SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

We have received late solutions to problem 3221 from the following solvers: Ateneo Problem Solving Group; John G. Heuer; Thanos Magkos; and Pavlos Maragoudakis. Our apologies to Chip Curtis, Missouri Southern State University, Joplin, MO, USA, for a lost batch of correct solutions to problems 3239, 3241, and 3243–3248.


Let \( u_1, u_2, \) and \( u_3 \) be any real numbers. Prove that
\[
\frac{1}{6} \sum_{i=1}^{3} \left[ \cos^2(u_i - u_{i+1}) + \cos^2(u_i + u_{i+1}) \right] \\
\geq (\cos u_1 \cos u_2 \cos u_3)^2 + (\sin u_1 \sin u_2 \sin u_3)^2,
\]
where the subscripts in the summation are taken modulo 3.

Solution by Daniel Tsai, student, Taipei American School, Taipei, Taiwan.

Since \( \cos(x + y) = \cos x \cos y - \sin x \sin y \) for real numbers \( x \) and \( y \), it follows that for each \( i \) we have
\[
\cos^2(u_i - u_{i+1}) + \cos^2(u_i + u_{i+1}) = 2(\cos^2 u_i \cos^2 u_{i+1} + \sin^2 u_i \sin^2 u_{i+1}),
\]
where subscripts are taken modulo 3. Since \( 0 \leq \cos^2 u_i \leq 1 \) and also \( 0 \leq \sin^2 u_i \leq 1 \), for each \( i \), we have
\[
\frac{1}{6} \sum_{i=1}^{3} \left[ \cos^2(u_i - u_{i+1}) + \cos^2(u_i + u_{i+1}) \right] \\
= \frac{1}{6} \sum_{i=1}^{3} 2(\cos^2 u_i \cos^2 u_{i+1} + \sin^2 u_i \sin^2 u_{i+1}) \\
= \frac{1}{3} \sum_{i=1}^{3} (\cos^2 u_i \cos^2 u_{i+1} + \sin^2 u_i \sin^2 u_{i+1}) \\
\geq \frac{1}{3} \sum_{i=1}^{3} (\cos^2 u_i \cos^2 u_{i+1} \cos^2 u_{i+2} + \sin^2 u_i \sin^2 u_{i+1} \sin^2 u_{i+2}) \\
= \frac{1}{3} \cdot 3(\cos^2 u_1 \cos^2 u_2 \cos^2 u_3 + \sin^2 u_1 \cos^2 u_2 \sin^2 u_3) \\
\geq (\cos u_1 \cos u_2 \cos u_3)^2 + (\sin u_1 \cos u_2 \sin u_3)^2.
\]
Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; IAN JUNE L. GARCES and WINFER C. TABARES, Ateneo de Manila University, Quezon City, The Philippines; ATENEÓ PROBLEM SOLVING GROUP, Ateneo de Manila University, Quezon City, The Philippines; DIONNE BAILEY, ELISIE CAMPBELL, and CHARLES R. DIMITRIE, Angelo State University, San Angelo, TX, USA; ROY BARBARA, Lebanese University, Farar, Lebanon; MANUEL BENITO, OSCAR CIAURI, EMILIO FERNANDEZ, and LUZ RONCAL, Logroño, Spain; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakio High School, Athens, Greece; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinen-gymnasium, Innsbruck, Austria; TOM LEO NG, Brooklyn, NY, USA; SALEM MALIK, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; PAVLOS MARAGOUÐAKIS, Pireas, Greece; ANDREA MONARO, student, University of Trento, Trento, Italy; JOSE H. NIETO, Universidad del Zulia, Maracaibo, Venezuela; XAVIER ROS, student, Universitat Politècnica de Catalunya, Barcelona, Spain; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

Ros proved two generalizations of this inequality. For \( n > 2 \) and \( u_1, u_2, \ldots, u_n \) real numbers,

\[
\frac{1}{2n} \sum_{i=1}^{n} \left[ \cos^2(u_i - u_{i+1}) + \cos^2(u_i + u_{i+1}) \right] \geq \left( \prod_{i=1}^{n} \cos u_i \right)^2 + \left( \prod_{i=1}^{n} \sin u_i \right)^2
\]

and

\[
\frac{1}{n(n-1)} \sum_{1 \leq i < j \leq n} \left[ \cos^2(u_i - u_j) + \cos^2(u_i + u_j) \right] \geq \left( \prod_{i=1}^{n} \cos u_i \right)^2 + \left( \prod_{i=1}^{n} \sin u_i \right)^2,
\]

where subscripts in the summation are taken modulo \( n \). Equality holds if and only if \( u_1 \equiv \cdots \equiv u_n \equiv 0 \) (mod \( \pi \)), or \( u_1 \equiv \cdots \equiv u_n \equiv \frac{\pi}{2} \) (mod \( \pi \)).

Benito, CIAURI, Fernandez, and Roncal proved another generalization. For \( n \) and \( u_i \) (\( 1 \leq i \leq n \)) as above,

\[
\frac{1}{2n} \sum_{i=1}^{n} \left[ \cos^2(u_i - u_{i+1}) + \cos^2(u_i + u_{i+1}) \right] \geq \left( \prod_{i=1}^{n} \cos u_i \right)^{4/n} + \left( \prod_{i=1}^{n} \sin u_i \right)^{4/n},
\]

where subscripts in the summation are taken modulo \( n \).


Let \( S \) be a set of complex \( 2 \times 2 \) matrices such that, for all \( A, B, C \in S \), we have \( ABCAB = C \).

(a) Show that \((ABC)^n = A^nB^nC^n\) for all positive integers \( n \) and all matrices \( A, B, C \in S \).

(b) Give an example of such a set \( S \) containing at least three matrices with two of them non-commuting.

Solution by Michael Parmenter, Memorial University of Newfoundland, St. John's, NL.

(a) The condition \( ABCAB = C \) implies \( ABCABC = C^2 \), which, up to symmetry, is the same as \( CABCA = B^2 \). Also, \( ABCAB = C \) implies
CABCA = C², so that B² = C² for any two matrices B, C ∈ S. Now, B² = C² implies B²C = C³ and CB² = C³, so that B² commutes with C for any two matrices B, C ∈ S. Applying the condition ABCAB = C for A = B = C, we obtain A^5 = A for every A ∈ S.

