M347. Proposed by the Mayhem Staff.

Four positive integers $a$, $b$, $c$, and $d$ are such that
\[
(a + b + c)d = 420, \\
(a + c + d)b = 403, \\
(a + b + d)c = 363, \\
(b + c + d)a = 228.
\]

Find the four integers.

M348. Proposed by the Mayhem Staff.

The perimeter of a sector of a circle is 12 (the perimeter includes the two radii and the arc). Determine the radius of the circle that maximizes the area of the sector.

M349. Proposed by the Mayhem Staff.

(a) Find all ordered pairs of integers $(x, y)$ with \(\frac{1}{x} + \frac{1}{y} = \frac{1}{5}\).

(b) How many ordered pairs of integers $(x, y)$ are there with \(\frac{1}{x} + \frac{1}{y} = \frac{1}{1200}\)²?

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Mayhem Solutions

M294. Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John’s, NL.

Nine circles of radius 1/2 are externally tangent to a circle of radius 1 and are tangent to one another, as shown.

Determine the distance between the centres of the first and last of the circles of radius 1/2.

Solution by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam, adapted by the editor.

Label the centre of the first circle of radius $\frac{1}{2}$ with $A$, the centre of the second circle with $B$, the centre of the last (ninth) circle with $C$, and the centre of the circle of radius 1 with $O$. We need to determine the length of $AC$. We know that $AO$, $BO$, and $CO$ are each equal to $1 + \frac{1}{2} = \frac{3}{2}$, because the smaller circles are tangent to the larger circle, and $AB = \frac{1}{2} + \frac{1}{2} = 1$, because the smaller circles with centres $A$ and $B$ are tangent.
Let $\angle AOB = \alpha$. Then $\angle AOC = 360^\circ - 8\alpha$, because there are 8 pairs of circles determining an angle of $\alpha$ at $O$. Applying the Law of Cosines to $\triangle AOC$ and using $\cos \theta = \cos(360^\circ - \theta)$, we have

$$AC^2 = AO^2 + CO^2 - 2AO \cdot CO \cos \angle AOC$$

$$= \left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right)^2 - 2 \left(\frac{3}{2}\right) \left(\frac{3}{2}\right) \cos(360^\circ - 8\alpha)$$

$$= \frac{9}{2}(1 - \cos 8\alpha).$$

To find $\cos 8\alpha$, we first find $\cos \alpha$ using the Law of Cosines in $\triangle AOB$:

$$AB^2 = AO^2 + BO^2 - 2AO \cdot BO \cos \angle AOB,$$

$$1^2 = \left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right)^2 - 2 \left(\frac{3}{2}\right) \left(\frac{3}{2}\right) \cos \alpha,$$

and solving for $\cos \alpha$ gives $\cos \alpha = \frac{7}{9}$. Therefore,

$$\cos 2\alpha = 2 \cos^2 \alpha - 1 = 2 \left(\frac{7}{9}\right)^2 - 1 = \frac{17}{81},$$

$$\cos 4\alpha = 2 \cos^2 2\alpha - 1 = 2 \left(\frac{17}{81}\right)^2 - 1 = -\frac{5983}{6561},$$

$$\cos 8\alpha = \cos^2 4\alpha - 1 = 2 \left(\frac{5983}{6561}\right)^2 - 1 = \frac{28545857}{43046721},$$

and hence

$$AC = \sqrt{\frac{9}{2}(1 - \cos 8\alpha)} = \sqrt{\frac{9}{2} \left(1 - \frac{28545857}{43046721}\right)} = \frac{1904\sqrt{2}}{2187}.$$

Also solved by HASAN DENKER, Istanbul, Turkey; ANGELA DREL, Rioio Terme, Italy; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; R. LAUMEN, Deurne, Belgium; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; RICARD PEIRO, IES "Abastos", Valencia, Spain; KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India; and J. SUCK, Essen, Germany. There was 1 incorrect solution submitted.

M295. Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John’s, NL.

