula) A quadrilateral $ABCD$ is inscribed in a circle of radius $r$ such that there exists a point $P$ on $CD$ satisfying $CB = BP = PA = AB$.

(a) Show that such points $A$, $B$, $C$, $D$, and $P$ do indeed exist.

(b) Prove that $PD = r$.

4. (Radu Gologan) A tennis tournament with $2n$ players ($n \geq 5$) takes place over 4 days. Each player has exactly one match a day, but it is possible that two players may play each other more than once. Prove that such a tournament can end with exactly one winner, exactly three players in second place, and no player losing all four matches. In that case, how many players won a single match and how many won exactly two matches?

**Final Round — 10th Grade**

1. (Vasile Pop) Let $M$ be a set with $n$ elements and let $P(M)$ denote the set of all subsets of $M$. Find all functions $f : P(M) \to \{0, 1, 2, \ldots, n\}$, with the following two properties:

(a) $f(A) \neq 0$ for any $A \neq \emptyset$, and

(b) $f(A \cup B) = f(A \cap B) + f(A \triangle B)$, for all $A$, $B \in P(M)$, where $A \triangle B = (A \cup B) \setminus (A \cap B)$.

2. (Iurie Boreico) Prove that for all integers $n > 0$ and all $a$, $b \in (0, \frac{\pi}{4})$ we have

$$\frac{\sin^n a + \sin^n b}{(\sin a + \sin b)^n} \geq \frac{\sin^2 a + \sin^2 b}{(\sin 2a + \sin 2b)^n}.$$  

3. (Marius Cavachi) For a real number $x$ let $[x]$ be the greatest integer not exceeding $x$. Prove that the sequence given by $a_n = [n\sqrt{2}] + [n\sqrt{3}]$, where $n$ is a non-negative integer, contains infinitely many odd numbers and infinitely many even numbers.

4. (Severius Moldoveanu and Costel Chiteş) Let $n \geq 2$ be an integer. Find $n$ pairwise disjoint sets $A_1$, $A_2$, \ldots, $A_n$ in the Euclidean plane such that

(a) For each circle $C$ in the plane $A_i \cap \text{Interior}(C) \neq \emptyset$, $1 \leq i \leq n$.

(b) For each line $d$ in the plane and each $A_i$, the projection of $A_i$ on $d$ is not all of $d$. 
We now look at solutions from our readers to problems of the 2004 Taiwanese Mathematical Olympiad given at [2007: 339].

1. Let \( \mathbb{N}_0 \) denote the set of non-negative integers. Find all functions \( f : \mathbb{N}_0 \rightarrow \mathbb{N}_0 \) such that \( f(3m + 2n) = f(m) \cdot f(n) \) for all \( m, n \in \mathbb{N}_0 \).

**Solution by Michel Bataille, Rouen, France, modified by the editor.**

There are three solutions: the constant functions \( f_0 \) and \( f_1 \), which map each \( k \in \mathbb{N}_0 \) to 0 and 1, respectively; and the function \( f_2 \) defined by \( f_2(0) = 1 \) and \( f_2(k) = 0 \) for all \( k \in \mathbb{N}_0 \setminus \{0\} \).

It is easy to check that \( f_0 \), \( f_1 \), and \( f_2 \) are solutions. To show the converse, we will use the following fact: any \( u \in \mathbb{N}_0 \) with \( u \neq 1 \) can be expressed as \( u = 3m + 2n \) for some \( m, n \in \mathbb{N}_0 \). If \( u = 0 \), 2, or 5, then this is clear; for all other \( u \) let \( u = 3p + r \), \( 0 \leq r \leq 2 \), and write \( u = 3(p - r) + 2r \).