Let k be any positive integer. Using the above observations, we have

\[(ABC)^2k = ((ABC)^2)^k = (C^2)^k = C^{2k}\]

and

\[A^{2k}B^{2k}C^{2k} = (A^2)^k(B^2)^k(C^2)^k = (C^2)^{3k} = C^{6k}\]

\[= C^kC^{5k} = C^k(C^5)^k = C^kC^k = C^{2k}\].

Thus, \((ABC)^2k = A^{2k}B^{2k}C^{2k}\). Using this result and the established commutativity, we obtain

\[(ABC)^{2k+1} = (ABC)^{2k}ABC = A^{2k}B^{2k}C^{2k}ABC = A^{2k}AB^{2k}BC^{2k}C = A^{2k+1}B^{2k+1}C^{2k+1}\].

Therefore, \((ABC)^n = A^nB^nC^n\) for all positive integers n.

(b) We first give a general approach to finding such an example. Let A be any matrix with \(A^4 = I\) and \(A^2 \neq I\), where I is the identity matrix. Let \(B = A^3\) and let C be any matrix such that \(C^2 = A^2\) and \(AC \neq CA\). Then \(AB = BA = A^4 = I\). It is easy to show that the condition \(XYZY = Z\) holds for any permutation \((X, Y, Z)\) of the matrices A, B, and C:

\[
\begin{align*}
ABCAB &= ICI = C, \\
BACBA &= ICI = C, \\
ACBAC &= ACIC = AC^2 = A^3 = B, \\
CABCA &= CICA = C^2A = A^3 = B, \\
BCABC &= BCIC = BC^2 = A^3A^2 = A, \\
CBACB &= CICB = C^2B = A^2A^3 = A.
\end{align*}
\]

One particular example is

\[
A = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad B = A^3 = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} i & x \\ 0 & -i \end{bmatrix},
\]

where \(x \neq 0\). Clearly, A and C do not commute, because

\[
AC - CA = \begin{bmatrix} 0 & 2xi \\ 0 & 0 \end{bmatrix}.
\]

Also solved by ROY BARBARA, Lebanese University, Fanar, Lebanon; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; JOSE H. NIETO, Universidad del Zulia, Maracaibo, Venezuela; PETER Y. WOO, Biola University, La Mirada, CA, USA (part (a) only); and the proposer.

Both Barbara and Parmenter noted that the result in part (a) is true in any semigroup.
Proposed by Mihály Bencze, Brasov, Romania.

Prove that
\[
\log_e(e^n - 1) \log_e(e^n + 1) + \log_e(\pi^n - 1) \log_e(\pi^n + 1) < e^2 + \pi^2.
\]

1. Essentially similar solutions by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam; Tom Leong, Brooklyn, NY, USA; Andrea Munaro, student, University of Trento, Trento, Italy; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and the proposer.

By the AM–GM Inequality, we have
\[
\log_e(e^n - 1) \log_e(e^n + 1) \leq \left(\frac{\log_e(e^n - 1) + \log_e(e^n + 1)}{2}\right)^2
\]
\[
= \frac{1}{4} \left(\log_e((e^n - 1)(e^n + 1))\right)^2 = \frac{1}{4} (\log_e(e^{2n} - 1))^2
\]
\[
< \frac{1}{4} (\log_e(e^{2\pi}))^2 = \pi^2.
\]
Similarly, we have
\[
\log_e(e^n - 1) \log_e(e^n + 1) < e^2.
\]

The result follows by adding (1) and (2).

II. Solution by Manuel Benito, Óscar Ciaurri, and Emilio Fernández, Logroño, Spain.

For \( a > 1 \) and \( b > 0 \), we have
\[
\log_a(a^b - 1) \log_a(a^b + 1)
\]
\[
= (b + \log_a(1 - a^{-b}))(b + \log_a(1 + a^{-b}))
\]
\[
= b^2 + b \log_a(1 - a^{-2b}) + \log_a(1 - a^{-b}) \log_a(1 + a^{-b}) < b^2,
\]
since \( b \) and \( 1 - a^{-2b} \) are positive, \( 1 - a^{-b} < 1 \), and \( 1 + a^{-b} > 1 \). From the inequality above, we conclude that
\[
\sum_{i=1}^{n} \log_{a_i}(a_i^{b_i} - 1) \log_{a_i}(a_i^{b_i} + 1) \leq \sum_{i=1}^{n} b_i^2,
\]
where \( a_i > 1 \) and \( b_i > 0 \) for \( i = 1, 2, \ldots, n \).

The proposed inequality is the special case when \( n = 2 \), \( a_1 = b_2 = e \) and \( a_2 = b_1 = \pi \).

Also solved by Dionne Bailey, Elsie Campbell, Charles Diminnie, Karl Haylak, Paul Koča, and Andrew Siefker, Angelo State University, San Angelo, TX, USA; Roy Barbara, Lebanese University, Fanar, Lebanon; Michel Bataille, Rouen, France;

Let \( C \) be a convex figure in the plane. A \textit{diametrical chord} \( AB \) of \( C \) parallel to the direction vector \( \mathbf{v} \) is a chord of \( C \) of maximal length parallel to the direction vector \( \mathbf{v} \).

Prove that if every diametrical chord of \( C \) bisects the area enclosed by \( C \), then \( C \) must be centro-symmetric.

Solution by P.C. Hammer and T. Jefferson Smith from 1964, adapted by the editor.

What follows is a simplified version of the proof of Theorem 2.4 in [3].
In that work the authors prove that any convex planar body is centrally symmetric provided that each line bisecting the area is a diametral line. (Hammer and Smith use the words \textit{diametral} and \textit{centrally symmetric} rather than the equivalent but less common \textit{diametrical} and \textit{centro-symmetric}).

Because in every direction there is exactly one line that bisects the given area, our assumption that every diametrical chord is area bisecting implies that there is a unique diametral chord in every direction. The Hammer and Smith result is therefore stronger since it applies also to centrally symmetric regions whose boundary contains line segments (for which points of parallel sides are joined by parallel diametral chords, one of which bisects the area).

