Mayhem Solutions

M288. Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John’s, NL.

The following figure can be cut into two pieces and reassembled into a square, by simply cutting off the ‘tab’ and placing it in the cutaway at the top, as shown in the second image.

Determine a method to cut the given figure into three pieces which can be reassembled to form a square. (Find a method which is essentially different from cutting it into two pieces; for example, cutting the tab into two pieces would not be considered different from the two-piece dissection.)

Solution by Richard I. Hess, Rancho Palos Verdes, CA, USA.

Also solved by Peter Hurthig, Columbia College, Vancouver, BC; Owen Ren, student, Magee Secondary School, Vancouver, BC; Kunal Singhal, student, Kendriya Vidyalaya School, Shillong, India; Mridul Singhal, student, Kendriya Vidyalaya School, Shillong, India; and Justin Yang, student, Lord Byng Secondary School, Vancouver, BC. There was also a correct solution that was not essentially different than the two-piece dissection. No two solutions to this problem were the same.

M289. Proposed by K.R.S. Sastry, Bangalore, India.

Solve the following equation for real $x$:

$$\log \left( x + \sqrt{5x - \frac{13}{4}} \right) = - \log \left( x - \sqrt{5x - \frac{13}{4}} \right).$$

Solution by Denise Cornwell, student, Angelo State University, San Angelo, TX, USA.

First we begin by bringing everything to one side to get

$$\log \left( x + \sqrt{5x - \frac{13}{4}} \right) + \log \left( x - \sqrt{5x - \frac{13}{4}} \right) = 0.$$

By the properties of logarithms, we can rewrite the equation as

$$\log \left[ \left( x + \sqrt{5x - \frac{13}{4}} \right) \left( x - \sqrt{5x - \frac{13}{4}} \right) \right] = 0.$$
We can now see that the product inside the logarithm must be 1, because
\[ \log y = 0 \] if and only if \( y = 1 \). Therefore, we successively obtain
\[
\left(x + \sqrt{5x - \frac{13}{4}}\right) \left(x - \sqrt{5x - \frac{13}{4}}\right) = 1, \\
x^2 - 5x + \frac{13}{4} = 1, \\
x^2 - 5x + \frac{9}{4} = 0, \\
(x - \frac{1}{2}) \left(x - \frac{9}{2}\right) = 0.
\]
Thus, \( x = \frac{1}{2} \) or \( x = \frac{9}{2} \).

We now must check for extraneous solutions. When \( x = \frac{1}{2} \), we get
\[
x + \sqrt{5x - \frac{13}{4}} = \frac{1}{2} + \sqrt{\frac{5}{2} - \frac{13}{4}} = \frac{1}{2} + \sqrt{\frac{3}{4}}.
\]
Since we are looking for real values \( x \), we can stop here and say that \( x = \frac{1}{2} \) is not a solution.

When \( x = \frac{9}{2} \), we have
\[
x + \sqrt{5x - \frac{13}{4}} = \frac{9}{2} + \sqrt{\frac{45}{2} - \frac{13}{4}} = \frac{9 + \sqrt{77}}{2}
\]
and
\[
x - \sqrt{5x - \frac{13}{4}} = \frac{9}{2} - \sqrt{\frac{45}{2} - \frac{13}{4}} = \frac{9 - \sqrt{77}}{2} = \frac{(9-\sqrt{77})(9+\sqrt{77})}{2(9+\sqrt{77})} = \frac{2}{9+\sqrt{77}}.
\]
When we substitute these back into the logarithm equation, we can see that \( x = \frac{9}{2} \) is a valid solution.

Also solved by HASAN DENKER, Istanbul, Turkey; JOSÉ HERNÁNDEZ SANTIAGO, student, Universidad Tecnológica de la Mixteca, Oaxaca, Mexico; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; SALEM MALIKIC, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; OWEN REN, student, Magee Secondary School, Vancouver, BC; NICK WILSON, student, Valley Catholic School, Beaverton, OR, USA; and the proposer. There were 3 incorrect solutions submitted.