Square $ABCD$ is inscribed in one-eighth of a circle of radius 1 so that there is one vertex on each radius and two vertices on the arc.

Determine the exact area of the square in the form $\frac{a + b\sqrt{c}}{d}$, where $a$, $b$, $c$, and $d$ are integers.
Solved independently by Hasan Denker, İstanbul, Turkey; Salem Malikič, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; R. Laumen, Deurne, Belgium; and Ricard Peiró, IES “Abastos”, Valencia, Spain.

Let \( N \) and \( M \) represent the midpoints of \( AD \) and \( BC \), respectively. In right triangle \( OAN \), we have \( AN = \frac{1}{2} BC \) and \( \angle AON = \frac{\pi}{8} \). Therefore, \( \frac{1}{2} BC \div ON = \tan \frac{\pi}{8} \). We calculate \( \tan \frac{\pi}{8} \).

Since
\[
1 = \tan \frac{\pi}{4} = \frac{2 \tan \frac{\pi}{8}}{1 - \tan^2 \frac{\pi}{8}},
\]
we let \( x = \tan \frac{\pi}{8} \) and we simplify to obtain \( x^2 + 2x - 1 = 0 \). Now \( x \) is positive, so by the quadratic formula \( \tan \frac{\pi}{8} = x = \frac{-2 + \sqrt{2}}{2} = \sqrt{2} - 1 \).

Thus \( \frac{1}{2} BC \div ON = \sqrt{2} - 1 \) and \( ON = \frac{BC}{2\sqrt{2} - 2} \), and by the Pythagorean Theorem in triangle \( BOM \) we have
\[
OB^2 = BM^2 + OM^2 = BM^2 + (ON + NM)^2.
\]

Since \( OB = 1 \), \( NM = BC \), \( BM = \frac{1}{2} BC \), and \( ON = \frac{BC}{2\sqrt{2} - 2} \), we obtain
\[
1 = \frac{1}{4} BC^2 + \left( \frac{BC}{2\sqrt{2} - 2} + BC \right)^2 = \frac{1}{4} BC^2 + \left( \frac{2\sqrt{2} - 1}{2\sqrt{2} - 2} \right)^2 BC^2
\]
\[
= \frac{1}{4} BC^2 + \left( \frac{2\sqrt{2} - 1)(2\sqrt{2} + 2)}{4} \right)^2 BC^2
\]
\[
= \frac{1}{4} BC^2 + \left( \frac{11 + 6\sqrt{2}}{4} \right) BC^2 = \left( \frac{6 + 3\sqrt{2}}{2} \right) BC^2,
\]
hence the area is \( BC^2 = \frac{2}{6 + 3\sqrt{2}} \), which upon rationalizing is \( \frac{2 - \sqrt{2}}{3} \).

Also solved by SAMUEL GÓMEZ MORENO, Universidad de Jaén, Jaén, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India; and J. SUCK, Essen, Germany.

M296. Proposed by Daniel Tsai, student, Taipei American School, Taipei, Taiwan.

Let \( n \) be a positive integer. In the Cartesian plane, consider the points \( a_k = (k, n) \) and \( b_k = (k, 0) \) for \( k = 1, 2, \ldots, n \). We connect each pair \( a_k, b_k \) by a straight (vertical) line segment. Then we draw an arbitrary finite number of horizontal line segments, each connecting two adjacent vertical line segments, such that no one point on any vertical segment is the endpoint of two horizontal segments.
Let $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_n\}$. Define a map from $A$ to $B$ as follows: starting from $a_i$, travel down the segment until you meet the end-point of a horizontal segment, go to the other end-point of that segment, and continue on down the new vertical line, repeating this until there are no more horizontal segments to meet, finally ending at $b_j$ for some $j$. Show that no two points of $A$ map to the same point of $B$.

Solution by Richard I. Hess, Rancho Palos Verdes, CA, USA.