Let \( f \) be a solution. For each \( u \geq 1 \) we have

\[
\begin{align*}
f(0) &= f(1) = \cdots = f(u) = 1 \implies f(u + 1) = 1; \quad (1) \\
f(1) &= \cdots = f(u) = 0 \implies f(u + 1) = 0. \quad (2)
\end{align*}
\]

To see these write \( f(u + 1) = f(3m + 2n) = f(m)f(n) \), where \( m, n \in \mathbb{N}_0 \) are such that \( u + 1 = 3m + 2n \). Then (1) follows since \( m \leq u \) and \( n \leq u \), so that \( f(m)f(n) = 1^2 = 1 \), and (2) follows since \( 1 \leq m \leq u \) or \( 1 \leq n \leq u \), so that \( f(m) = 0 \) or \( f(n) = 0 \), respectively.

From \( f(0) = f(3 \cdot 0 + 2 \cdot 0) = f(0)^2 \), we obtain \( f(0) = 0 \) or 1.

Let \( a = f(1) \). We have \( f(5) = f(3 \cdot 1 + 2 \cdot 1) = a^2 \), from which we deduce that \( f(25) = f(3 \cdot 5 + 2 \cdot 5) = a^4 \). Also

\[
\begin{align*}
f(25) &= f(3 \cdot 7 + 2 \cdot 2) = f(7)f(2) = f(3 \cdot 1 + 2 \cdot 2)f(3 \cdot 0 + 2 \cdot 1) \\
&= a^2f(0)f(2) = a^2f(0)f(2) = a^3f(0) ,
\end{align*}
\]

hence \( a^4 = a^3f(0) \).

Now if \( f(0) = 0 \), then \( f(1) = a = 0 \), and \( f = f_0 \) follows by induction using (2). If \( f(0) = 1 \), then \( a^4 = a^3 \), hence either \( f(1) = 1 \) or \( f(1) = 0 \). In the first case \( f = f_1 \) follows by induction using (1), in the second case \( f = f_2 \) follows by induction using (2).

3. Suppose that the points \( D \) and \( E \) lie on the circumcircle of \( \triangle ABC \), ray \( \overline{AD} \) is the interior angle bisector of \( \angle BAC \), and ray \( \overline{AE} \) is the exterior angle bisector of \( \angle BAC \). Let \( F \) be the symmetrical point of \( A \) with respect to \( D \), and let \( G \) be the symmetrical point of \( A \) with respect to \( E \). Prove that, if the circumcircle of \( \triangle ADG \) and the circumcircle of \( \triangle AEF \) intersect at \( P \), then \( AP \) is parallel to \( BC \).

**Solution by Titu Zvonaru, Comănăști, Romania.**

Suppose that the parallel to \( AD \) and passing through \( E \) intersects the parallel to \( AE \) and passing through \( D \) at the point \( T \). Let \( O_1 \) be the circumcentre of triangle \( ADG \) and let \( O_2 \) be the circumcentre of triangle \( AEF \).
Since $O_1$ is on the perpendicular bisector of $AP$, and $O_2$ is on the perpendicular bisector of $AP$, it follows that $O_1O_2 \perp AP$.

Since $AE \perp AD$, the point $O_1$ is the mid-point of $DG$ and the mid-point of $ET$; the point $O_2$ is the mid-point of $EF$ and the mid-point of $DT$. It follows that $O_1O_2 \parallel DE$.

Since $AD$ is the internal bisector of $\angle BAC$, the point $D$ is the mid-point of arc $BC$; we deduce that $DE$ is a diameter of the circumcircle of $\triangle ABC$, hence $DE \perp BC$.

We have now shown that $O_1O_2 \perp AP$, $O_1O_2 \parallel DE$, and $DE \perp BC$, hence, $AP$ is parallel to $BC$.

4. Let $O$ and $H$ be the circumcentre and orthocentre of an acute triangle $ABC$. Suppose that the bisectrix of $\angle BAC$ intersects the circumcircle of $\triangle ABC$ at $D$, and that the points $E$ and $F$ are symmetrical points of $D$ with respect to $BC$ and $O$, respectively. If $AE$ and $FH$ intersect at $G$ and if $M$ is the mid-point of $BC$, prove that $GM$ is perpendicular to $AF$.

Solution by Titu Zvonaru, Comănești, Romania.