In order to avoid a page of technical arguments, we will further restrict our result by assuming that the boundary of the convex region, denoted by \( C \), is a differentiable curve. For such boundaries our assumption that an area-bisecting chord is diametral implies that the tangent lines at its ends are parallel. We treat the plane as a vector space and let \( u(\theta) = (\cos \theta, \sin \theta) \) be a unit vector function; then \( u'(\theta) = (-\sin \theta, \cos \theta) = u(\theta) + \frac{1}{2}\pi \). Let \( m(\theta) \) be the unique diametral line parallel to the direction of \( u(\theta) \).

Then there exists a unique real number \( p(\theta) \) such that \( m(\theta) \) is representable as

\[
\{ x : x \cdot u'(\theta) = p(\theta) \} = \{ x : x = p(\theta) u'(\theta) + t u(\theta), \ t \in \mathbb{R} \}.
\]

Note that \( p(\theta) u'(\theta) \) is the foot of the perpendicular from the origin to \( m(\theta) \).
Because we assume that \( C \) is differentiable, it follows easily that so is \( p(\theta) \).
We now represent \( C \) by a function \( x(\theta) \) in the following way. Each line \( m(\theta) \) intersects the boundary in two points, one of which is given by

\[
x(\theta) = p(\theta) u'(\theta) + f(\theta) u(\theta),
\]
where \( f(\theta) \) is assigned a value that makes it continuous. With that assignment then, since \( p(\theta + \pi) = -p(\theta) \) while \( u(\theta + \pi) = -u(\theta) \), the other point of \( m(\theta) = m(\theta + \pi) \) on the boundary would be

\[
x(\theta + \pi) = p(\theta)u'(\theta) - f(\theta + \pi)u(\theta). \tag{3}
\]

Subtracting (3) from (2) gives us \( x(\theta) - x(\theta + \pi) = (f(\theta) + f(\theta + \pi))u(\theta) \). Therefore, we choose the function \( f(\theta) \) so that \( f(\theta) + f(\theta + \pi) > 0 \), whence

\[
f(\theta) + f(\theta + \pi) = |x(\theta) - x(\theta + \pi)| \tag{4}
\]

for this choice. With this notation the condition that the tangents at the ends of a diametral chord are parallel becomes

\[
|x'(\theta + \pi)| \cdot x'(\theta) = -|x'(\theta)| \cdot x'(\theta + \pi). \tag{5}
\]

We must prove that \( x'(\theta + \pi) = -x'(\theta) \). This will require two simple observations about neighbouring diametral chords. First, in the limit, they intersect at

\[
s(\theta) = p(\theta)u'(\theta) - p'(\theta)u(\theta). \tag{6}
\]

More precisely, the diametral line \( m(\theta) \) (whose points, according to (1) with \( x = (x_1, x_2) \), satisfies the equation \( x_1 \sin \theta - x_2 \cos \theta + p(\theta) = 0 \)) intersects the diametral line \( m(\theta + h) \) in the point

\[
\left( \frac{p(\theta) \cos(\theta + h) - p(\theta + h) \cos \theta}{\sin h}, \frac{p(\theta) \sin(\theta + h) - p(\theta + h) \sin \theta}{\sin h} \right).
\]

The limit of these intersection points on \( m(\theta) \) as \( h \to 0 \) is \( s(\theta) \) in (6). Alternatively, one can avoid such calculations by referring to the diagram below,

\[\text{Diagram}\]

where \( m(\theta + h) = SR \) intersects \( m(\theta) = SP \) at \( S \), while \( R \) and \( P \) are the feet of the perpendiculars from the origin \( O \). The claim to be verified...
is that the distance \( PS \) from the foot of the perpendicular \( P \) to the point of intersection \( S \) approaches \( p'(\theta) \): Because of the right angles at \( R \) and \( P \), these points lie on the circle whose diameter is \( OS \). The circle with centre \( O \) and radius \( p = p(\theta) = OP \) (which is tangent to \( SP \) at \( P \)) intersects \( OR \) at \( Q \) (as in the figure). Thus, for small values of \( h = d\theta \) we see that \( PQ \) is approximately \( pd\theta \), and \( QR = dp \). As \( h \to 0 \), \( RP \) approaches the tangent to circle \( OPS \) at \( P \), so that \( \angle RPS \to \angle POR \). Consequently, \( \frac{dp}{pd\theta} \to \frac{SP}{p} \), so that the limit satisfies

\[
SP = \frac{dp}{d\theta} = p'(\theta),
\]
as claimed.

The second required observation is that as \( h \to 0 \), the point where \( m(\theta + h) \) intersects \( m(\theta) \) approaches the midpoint \( \frac{1}{2}(x(\theta) + x(\theta + \pi)) \) of the chord of \( C \) along \( m(\theta) \). This is an immediate consequence of the fact that two area bisecting chords, namely \( m(\theta + h) \) and \( m(\theta) \), divide the region bounded by \( C \) into four sectors such that opposite sectors have the same area. [Ed.: The intersecting lines are becoming the sides of isosceles triangles with equal areas and vertex angles: If the area bisecting chords \( AB \) and \( A'B' \) intersects at \( X \), then \( \angle A'XA = \angle B'XB \) while \( XA' \to XA \) as \( A' \to A \) and \( XB' \to XB \) as \( B' \to B \).] This observation and equations (2), (3), and (5) yield

\[
s(\theta) = \frac{1}{2}(x(\theta) + x(\theta + \pi)) = p(\theta)u'(\theta) + \frac{1}{2}(f(\theta) - f(\theta + \pi))u(\theta) = p(\theta)u'(\theta) - p'(\theta)u(\theta),
\]
from which we conclude that

\[
p'(\theta) + f(\theta) = f(\theta + \pi) - p'(\theta). \tag{7}
\]