The process described is uniquely reversible, so that starting from $b_j$ gets you to $a_i$ if going from $a_i$ got you to $b_j$. Therefore, when going in reverse, there is no possibility of splitting paths from $b_j$ to $a_i$, and therefore no two points in $A$ can map to the same point in $B$.

Also solved by JACLYN CHANG, student, Western Canada High School, Calgary, AB; and the proposer.

M297. Proposed by John Grant McLoughlin, University of New Brunswick, Fredericton, NB.

Numbers such as 34543 and 713317 whose digits can be reversed without changing the number are called palindromes. Show that all four-digit palindromes are multiples of 11.

Solved independently by Angela Drei, Riolo Terme, Italy; Samuel Gómez Moreno, Universidad de Jaén, Jaén, Spain; Richard I. Hess, Rancho Palos Verdes, CA, USA; R. Laumen, Deurne, Belgium; Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; and Kunal Singh, student, Kendriya Vidyalaya School, Shillong, India.

Every four-digit palindrome can be written in the form $abba$, where $a$ and $b$ are integers, such that $1 \leq a \leq 9$ and $0 \leq b \leq 9$. We can then conclude that

$$abba = 1000a + 100b + 10b + a = 1001a + 110b = 11(91a + 10b).$$

Thus, every four-digit palindrome is a multiple of 11.

Also solved by JACLYN CHANG, student, Western Canada High School, Calgary, AB; and DANIEL TSAI, student, Taipei American School, Taipei, Taiwan.

M298. Proposed by John Grant McLoughlin, University of New Brunswick, Fredericton, NB.

(a) Given that a number is a four-digit palindrome, what is the probability that the number is a multiple of 99?

(b) Given that a four-digit number is a multiple of 99, what is the probability that the number is a palindrome?
Solution by Samuel Gómez Moreno, Universidad de Jaén, Jaén, Spain.

(a) A four-digit palindrome has the form \(abba\), where \(a\) and \(b\) are integers with \(1 \leq a \leq 9\) and \(0 \leq b \leq 9\). There are therefore \(9 \times 10 = 90\) four-digit palindromes. By Problem M297, we know that all four-digit palindromes are multiples of 11. For a four-digit palindrome, being a multiple of 99 is therefore equivalent to being a multiple of 9.

A positive integer is a multiple of 9 if and only if the sum of its digits is also a multiple of 9, so the four-digit palindromes which are multiples of 99 are precisely those for which \(2(a + b)\) is a multiple of 9, that is, \(2(a + b)\) equals 18 or 36. A complete list of these palindromes is 1881, 2772, 3663, 4554, 5445, 6336, 7227, 8118, 9009, and 9999. Thus the probability is \(\frac{10}{99} = \frac{1}{9}\).

(b) There are a total of 91 four-digit multiples of 99, namely the multiples from 99(11) = 1089 to 99(101) = 9999. By part (a), we know that among these there are ten four-digit palindromes. Therefore the required probability is \(\frac{10}{91}\).

Also solved by ANGELA DREI, Rioio Terme, Italy; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; R. LAUMEN, Deurne, Belgium and DANIEL TSAI, student, Taipei American School, Taipei, Taiwan. There were 2 incorrect solutions submitted.

M299. Proposed by Titu Zvonaru, Comănești, Romania.

Let \(a, b,\) and \(c\) be positive real numbers with \(ab + bc + ca = 3\). Prove that
\[
\frac{ab}{c^2+1} + \frac{bc}{a^2+1} + \frac{ca}{b^2+1} \geq \frac{3}{2}.
\]


Using the Arithmetic Mean-Geometric Mean Inequality, we have
\[
(abc)^{2/3} = \sqrt[3]{(ab)(bc)(ac)} \leq \frac{ab + bc + ca}{3} = 1,
\]

hence \(abc \leq 1\). The desired inequality is equivalent to
\[
\left(\frac{ab}{c^2+1} - \frac{ab}{c^2+1}\right) + \left(\frac{bc}{a^2+1} - \frac{bc}{a^2+1}\right) + \left(\frac{ca}{b^2+1} - \frac{ca}{b^2+1}\right) \leq \frac{3}{2},
\]

which is the same as
\[
\frac{abc^2}{c^2+1} + \frac{bca^2}{a^2+1} + \frac{cab^2}{b^2+1} \leq \frac{3}{2}.
\]