The point $D$ is the mid-point of the arc $BC$, and $FD$ is the perpendicular bisector of $BC$. If $AB = AC$, then points $D$, $M$, $O$, $E$, $G$, $A$, and $F$ are collinear, so we assume $AB \neq AC$.

We will prove that $AHEF$ is a parallelogram. In that case it follows that $G$ is the mid-point of $AE$, and since $M$ is the mid-point of $DE$, we see (in $\triangle ADE$) that $GM \parallel AD$. Since $DF$ is a diameter, $AD \perp AF$, hence $GM \perp AF$. We offer two demonstrations.

Proof 1: As usual, $a = BC$ and $R$ is the circumradius of $\triangle ABC$. Since $\angle MBD = \frac{1}{2}\angle BAC$, we have that $MD = \frac{a}{2} \tan \frac{A}{2}$. We obtain

\[
EF = 2R - 2MD = 2R - a \tan \frac{A}{2} = 2R - 2R \cdot \sin A \tan \frac{A}{2} = 2R \left( 1 - 2 \sin \frac{A}{2} \cos \frac{A}{2} \tan \frac{A}{2} \right) = 2R \left( 1 - 2 \sin^2 \frac{A}{2} \right) = 2R \cos A.
\]
It is known that $AH = 2R \cos A$; because $AH \perp BC$, $EF \perp BC$, and $AH = EF$, it follows that $AHEF$ is a parallelogram.

**Proof 2:** Let $AH$ intersect the circumcircle at $T$.

It is known that $T$ is the symmetric point of $H$ with respect of $BC$. It follows that the trapezoid $HTDE$ is isosceles. Since the trapezoid $ATDF$ is isosceles, we have

$$\angle HED = \angle EDT = \angle AFE,$$

hence $AF \parallel HE$. However, $AH \parallel EF$, therefore the quadrilateral $AHEF$ is a parallelogram.

Next, an apology. When I was writing up the material for the May number of the *Corner* I somehow left off Michel Bataille’s name in the list of solvers for Problem 5 of the Albanian Mathematical Olympiad (Test 2) discussed at [2007: 278–279, 2008: 222–224]. We recently received an alternate solution by Li Zhou, which we give now.

**5.** In an acute-angled triangle $ABC$, let $H$ be the orthocentre, and let $d_a$, $d_b$, and $d_c$ be the distances from $H$ to the sides $BC$, $CA$, and $AB$, respectively. Prove that $d_a + d_b + d_c \leq 3r$, where $r$ is the radius of the incircle of triangle $ABC$.

**Alternate solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.**

Let $a = BC$, $b = CA$, and $c = AB$, and $s = \frac{1}{2}(a + b + c)$. Without loss of generality, assume that $a \leq b \leq c$. Then

$$d_a = CH \cos B \leq CH \cos A = d_b$$
$$d_b = AH \cos C \leq AH \cos B = d_c.$$

Denote by $[ABC]$ the area of triangle $ABC$. By Chebyshev’s Inequality,

$$sr = [ABC] = \frac{1}{2}(ad_a + bd_b + cd_c) \geq \frac{1}{3}s(d_a + d_b + d_c),$$

completing the proof.

Now we turn to our files of solutions from our readers and to the XXV Brazilian Mathematical Olympiad 2003 given at [2007: 410].

**1.** Find the smallest positive prime that divides $n^2 + 5n + 23$ for some integer $n$. 
Solved by Geoffrey A. Kandall, Hamden, CT, USA; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Kandall's solution.

The answer is 17.

Let \( P(n) = n^2 + 5n + 23 \). Since \( P(-2) = 17 \), we need only show that \( P(n) \not\equiv 0 \pmod{p} \) if \( p \in \{2, 3, 5, 7, 11, 13\} \).

For instance, if \( n \equiv 0, 1, 2, 3, 4, 5, 6 \pmod{7} \), then we have respectively \( P(n) \equiv 2, 1, 2, 5, 3, 3, 5 \pmod{7} \). Thus, the desired conclusion is true for \( p = 7 \).