By taking the derivatives of the expressions in equations (2) and (3), we note that the coefficient of \( u'(\theta) \) in \( x'(\theta) \) is \( p'(\theta) + f(\theta) \), while in \( x'(\theta + \pi) \) it is \( -(f(\theta + \pi) - p'(\theta)) \). Setting the coefficients of \( u'(\theta) \) equal in equation (5) therefore gives us

\[
(p'(\theta) + f(\theta))|x'(\theta + \pi)| = (f(\theta + \pi) - p'(\theta))|x'(\theta)|.
\]

Because \( p'(\theta) + f(\theta) \) is strictly positive (compare formula (2) with (6) and recall that the midpoint of each chord is interior to \( C \)), this last equation together with the equality in (7) implies that

\[
|x'(\theta + \pi)| = |x'(\theta)|.
\]

Equation (5) now says that \( x'(\theta) = -x'(\theta + \pi) \). On integration we find that \( \frac{1}{2}(x(\theta) + x(\theta + \pi)) \) is a constant, namely the midpoint of every diametral chord, which completes the proof.
Also solved by PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.
The editors are grateful for the help and the references provided by Paul Goodey, University of Oklahoma, Norman, OK; and Horst Martini University of Technology, Chemnitz, Germany. By coincidence, each has recently published a paper that discusses the claim in our problem, see [1] and [2] where related theorems are proved and further references provided. In [3] the authors prove that G will also be centrally symmetric if each diametral line bisects the circumference. They trace their theorems back to a 1921 work of Konrad Zindler [4].

References


Prove that, as the points A, B, C move over the surface of an ellipsoid centred at O while the lines OA, OB, OC stay mutually perpendicular, the plane ABC remains tangent to a fixed sphere.

Solution by Apostolis K. Demis, Varvakeio High School, Athens, Greece.

Let \( \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \) be the equation of the given ellipsoid in an orthogonal normal coordinate system; that is, the centre of the ellipsoid is O(0, 0, 0) and its axes are along the x-, y-, and z-axes. If the vectors from the origin to the moving points are \( \overrightarrow{OA} = |\overrightarrow{OA}|(a_1, a_2, a_3) \), \( \overrightarrow{OB} = |\overrightarrow{OB}|(b_1, b_2, b_3) \), and \( \overrightarrow{OC} = |\overrightarrow{OC}|(c_1, c_2, c_3) \), then the condition that the lines OA, OB, and OC stay mutually perpendicular implies the equations
\[
\begin{align*}
\sum_{i=1}^{3} a_i^2 &= \sum_{i=1}^{3} b_i^2 = \sum_{i=1}^{3} c_i^2 = 1, \\
\sum_{i=1}^{3} a_i b_i &= \sum_{i=1}^{3} b_i c_i = \sum_{i=1}^{3} c_i a_i = 0. 
\end{align*}
\]
(1)

The points A, B, and C lie on the ellipse, hence, their coordinates satisfy the equation of the ellipse:
\[
\begin{align*}
\frac{1}{|\overrightarrow{OA}|^2} &= \frac{a_1^2}{a^2} + \frac{a_2^2}{b^2} + \frac{a_3^2}{c^2}, \\
\frac{1}{|\overrightarrow{OB}|^2} &= \frac{b_1^2}{a^2} + \frac{b_2^2}{b^2} + \frac{b_3^2}{c^2}, \\
\frac{1}{|\overrightarrow{OC}|^2} &= \frac{c_1^2}{a^2} + \frac{c_2^2}{b^2} + \frac{c_3^2}{c^2}.
\end{align*}
\]
(2)
If we define the matrix

\[ M = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}, \]

then the first set of equations in (1) imply that \( MM^t = I \). That is, \( M \) is an orthogonal matrix. This means that we also have \( M^tM = I \), so that

\[ a_1^2 + b_1^2 + c_1^2 = a_2^2 + b_2^2 + c_2^2 = a_3^2 + b_3^2 + c_3^2 = 1. \]  

(3)

Combining the equations in (2) and (3), we deduce that

\[
\frac{1}{|OA|^2} + \frac{1}{|OB|^2} + \frac{1}{|OC|^2} = \frac{a_1^2 + b_1^2 + c_1^2}{a^2} + \frac{a_2^2 + b_2^2 + c_2^2}{b^2} + \frac{a_3^2 + b_3^2 + c_3^2}{c^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}.
\]

Letting \( d \) denote the distance from \( O \) to the plane \( ABC \), we recognize the left side of the preceding set of equations to be \( \frac{1}{d^2} \). [Ed.: Although Demis includes a short proof of this claim, it can be found in standard references; the distance from the origin to the plane \( px + qy + rz = 1 \) is \( \frac{1}{\sqrt{p^2 + q^2 + r^2}} \), where \( \frac{1}{p}, \frac{1}{q}, \) and \( \frac{1}{r} \) are the distances from the origin to the points where the coordinate axes intersect the plane.] The equation tells us that \( \frac{1}{d^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \) is a constant. In other words,

\[ d = \frac{abc}{\sqrt{a^2b^2 + b^2c^2 + c^2a^2}}, \]

and we conclude that the plane \( ABC \) remains tangent to the sphere with centre \( O \) and radius \( d \).

Also solved by MICHEL BATAILLE, Rouen, France; MANUEL BENITO, OSCAR CIAURRI, EMILIO FERNANDEZ and LUZ RONCAL, Logroño, Spain; JOEL SCHLOSBERG, Bayside, NY, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

Although all the submitted solutions were essentially the same, only the treatment of Benito, CIAURRI, Fernandez, and Roncal explicitly dealt with the \( n \)-dimensional version of the problem - the result (and the proof) is independent of dimension; it applies to an \( n \)-dimensional ellipsoid. The \( n \)-dimensional version of our problem can be found as exercise 15.7.20 in Marcel Berger’s Geometry, volume 2 (Springer 1987). His solution, which is also similar to our flattened solution, appears in the accompanying volume by Berger et al., Problems in Geometry (Springer Problem Books in Mathematics, Springer 1984), problem 15.4, page 227-229. After their proof the authors investigate the dual problem: the polarity with respect to the fixed sphere in our problem takes \( A, B, C \) to mutually orthogonal planes that intersect in the pole of the plane \( ABC \) (which is the point where the plane is tangent to the sphere). Because \( A, B, C \) lie on an ellipsoid, their polar planes are tangent to an ellipsoid, the polar body of the ellipsoid we started with. If, instead, we start with that new ellipsoid, we get the dual result.
The locus of those points common to three mutually orthogonal planes that are
tangent to an ellipsoid is a sphere, concentric with the ellipsoid.
The dual result seems to be better known, and it comes with a variety of proofs. The sphere so
obtained is variously known as the orthoptic sphere, Monge's sphere, or the director sphere.

