Problem of the Month

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As spring arrives in most parts of the country, it's time to dust off our bicycles. This problem is a bit reminiscent of the gears and wheels of a bicycle.

Problem (1996 Cayley Contest)

Two circular discs have radii 8 and 28. The larger disc is fixed while the smaller disc rolls around the outside of the larger disc. In their original positions, point \( A \) on the smaller disc coincides with point \( B \) on the larger disc. The least number of rotations that the small disc makes about its centre before \( A \) and \( B \) again coincide is

(A) 5  (B) 2  (C) 9  (D) 4.5  (E) 3.5

A friend of mine asked me about this problem from the archives a couple of months ago. It stumped me! Here's what I did initially:

As the smaller circle rolls around the larger circle, we assume that no slipping occurs. As a result, the length of circumference travelled as measured on the smaller circle equals that as measured on the larger circle.

Let's suppose that the smaller circle travels \( m \) times around its circumference while rolling \( n \) times around the larger circle. Since we want points \( A \) and \( B \) to line up at the end of the process, then each of \( m \) and \( n \) should be integers.

The length of circumference travelled along the smaller circle is \( m2\pi(8) = 16\pi m \) and the length travelled along the larger circle is \( n2\pi(28) = 56\pi n \).

For these lengths to be equal, \( 16\pi m = 56\pi n \) or \( 2m = 7n \). Since \( m \) and \( n \) are positive integers and we want \( m \) to be as small as possible, then \( m = 7 \) and \( n = 2 \).

Therefore, the smaller circle rotates 7 times around its centre before the first time that \( A \) and \( B \) coincide.

Wait... 7 is not one of the possible answers! Is the question wrong? Did we do something wrong mathematically? Is our logic wrong?

The question is correct. In fact, this is indeed the first time that \( A \) and \( B \) coincide again, but somehow 7 is the wrong answer. The thing that we haven’t taken into account is that as the smaller circle rolls, its motion around the larger circle produces extra rotations around its centre.

Here are three ways to look at this problem. It’s up for you to decide how valid these solutions are and which convinces you the most.
Solution 1. $7 + 2 = 9$.

That's the shortest solution to a POTM in my time here! What is this trying to say? The smaller circle rotates 7 times while rolling 2 times about the centre of the larger circle. So perhaps the smaller circle rotates an extra 2 times around its centre, for a total of 9 times. Do you buy this? (Actually, a good point here is that the total number of rotations of the smaller circle should be at least 7, so given the possible answers, (C) 9 must be correct.)

Solution 2. When two circles are tangent (like the smaller and larger circles), the line joining their centres passes through the point of tangency. Thus, the distance between the centres is the sum of the radii, or $8 + 28 = 36$. Therefore, the centre of the smaller circle is always a distance of 36 from the centre of the larger circle. Hence, the centre of the smaller circle travels around a circle of radius 36. The centre of the smaller circle moves 2 times around this circle, so travels a total distance of $2(2\pi(36)) = 144\pi$.

The circumference of the smaller circle is $2\pi(8) = 16\pi$, and so while the centre of this circle moves a distance of $144\pi$ cm, it must rotate $\frac{144\pi}{16\pi} = 9$ times.

This feels better than Solution 1, but still leaves a somewhat uncertain feeling. The ratio of distance travelled by the centre to the number of rotations works on a flat surface, but should it work rolling around a circular surface?

The last solution is slightly less intuitive, but a bit more mathematical.

Solution 3. First, we turn the diagram so that the line joining the centres is horizontal. We don't really have to do this, but you might find it easier to visualize.

Next let's suppose that the smaller circle has rolled so that the point of contact between the circles is at an angle of $x$ below the horizontal. (We'll treat the angles as measured in radians, but feel free to think in degrees if that's easier for you.) Call the centre of the larger circle $O$, the centre of the smaller circle $C$, the point of tangency $T$; we label the end-point of the radius to the left of $C$ as $P$. We have defined $\angle TOB = x$. Since $OB$ is parallel to $PC$, then we also have $\angle TCP = x$.

We now use the fact that the length of arc $BT$ equals the length of arc $TA$ (by equal distances rolled). The length of arc $BT$ as a fraction of the entire circumference of the circle with centre $O$ equals the fraction that $\angle TOB$ is of the angle associated with a full revolution. Thus, the length of arc $BT$ is

$$\frac{\angle TOB}{2\pi} (2\pi(TO)).$$
Similarly, the length of arc $TA$ is
\[ \frac{\angle TCA}{2\pi}(2\pi(TC)) . \]
Since these arcs are equal in length
\[ \frac{\angle TOB}{2\pi}(2\pi(TO)) = \frac{\angle TCA}{2\pi}(2\pi(TC)) , \]
while in degrees this equation is
\[ \frac{\angle TOB}{360^\circ}(2\pi(TO)) = \frac{\angle TCA}{360^\circ}(2\pi(TC)) . \]
The radian version of this equation becomes
\[ \frac{x}{2\pi}(2\pi(28)) = \frac{\angle TCA}{2\pi}(2\pi(8)) , \]
while the degree version becomes
\[ \frac{x}{360^\circ}(2\pi(28)) = \frac{\angle TCA}{360^\circ}(2\pi(8)) . \]
Both of these give $28x = 8\angle TCA$, whence $\angle TCA = \frac{7}{4}x$. Therefore, we have
\[ \angle PCA = \angle TCP + \angle TCA = \frac{9}{2}x . \]
What does this tell us? When the smaller circle has rolled twice around the circumference of the larger circle, we'll have $x = 4\pi$, so this tells us that the total angle travelled by $B$ around $C$ is $\frac{9}{4}(4\pi) = 18\pi$. Thus, the smaller circle makes 9 complete rotations around its center.

I find this problem pretty fascinating because of its counterintuitive nature. I hope that these different solutions gave you something to think about!

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**Notes from the Mayhem Editor**

Calling all proposers! Do you have a neat problem that you’d like to see in Mayhem? We would appreciate receiving your problem proposals. They can be sent by email to: mayhem-editors@cms.math.ca or by regular mail to: Ian VanderBurgh; CEMC, University of Waterloo; 200 University Ave. W.; Waterloo, ON, Canada; N2L 3G1. Please keep in mind that we are looking for problems at the level of those that have appeared so far in Volume 34.

Some of Mayhem’s loyal followers have noticed that a few favourite problems from the past have reappeared in recent issue(s). While this may have been accidental, we believe that this is perfectly acceptable in Mayhem, as its audience is intended to be students in secondary and post-secondary schools, who tend to move on after a few years.