The complete calculation is summarized in the table below.

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3. Let \( ABCD \) be a rhombus. Let \( E, F, G, \) and \( H \) be points on the sides \( AB, BC, CD, \) and \( DA \), respectively, so that \( EF \) and \( GH \) are tangent to the incircle of \( ABCD \). Show that \( EH \) and \( FG \) are parallel.

Solved by Michel Bataille, Rouen, France.

The hexagon \( AEFGCH \) circumscribes the incircle of the rhombus, so Brianchon's theorem implies that the lines \( AC, EG, \) and \( FH \) are concurrent. Let \( U \) be their common point and let \( h \) denote the homothety with centre \( U \) which transforms \( A \) into \( C \). Since \( AE \) and \( CG \) are parallel and \( E, U, \) and \( G \) are collinear, we see that \( h(E) = G \). Similarly, \( h(H) = F \) and \( EH \parallel FG \) follows.
5. Let $f(x)$ be a real-valued function defined on the positive reals such that

(i) $f(x) < f(y)$ if $x < y$, and

(ii) $f \left( \frac{2xy}{x+y} \right) = \frac{f(x) + f(y)}{2}$ for all $x$.

Show that $f(x) < 0$ for some value of $x$.

**Solution by Michel Bataille, Rouen, France.**

As $f$ is an increasing function on $(0, \infty)$, either $\lim_{x \to 0^+} f(x) = -\infty$ or $\lim_{x \to 0^+} f(x) = a$ for some real number $a$. Assume that the latter holds. In the relation

$$f \left( \frac{2xy}{x+y} \right) = \frac{f(x) + f(y)}{2},$$

fix $x > 0$ and let $y$ approach $0^+$. Since $\lim_{y \to 0^+} \frac{2xy}{x+y} = 0$, it follows that

$$a = \frac{f(x) + a}{2},$$

hence, $f(x) = a$. Consequently, $f$ would be a constant function, contrary to (i). Thus, $\lim_{x \to 0^+} f(x) = -\infty$ and certainly $f(x) < 0$ for some positive $x$.

Next we turn to solutions of problems of the Second and Third Selection Tests of the 2004 Republic of Moldova, given at [2007: 411-412].

6. Find all functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy the relation

$$f(x^3) - f(y^3) = (x^2 + xy + y^2)(f(x) - f(y))$$

for all real numbers $x$ and $y$.

**Solved by Michel Bataille, Rouen, France; and Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina. We give the solution of Malikić.**

Taking $y = 0$ in the identity yields $f(x^3) - f(0) = x^2(f(x) - f(0))$. Setting $g(x) = f(x) - f(0)$ we have $g(x^3) = x^2g(x)$. The following are then equivalent

$$f(x^3) - f(y^3) = (x^2 + xy + y^2)(f(x) - f(y)),
\quad
f(x^3) - f(y^3) = (x^2 + xy + y^2)(g(x) - g(y)),
\quad
x^2g(x) - y^2g(y) = x^2g(x) + xyg(x) + y^2g(x)
\quad
- x^2g(y) - xyg(y) - y^2g(y),
\quad
0 = xyg(x) + y^2g(x) - x^2g(y) - xyg(y)
\quad
0 = (x + y)(yg(x) - xg(y)).
$$

(1)
Taking \( y = 1 \) in equation (1), we must have \((x + 1)(g(x) - x \cdot g(1)) = 0\). Thus, for all \( x \in \mathbb{R} \setminus \{-1\} \), we must have \( g(x) = xg(1) \), or equivalently \( f(x) - f(0) = x(f(1) - f(0)) \). This means that \( f(x) = kx + c \) for all \( x \in \mathbb{R} \setminus \{-1\} \), where \( k = f(1) - f(0) \) and \( c = f(0) \).

By what we have just done, \( f(2^3) = f(8) = 8k + c \) and \( f(2) = 2k + c \), thus, taking \( x = 2 \) and \( y = -1 \) in the identity for \( f \) yields

\[
8k + c - f(-1) = 3(2k + c - f(-1)).
\]

Solving for \( f(-1) \) we obtain \( f(-1) = k(-1) + c \). Finally, we conclude that \( f(x) = kx + c \), where \( k \) and \( c \) are constants.