3256. [2007 : 298, 300] Proposed by Václav Konečný, Big Rapids, MI, USA.

A bicentric quadrilateral (also called a chord-tangent quadrilateral) is a
quadrilateral that is simultaneously inscribed in one circle and circumscribed
about another.

Let \(ABCD\) be a bicentric quadrilateral in which there are no parallel
sides. Suppose that the circumscribed and inscribed circles of \(ABCD\) have
centres \(O\) and \(I\), respectively. Let \(AC\) meet \(BD\) at \(E\). Join the points of
tangency on the opposite sides of the quadrilateral, thus obtaining two lines
which intersect at a point \(T\).

Prove that \(O, E, T,\) and \(I\) are collinear. When do the points \(E\) and \(T\)

Solution by Michel Bataille, Rouen, France.

First we show the following:

If the points \(A, B, C,\) and \(D\) are arranged so that the lines \(AB,\)
\(BC, CD,\) and \(DA\) are tangent to a circle \([\text{Ed.}: \text{or, more generally,}
to a conic}] \(\gamma\) at \(P, Q, R,\) and \(S,\) respectively, then the point of
intersection \(T\) of \(PR\) and \(QS\) is also the point of intersection of
\(AC\) and \(BD.\)

This result is just a special case of Brianchon's Theorem. \([\text{Ed.}: \text{See the com-
ments after the list of solvers}].\) However, the following proof seems more
appropriate here. Let \(AB\) and \(CD\) meet at \(U,\) and let \(AD\) and \(BC\) meet at
\(V.\) Then \(PR\) is the polar of \(U\) with respect to \(\gamma,\) and \(QS\) is the polar of \(V.\) It
follows that \(UV\) is the polar of \(T.\) But \(X = PS \cap QR\) and \(W = RS \cap PQ\) are
conjugates of \(T,\) hence, the polar \(UV\) of \(T\) passes through \(W\) and \(X.\) More-
over, the polar \(RS\) of \(D\) and the polar \(PQ\) of \(B\) pass through \(W,\) hence, the
polar of \(W\) is \(BD,\) and so \(BD\) passes through \(T.\) Similarly, the polar of \(X\)
is \(AC\) and passes through \(T.\) The result follows. Note that the argument is
projective, so that it is easily adapted if any of \(U, V, X,\) or \(W\) is at infinity;
in other words, the proof remains valid should any or all of the sides of the
quadrilaterals \(ABCD\) and \(PQRS\) be parallel.

We now suppose that \(\gamma\) is a circle with centre \(I\) and that \(A, B, C,\) and
\(D\) are, in addition, on a circle \(\Gamma\) with centre \(O.\) Since \(AC\) and \(BD\) meet at
\(T,\) the polar of \(T\) with respect to \(\Gamma\) passes through \(U\) and \(V,\) hence, is the
line \(UV.\) As a result, \(UV\) is the polar of \(T\) with respect to \(\Gamma\) as well as to \(\gamma.\)
Because lines that are conjugate with respect to a circle are perpendicular,
we conclude that \(E\) (which, as we have seen, coincides with \(T),\) \(O,\) and \(I\) are
on the perpendicular to \(UV\) through \(T.\)
Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; JOHN G. HEUVER, Grande Prairie, AB; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; ANDREA MUNARO, student, University of Trento, Trento, Italy; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

Leon Anne served as rapporteur at the Collège Louis-le-Grand. His proof that $E = T$ appeared in *Les Nouvelles Annales de 1842, page 186-1844, pages 28 and 455; it was reproduced in paragraph 1274 (pages 563-564) in [4].

Our problem appears in paragraph 1275 (page 565) in [4] under the heading “Newton’s Theorem,” with a proof similar to that of our first featured solution. No explicit reference to Newton’s work appears there. The result appeared before in *Crux Mathematicorum* [1999, 226], as an unused I.M.O. proposal. Other references, such as [3], page 2, seem to refer to the (restricted) assertion that $E = T$ for a circle as Newton’s theorem. Proofs of this restricted theorem in [1], page 79; and [5], page 100, apply Brianchon’s theorem (if a hexagon is circumscribed about a circle, the three diagonals are concurrent) to the degenerate hexagons $ABCD$ and $ABCD$ to conclude that $AC$, $PR$, $BD$, and $QA$ all pass through the same point $E = T$. In Problem 39 of [2], pages 188-191, there is a discussion of properties of chord-tangent quadrilaterals. We thank Curtis, Demis, Heuver (he refers to problem 40 of [6]), Malikic, and Konceny for the helpful references.

References


3257. [2007 : 298, 300] Proposed by Bill Sands, University of Calgary, Calgary, AB.

Find the number of ordered pairs $(A, B)$ of subsets of $\{1, 2, \ldots, 13\}$ such that $|A \cup B|$ is even.

Essentially similar solutions by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; Michel Bataille, Rouen, France; José H. Nieto, Universidad del Zulia, Maracaibo, Venezuela; and Xavier Ros, student, Universitat Politècnica de Catalunya, Barcelona, Spain; modified by the editor.

The solution provided is for $S = \{1, 2, \ldots, n\}$, the given problem being the special case $n = 13$. If $A$ is a subset of $S$ with $|A| = k$, $0 \leq k \leq n$, and if $B$ is a subset of $S$ such that $A \cup B = S$, then $B$ is the union of $S - A$ and an arbitrary subset $C$ of $A$. There are $\binom{n}{k}$ such subsets $A$ and there are...
\[ 2^k \text{ possibilities for the subset } C, \text{ hence, the number of such pairs } (A, B) \text{ is} \]
\[ \sum_{k=0}^{n} \binom{n}{k} 2^k = 3^n. \]

It is clear that the number of ordered pairs \((A, B)\) of subsets of \(S\) such that \(|A \cup B|\) is even, is \(E_n = \sum_{k \text{ even}} \binom{n}{k} 3^k\). To evaluate \(E_n\) observe that
\[ \sum_{k=0}^{n} \binom{n}{k} 3^k = (1 + 3)^n = 4^n, \quad (1) \]
\[ \sum_{k=0}^{n} (-1)^k \binom{n}{k} 3^k = (1 - 3)^n = (-2)^n. \quad (2) \]

Adding \((1)\) and \((2)\) yields \(2E_n = 4^n + (-2)^n\), hence, \(E_n = 2^{2n-1} + (-1)^n 2^{-n-1}\).