We have \((x-1)^2 \geq 0\), hence \(x^2 + 1 \geq 2x\), and hence \(\frac{1}{x^2+1} \leq \frac{1}{2x}\) when \(x\) is positive. Thus,
\[
\frac{abc^2}{c^2+1} + \frac{bca^2}{a^2+1} + \frac{cab^2}{b^2+1} \leq \frac{abc^2}{2c} + \frac{bca^2}{2a} + \frac{cab^2}{2b} = \frac{3}{2} abc \leq \frac{3}{2}.
\]
Equality holds if and only if \( a = b = c = \frac{1}{2} \).

Also solved by ARKADY ALT, San Jose, CA, USA (two solutions); JOE HOWARD, Portales, NM, USA; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam (second solution); and SHI CHANGWEI, Xi’an City, Shaan Xi Province, China. There were 2 incorrect solutions submitted.

**M300. Proposed by Geoffrey A. Kandall, Hamden, CT, USA.**

Let \( \triangle ABC \) be an arbitrary triangle. Let \( D \) and \( E \) be points on the sides \( AC \) and \( AB \), respectively, and let \( P \) be the point of intersection of \( BD \) and \( CE \). If \( AE : EB = r \) and \( AD : DC = s \), determine the ratio of areas \([\triangle ABC] : [\triangle PBC]\) in terms of \( r \) and \( s \).

**Solution by Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina.**

Let \( AP \) meet \( BC \) at point \( K \). Starting with Ceva’s Theorem applied to \( \triangle ABC \) and substituting the given ratios, we obtain

\[
\frac{AE}{EB} \cdot \frac{BK}{KC} \cdot \frac{CD}{DA} = 1 \quad \Longrightarrow \quad r \cdot \frac{BK}{KC} \cdot \frac{1}{s} = 1 \quad \Longrightarrow \quad \frac{KC}{BK} = \frac{r}{s},
\]

and hence

\[
\frac{CB}{BK} = \frac{KC + BK}{BK} = \frac{r}{s} + 1 = \frac{r + s}{s}.
\]

Applying Menelaus’ Theorem to the transversal \( BPD \) and \( \triangle AKC \), and substituting the expression just obtained, we have

\[
\frac{AD}{DC} \cdot \frac{CB}{BK} \cdot \frac{KP}{PA} = 1 \quad \Longrightarrow \quad s \cdot \frac{r + s}{s} \cdot \frac{KP}{PA} = 1
\]

\[
\Longrightarrow \quad \frac{PA}{KP} = \frac{r + s}{s},
\]

so it follows that

\[
\frac{KA}{KP} = \frac{PA + KP}{KP} = \frac{r + s + 1}{s}.
\]

Since \( \frac{x}{y} = \frac{z}{t} \) implies \( \frac{x}{y} = \frac{z}{t} = \frac{x + z}{y + t} \), we combine this with the previous results to obtain

\[
r + s + 1 = \frac{KA}{KP} = \frac{[\triangle BKA]}{[\triangle BKP]} = \frac{[\triangle KAC]}{[\triangle KPC]}
\]

\[
= \frac{[\triangle BKA] + [\triangle KAC]}{[\triangle BKP] + [\triangle KPC]} = \frac{[\triangle ABC]}{[\triangle PBC]},
\]

where \([\triangle BKA]\) denotes the area of \( \triangle BKA \), and so forth. This shows that \([\triangle ABC] : [\triangle PBC] = r + s + 1 \).

Also solved by HASAN DENKER, Istanbul, Turkey; RICARD PEIRO, IES “A bastos”, Valencia, Spain; KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India; J. SUCK, Essen, Germany; and TITU ZVONARU, Comănești, Romania. There was 1 incorrect solution submitted.