Conversely, if \( f(x) = kx + c \) where \( k \) and \( c \) are arbitrary constants, then one readily checks that this \( f \) satisfies the required identity for all reals \( x \) and \( y \).

7. Let \( ABC \) be an acute-angled triangle with orthocentre \( H \) and circumcentre \( O \). The inscribed and circumscribed circles have radii \( r \) and \( R \), respectively. If \( P \) is an arbitrary point of the segment \([OH]\), prove that \( 6r \leq PA + PB + PC \leq 3R \).

Solution by Arkady Alt, San Jose, CA, USA.

Let \( \overline{PO} = t\overline{HO} \), \( t \in [0, 1] \) and let \( X \in \{A, B, C\} \). Then

\[
\overline{PX} = \overline{PO} + \overline{OX} = t\overline{HO} + \overline{OX} = t(\overline{HX} + \overline{XO}) + \overline{OX} = (1-t)\overline{OX} + t\overline{HX}.
\]

Since \( |\overline{PX}| = |(1-t)\overline{OX} + t\overline{HX}| \leq |(1-t)|\overline{OX}| + t|\overline{HX}| \), we have

\[
PA + PB + PC = \sum_{\text{cyclic}} |\overline{PA}| \leq \sum_{\text{cyclic}} ((1-t)|\overline{OA}| + t|\overline{HA}|)
= 3(1-t)R + t\sum_{\text{cyclic}} HA.
\]

For any vertex \( X \), \( HX = 2R \cos X \). Also, \( \cos A + \cos B + \cos C = 1 + \frac{r}{R} \) and Euler’s Inequality, \( R \geq 2r \), holds. Thus,

\[
PA + PB + PC \leq 3(1-t)R + 2Rt(\cos A + \cos B + \cos C) = 3(1-t)R + t(2R + 2r) \leq 3(1-t)R + t(2R + R) = 3R.
\]

Next we prove the inequality \( 6r \leq PA + PB + PC \) for any interior point \( P \) in the acute-angled triangle \( ABC \).

For each vertex \( X \) let \( R_X \) be the distance from \( P \) to \( X \). Let \( h_a, h_b, \) and \( h_c \) be the heights of the triangle to the corresponding side, and let \( d_a, d_b, \) and \( d_c \) be the distances from \( P \) to the corresponding side.
Since $R_A + d_a \geq h_a$ we have $\sum_{\text{cyclic}} (R_a + d_a) \geq \sum_{\text{cyclic}} h_a$. By the Erdös-Mordell Inequality in the form $\sum_{\text{cyclic}} d_a \leq \frac{1}{2} \sum_{\text{cyclic}} R_A$ and the preceding inequality we have $\frac{3}{2} \sum_{\text{cyclic}} R_A \geq \sum_{\text{cyclic}} h_a$, or equivalently $\frac{3}{2} \sum_{\text{cyclic}} h_a \leq \sum_{\text{cyclic}} R_A$.

Since

$$h_a + h_b + h_c = 2F \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \geq 2F \left( \frac{9}{a + b + c} \right) = \frac{9F}{2s} = 9r,$$

where $F$ is the area of triangle $ABC$, we finally obtain

$$6r \leq \frac{2}{3}(h_a + h_b + h_c) \leq R_a + R_b + R_c.$$

Equality occurs if and only if $P$ is the circumcenter and $a = b = c$.

9. For all positive real numbers $a$, $b$, and $c$, prove the inequality

$$\left| \frac{4(a^3 - b^3)}{a + b} + \frac{4(b^3 - c^3)}{b + c} + \frac{4(c^3 - a^3)}{c + a} \right| \leq (a - b)^2 + (b - c)^2 + (c - a)^2.$$

Solved by Arkady Alt, San Jose, CA, USA.