The required number of pairs is \(E_{13} = 2^{25} - 2^{12} = 33550336\).

Also solved by ROY BARBARA, Lebanese University, Fanar, Lebanon; MANUEL BENITO, OSCAR CIAUERRI, EMILIO FERNANDEZ, and LUZ RONCAL, Logroño, Spain; WHITNEY BULLOCK and DEBORAH SALAS-SMITH, students, California State University, Fresno, CA, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brahl, NRW, Germany; JOHN HAWKINS and DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; WALther JANOUS, Ursulinengymnasium, Innsbruck, Austria; Tom LEONG, Brooklyn, NY, USA; KATHERINE E. LEWIS, SUNY Oswego, Oswego, NY, USA; SALEM MALIK, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; EDMUND SWYLAN, Riga, Latvia; BINGJIE WU, student, High School Affiliated to Fudan University, Shanghai, China; and the proposer.

Guepel generalized the result by showing that the number of \(q\)-tuples \((B_1, \ldots, B_q)\) of subsets of \(\{1, \ldots, N\}\) such that \(|B_1 \cup \cdots \cup B_q|\) is even is \(\frac{1}{2} \left(2^q N + (-1)^q \left(2 - 2^q\right)^N\right)\). The Missouri State University Problem Solving Group showed that if \(|B_1 \cup \cdots \cup B_q|\) is restricted to be a multiple of \(d\), then the number of \(q\)-tuples is \(\frac{d}{d-1} \left(1 + (2^q - 1)d^{q-1}\right)^N\), where \(\omega\) is a primitive \(d^{th}\) root of unity.

\[ 3258 \star. \quad [2007 : 298, 300] \text{ Proposed by Alper Cay, Uzman Private School, Kayseri, Turkey.} \]

Let \(ABC\) be a triangle with \(\angle ABC = 80^\circ\). Let \(BD\) be the angle bisector of \(\angle ABC\) with \(D\) on \(AC\). If \(AD = DB + BC\), determine \(\angle A\), using a purely geometric argument.

**Solution.** The Problem Editor of *Math Horizons*, Andy Liu, informed us that this problem appeared recently as problem 201 in the April 2006 issue of *Math Horizons*, page 32. A solution (showing that \(\angle A = 20^\circ\)) by David Rhee and Jerry Lo was published in the November 2006 issue, pages 41-42.

Also solved by MANUEL BENITO, OSCAR CIAUERRI, EMILIO FERNANDEZ, and LUZ RONCAL, Logroño, Spain; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; VAeLAV KONECNY, Big Rapids, MI, USA; JOSE H. NIETO, Universidad del Zulia, Maracaibo, Venezuela; and PETER Y. WOO, Biola University, La Mirada, CA, USA.
Proposed by Neven Jurić, Zagreb, Croatia.

Is it possible to find a cubic polynomial $P$ such that, for any positive integer $n$, the polynomial $P \circ P \circ \cdots \circ P$ has exactly $3^n$ distinct real roots?

Find one, if possible, or show that none exists.

I. Solution by José H. Nieto, Universidad del Zulia, Maracaibo, Venezuela, modified slightly by the editor.

Let $P(x) = x^3 - 3x$. Then $P$ has three distinct real roots, namely 0 and $\pm \sqrt{3}$. Since $P'(x) = 3x^2 - 3 = 3(x+1)(x-1)$ we see that $P$ is strictly increasing on $(-\infty, -1]$ and $[1, \infty)$, and strictly decreasing on $[-1, 1]$.

Note that $P(-2) = -2$, $P(-1) = 2$, $P(1) = -2$, and $P(2) = 2$, so it follows that $P([-2, -1]) = P([-1, 1]) = P([1, 2]) = [-2, 2]$.

Now we prove by induction that $P^n = P \circ P \circ \cdots \circ P$ (the $n$-fold composite) has exactly $3^n$ distinct real roots, all of which lie in the interval $(-2, 2)$. This is clearly true for $n = 1$. Suppose the claim holds for some $n \geq 1$. Note that for $x \in (-2, -1)$, the polynomial $P$ takes each value in $(-2, 2)$ exactly once. The same is true for $x \in (-1, 1)$ and for $x \in (1, 2)$. Therefore, $P^{n+1}(x) = P^n(P(x))$ has exactly $3^n$ distinct real roots in each of the intervals $(-2, -1)$, $(-1, 1)$, and $(1, 2)$, hence, $P^{n+1}$ has $3^{n+1}$ distinct real roots in the interval $(-2, 2)$. These are all the roots of $P^{n+1}$, because $P^{n+1}$ is of degree $3^{n+1}$. The induction is complete.

II. Solution by the Missouri State University Problem Solving Group.

More generally, we show that for any positive integer $d$, there exists a polynomial $P$ of degree $d$ such that for any positive integer $n$, the $n$-fold composite $P^n = P \circ P \circ \cdots \circ P$ has exactly $d^n$ distinct real roots.

Let $P(x) = T_d(x) = \cos(d \cos^{-1} x)$ be the Chebyshev Polynomial of degree $d$ of the first kind. It follows from the definition (and is well known) that $T_d$ has the $d$ distinct real roots $\cos \left( \frac{(2k+1)\pi}{2d} \right)$, $k = 0, 1, \ldots, d-1$, and that $T_a \circ T_b = T_{ab}$. Therefore, $T_{d^n} = T_d \circ T_d \circ \cdots \circ T_d$ has $d^n$ distinct real roots. In particular, if we take $d = 3$, then

$$P(x) = T_3(x) = \left( x - \cos \left( \frac{\pi}{6} \right) \right) \left( x - \cos \left( \frac{\pi}{2} \right) \right) \left( x - \cos \left( \frac{5\pi}{6} \right) \right)$$

$$= x \left( x - \frac{\sqrt{3}}{2} \right) \left( x + \frac{\sqrt{3}}{2} \right) = \frac{1}{4} (4x^3 - 3x)$$

is a solution to the proposed problem.

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; SALEM MALIKIC, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; XAVIER ROS, student, Universitat Politècnica de Catalunya, Barcelona, Spain; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

Let $a$, $b$ be distinct positive real numbers such that $(a - 1)(b - 1) \geq 0$. Prove that

$$a^b + b^a \geq 1 + ab + (1 - a)(1 - b) \cdot \min\{1, ab\}.$$  

Solution by Tom Leong, Brooklyn, NY, USA.

The numbers $a$ and $b$ need not be distinct, as the proof will show.

First suppose that $0 < a < 1$. Then we also have $0 < b < 1$. Put $a = 1 - r$ and $b = 1 - s$, where $0 < r < 1$ and $0 < s < 1$. Since in this case $\min\{1, ab\} = ab$, the right-hand side of the inequality is

$$1 + (1 - r)(1 - s) + rs(1 - r)(1 - s) = 1 + (1 - r)(1 - s)(1 + rs).$$

We have $(1 - r)^s \leq 1 - rs$ and $(1 - s)^r \leq 1 - rs$ by Bernoulli's Inequality. Since $\frac{1}{1 - rs} > 1 + rs$, we obtain

$$a^b + b^a = \frac{1 - r}{(1 - r)^s} + \frac{1 - s}{(1 - s)^r} \geq \frac{1 - r}{1 - rs} + \frac{1 - s}{1 - rs} = 1 + \frac{(1 - r)(1 - s)}{1 - rs} \geq 1 + (1 - r)(1 - s)(1 + rs) = 1 + ab + (1 - a)(1 - b)ab.$$  

Next suppose that $a \geq 1$. Then we also have $b \geq 1$. Put $a = 1 + r$ and $b = 1 + s$, where $r \geq 0$ and $s \geq 0$. Since in this case $\min\{1, ab\} = 1$, the right-hand side of the inequality is

$$1 + (1 + r)(1 + s) + rs = 2rs + r + s + 2.$$  

We have $(1 + r)^{1+s} \geq 1 + r(1+s)$ and $(1 + s)^{1+r} \geq 1 + s(1+r)$ by Bernoulli's Inequality. Adding the two inequalities, we have $a^b + b^a \geq 2rs + r + s + 2$. Equality holds for $a = b = 1$.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; THANOS MAGKOS 3rd High School of Kozani, Kozani, Greece; SALEM MALIKIC, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; XAVIER ROS, student, Universitat Politècnica de Catalunya, Barcelona, Spain; and the proposer.
The Fibonacci numbers $F_n$ and Lucas numbers $L_n$ are defined by the following recurrences:

$$F_0 = 0, \quad F_1 = 1,$$
$$L_0 = 2, \quad L_1 = 1,$$
and
$$F_{n+1} = F_n + F_{n-1}, \quad \text{for } n \geq 1;$$
$$L_{n+1} = L_n + L_{n-1}, \quad \text{for } n \geq 1.$$

Prove that

$$\sum_{n=1}^{\infty} \frac{\arctan \left( \frac{1}{L_{2n}} \right) \arctan \left( \frac{1}{L_{2n+2}} \right)}{\arctan \left( \frac{1}{F_{2n+1}} \right)} \leq \frac{4}{\pi} \arctan(\beta) \left( \arctan(\beta) + \frac{1}{3} \right),$$

where $\beta = \frac{1}{2}(\sqrt{5} - 1)$.

Solution by Manuel Benito, Óscar Ciaurri, Emilio Fernández, and Luz Roncal, Logroño, Spain; and Chip Curtis, Missouri Southern State University, Joplin, MO, USA.

The following relations between the Fibonacci and Lucas numbers

$$L_{2n} + L_{2n+2} = 5F_{2n+1} \quad \text{and} \quad L_{2n}L_{2n+2} - 1 = 5F_{2n+1}^2,$$

are well known and easy to check. From these we have

$$\frac{1}{F_{2n+1}} = \frac{L_{2n} + L_{2n+2}}{L_{2n}L_{2n+2} - 1} = \frac{1}{1} + \frac{1}{L_{2n}L_{2n+2}}.$$

so that

$$\arctan \left( \frac{1}{F_{2n+1}} \right) = \arctan \left( \frac{\frac{1}{L_{2n}} + \frac{1}{L_{2n+2}}}{1 - \frac{1}{L_{2n}L_{2n+2}}} \right) = \arctan \left( \frac{1}{L_{2n}} \right) + \arctan \left( \frac{1}{L_{2n+2}} \right).$$

Applying the inequality $xy \leq \frac{1}{4}(x + y)^2$, we obtain

$$\arctan \left( \frac{1}{L_{2n}} \right) \arctan \left( \frac{1}{L_{2n+2}} \right) \leq \frac{1}{4} \left[ \arctan \left( \frac{1}{F_{2n+1}} \right) \right]^2,$$
therefore,
\[
\sum_{n=1}^{\infty} \frac{\arctan \left( \frac{1}{L_{2n}} \right) \arctan \left( \frac{1}{L_{2n+2}} \right)}{\arctan \left( \frac{1}{F_{2n+1}} \right)} \leq \frac{1}{4} \sum_{n=1}^{\infty} \arctan \left( \frac{1}{F_{2n+1}} \right).
\]