Let $G(a, b, c) = (a - b)^2 + (b - c)^2 + (c - a)^2$ and

$$F(a, b, c) = \frac{4(a^3 - b^3)}{a + b} + \frac{4(b^3 - c^3)}{b + c} + \frac{4(c^3 - a^3)}{c + a}.$$

It suffices to prove that $F(a, b, c) \leq G(a, b, c)$ for all positive real numbers $a$, $b$, and $c$. Indeed, if $F(a, b, c) < 0$ then under this assumption we have

$$|F(a, b, c)| = -F(a, b, c) = F(b, a, c) \leq G(b, a, c) = G(a, b, c).$$

For positive $a$ and $b$ we have $\frac{4b^2}{a + b} \geq 3b - a$, since this is equivalent to $4b^2 \geq 3b^2 - a^2 + 2ab$, and hence to $(a - b)^2 \geq 0$. We now have

$$\sum_{\text{cyclic}} \frac{4(a^3 - b^3)}{a + b} = 4 \sum_{\text{cyclic}} \frac{a^3 + b^3}{a + b} + 2 \sum_{\text{cyclic}} \frac{4b^3}{a + b} \leq 4 \sum_{\text{cyclic}} (a^2 - ab + b^2) - 2 \sum_{\text{cyclic}} b(3b - a) = \sum_{\text{cyclic}} (4a^2 - 4ab + 4b^2 - 6b^2 + 2ab) = \sum_{\text{cyclic}} (4a^2 - 2ab - 2b^2) = \sum_{\text{cyclic}} (a - b)^2.$$. 
10. Determine all the polynomials $P(X)$ with real coefficients which satisfy the relation
\[(x^3 + 3x^2 + 3x + 2)P(x - 1) = (x^3 - 3x^2 + 3x - 2)P(x)\]
for every real number $x$.

Solved by Arkady Alt, San Jose, CA, USA. Comment by Michel Bataille, Rouen, France.

This problem was one of the problems of the 2003 Vietnamese Mathematical Olympiad. A solution appeared in this journal at [2007: 90–91].

11. Let $ABC$ be an isosceles triangle with $AC = BC$, and let $I$ be its incentre. Let $P$ be a point on the circumcircle of the triangle $AIB$ lying inside the triangle $ABC$. The straight lines through $P$ parallel to $CA$ and $CB$ meet $AB$ at $D$ and $E$, respectively. The line through $P$ parallel to $AB$ meets $CA$ and $CB$ at $F$ and $G$, respectively. Prove that the straight lines $DF$ and $GE$ intersect on the circumcircle of the triangle $ABC$.

Solved by Ricardo Barroso Campos, University of Seville, Seville, Spain.

For convenience let
\[\angle CAB = \alpha, \quad \angle ACB = \gamma, \quad \angle DFP = \xi, \quad \angle GPB = \omega.\]

We then have
\[\angle APB = \angle AIB = \alpha + \gamma, \quad \angle APF = 180^\circ - (\gamma + \alpha) - \omega = \alpha - \omega, \quad \angle GBP = \alpha - \omega, \quad \angle PAF = \omega.\]

Thus, $GBEP \sim FPDA$, so that
\[\angle EGP = \angle DFA = 180^\circ - \alpha - \xi = \alpha + \gamma - \xi.\]

Let $U$ be the intersection of the lines $EG$ and $FD$. We then have
\[\angle GUF = 180 - \angle DFP - \angle EGP = (2\alpha + \gamma) - \xi - (\alpha + \gamma - \xi) = \alpha.\]

Now, $UBGD$ is inscribable, since $\angle DUG = \angle DBG = \alpha$. Also, $PGBD$ is inscribable, since $\angle DPG + \angle GBD = 180^\circ$. Thus, $PGBUD$ is inscribable and $\angle GUB = \angle GPB = \omega$. Similarly, $\angle FUA = \angle FPA = \alpha - \omega$ and $\angle AUB = 2\omega$; hence, $U$ is on the circumcircle of triangle $ABC$.

That completes this number of the Corner. Send solutions soon!