Using the relation \( F_{2n+2} - F_{2n} = F_{2n+1} \) and the well known and easy to check formula \( F_{2n} F_{2n+2} + 1 = F_{2n+1}^2, \) we have
\[
\frac{1}{F_{2n+1}} = \frac{F_{2n+2} - F_{2n}}{F_{2n} F_{2n+2} + 1} = \frac{1}{1 + \frac{F_{2n}}{F_{2n+2}}},
\]
and then
\[
\arctan \left( \frac{1}{F_{2n+1}} \right) = \arctan \left( \frac{\frac{1}{F_{2n}} - \frac{1}{F_{2n+2}}}{1 + \frac{1}{F_{2n} F_{2n+2}}} \right)
= \arctan \left( \frac{1}{F_{2n}} \right) - \arctan \left( \frac{1}{F_{2n+2}} \right).
\]
Hence,
\[
\frac{1}{4} \sum_{n=1}^{\infty} \arctan \left( \frac{1}{F_{2n+1}} \right) = \frac{1}{4} \sum_{n=1}^{\infty} \left[ \arctan \left( \frac{1}{F_{2n}} \right) - \arctan \left( \frac{1}{F_{2n+2}} \right) \right]
= \frac{1}{4} \arctan \left( \frac{1}{F_2} \right) = \frac{1}{4} \arctan 1 = \frac{\pi}{16}.
\]
Thus, the sum of the given series does not exceed \( \frac{\pi}{16} \approx 0.196, \) which improves the proposed upper bound, because
\[
\frac{4}{\pi} \arctan(\beta) \left( \arctan(\beta) + \frac{1}{3} \right) \approx 0.625.
\]

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposer.
Janous also improved the proposed upper bound.

3262. [2007 : 299, 301] Proposed by Ovidiu Furdui, University of Toledo, Toledo, OH, USA.

Let \( m \) be an integer, \( m \geq 2, \) and let \( a_1, a_2, \ldots, a_m \) be positive real numbers. Evaluate the limit
\[
L_m = \lim_{n \to \infty} \frac{1}{n^m} \int_1^e \prod_{k=1}^m \ln(1 + a_k x^n) \, dx.
\]
Solution by Manuel Benito, Óscar Ciaurri, Emilio Fernández, and Luz Roncal, Logroño, Spain, modified by the editor.

For each integer \( m \geq 1 \) we will show that

\[
L_m = (-1)^{m+1}m! + e \sum_{k=0}^{m} (-1)^k \frac{m!}{(m-k)!}.
\] (1)

First note that for \( x \geq 1 \), we have

\[
xa_k^{1/n} \leq (1 + a_k x^n)^{1/n} \leq x(1 + a_k)^{1/n}.
\] (2)

Since \( a_k^{1/n} \) and \( (1 + a_k)^{1/n} \) each converge to 1 as \( n \to \infty \), it follows from the above that \( (1 + a_k x^n)^{1/n} \) converges to \( x \) as \( n \to \infty \), thus,

\[
\lim_{n \to \infty} \frac{\ln(1 + a_k x^n)}{n} = \lim_{n \to \infty} \ln(1 + a_k x^n)^{1/n} = \ln x.
\] (3)

Taking logarithms across the last inequality in (2), we obtain

\[
\frac{\ln(1 + a_k x^n)}{n} \leq \ln x + \frac{\ln(1 + a_k)}{n} \leq \ln x + \ln(1 + a_k),
\]

from which it follows that

\[
\prod_{k=1}^{m} \frac{\ln(1 + a_k x^n)}{n} \leq \prod_{k=1}^{n} (\ln x + \ln(1 + a_k)).
\]

By Lebesgue’s Dominated Convergence Theorem, we may bring the limit inside the integral; then we apply (3) as follows

\[
L_m = \int_1^e \lim_{n \to \infty} \prod_{k=1}^{n} \frac{\ln(1 + a_k x^n)}{n} \, dx
\]

\[
= \int_1^e \prod_{k=1}^{n} \lim_{n \to \infty} \frac{\ln(1 + a_k x^n)}{n} \, dx
\]

\[
= \int_1^e (\ln x)^m \, dx.
\] (4)

Next we integrate by parts to derive the recurrence relation

\[
L_m = e - mL_{m-1}.
\] (5)

Finally, we use induction on \( m \) to show that (with the appropriate initial condition) the solution to the recurrence in (5) is given by (1).

The case when \( m = 1 \) is clear, since the right side of (1) is 1 and from (4) we have \( L_1 = \int_1^e \ln x \, dx = 1 \).
Suppose (1) holds for some \( m \geq 1 \). Then using (5) we have

\[
L_m = e - m \left\{ (-1)^m (m - 1)! + e \sum_{k=0}^{m-1} (-1)^k \frac{(m - 1)!}{(m - 1 - k)!} \right\}
\]

\[
= e + (-1)^m + m! + e \sum_{k=0}^{m-1} (-1)^{k+1} \frac{m!}{(m - 1 - k)!}
\]

\[
= (-1)^m + m! + e \sum_{k=1}^{m} (-1)^k \frac{m!}{(m - k)!}
\]

\[
= (-1)^m + m! + e \sum_{k=0}^{m} (-1)^k \frac{m!}{(m - k)!}
\]

and our proof is complete.

Also solved by MICHEL BATAILLE, Rouen, France; PAUL BRACKEN and N. NADEAU, University of Texas, Edinburg, TX, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD L. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Urslinengymnasium, Innsbruck, Austria; XAVIER ROS, student, Universitat Politècnica de Catalunya, Barcelona, Spain; PETER Y. WOO, Boly University, La Mirada, CA, USA; and the proposer. There was 1 incorrect solution submitted.

Janous notes the interesting fact that \( L_m \) can be expressed in terms of \( D_m \), the number of derangements of 1, 2, . . ., \( m \). (A permutation \( \sigma \) of 1, 2, . . ., \( m \) is called a derangement if \( \sigma(i) \neq i \) for all \( i = 1, 2, \ldots, m \).) Since it is well known that \( D_m = m! \sum_{k=0}^{m} (-1)^k \frac{1}{k!} \), we see that \( L_m = (-1)^{m+1} m! + (-1)^m e D_m \).

The proposer marked that his proposal was a generalization of the following problem, which appeared in the Romanian journal Gazeta in 2000:

Compute \( \lim_{n \to \infty} \frac{1}{n^2} \int_1^n \ln(1 + x^n) \ln(1 + 2a^n) \, dx \).

Both he and Bracken and Nadeau pointed out the interesting fact that the answer is completely independent of the \( a_k \)'s given.

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