M290. Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.

Give a purely geometric proof that \( \tan^{-1} \left( \frac{1}{2} \right) + \tan^{-1} \left( \frac{1}{3} \right) = \frac{\pi}{4} \).

Solution by Nick Wilson, student, Valley Catholic School, Beaverton, OR, USA; and Titu Zvonaru, Comănești, Romania (independently).

In the diagram, each of the six squares is a unit square. Note that \( \angle CAD = \tan^{-1} \left( \frac{1}{2} \right) \),
since \( CD = 1 \) and \( AD = 3 \), and that
\( \angle BAC = \tan^{-1} \left( \frac{1}{3} \right) \), since \( AB \) and \( BC \) are perpendicular, \( BC = \sqrt{2} \), and \( AB = 2\sqrt{2} \).
Then \( \tan^{-1} \left( \frac{1}{2} \right) + \tan^{-1} \left( \frac{1}{3} \right) = \angle BAD = \frac{\pi}{4} \).

Also solved by DENISE CORNWELL, student, Angelo State University, San Angelo, TX, USA; HASAN DENKER, Istanbul, Turkey; SAMUEL GOMEZ MORENO, Universidad de Jaén,
M291. Proposed by Robert Bilinski, Collège Montmorency, Laval, QC.

The right triangle having sides 3, \(\sqrt{7}\), and 4, has the strange property that the two integer lengths sum to the value under the square root sign for the length of the third side.

1. Find another such triangle.

2. Prove that there are infinitely many such triangles, and show how to construct them.

3. Does the formula work only for integers?

Adapted from the solution by Denise Cornwell, student, Angelo State University, San Angelo, TX, USA.

(a) Another triangle with the same property is the right-angled triangle with legs 5 and \(\sqrt{11}\) and hypotenuse 6.

(b) To find the formula to construct infinitely many such triangles, we let the integer sides be called \(a\) and \(c\). Then the side under the square root is \(a + c\). We try to find an infinite family of triangles which are right-angled with legs of lengths \(a\) and \(\sqrt{a + c}\) and hypotenuse of length \(c\). For this to happen, we must have

\[
\begin{align*}
  a^2 + (\sqrt{a + c})^2 &= c^2, \\
  a^2 + (a + c) &= c^2, \\
  a^2 + a &= c^2 - c, \\
  a^2 + a + \frac{1}{4} &= c^2 - c + \frac{1}{4}, \\
  (a + \frac{1}{2})^2 &= (c - \frac{1}{2})^2, \\
  a + \frac{1}{2} &= c - \frac{1}{2} \quad \text{(since \(a\) and \(c\) are positive)}, \\
  c &= a + 1.
\end{align*}
\]

Therefore, we may construct infinitely many of these right-angled triangles by letting the legs of triangle be \(a\) and \(\sqrt{a + (a + 1)} = \sqrt{2a + 1}\) and letting the hypotenuse be \(a + 1\).

(c) The formula will work for any real number \(a > 0\) (to ensure the triangle does not have a negative side).

Also solved by Hasan Denker, Istanbul, Turkey; Samuel Gómez Moreno, Universidad de Jaén, Jaén, Spain; Richard I. Hess, Rancho Palos Verdes, CA, USA; Salem Malikic, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; Owen Ren, student, Magee Secondary School, Vancouver, BC; Kunal Singh, student, Kendriya Vidyalaya School, Shillong, India; D.J. Smeenk, Zaltbommel, the Netherlands; J. Suck, Essen, Germany; and Daniel Tsai, student, Taipei American School, Taipei, Taiwan.
There was some confusion about whether the word "construct" meant to demonstrate explicitly the infinite family or show how these triangles can be created using compass and straightedge.

**M292. Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.**

Let \( x \) be a positive number. Prove that \( \sqrt{\frac{[x]}{x+[x]}} + \sqrt{\frac{\{x\}}{x+[x]}} > 1 \), where \([x]\) and \(\{x\}\) represent the integer part and the fractional part of \(x\), respectively.

**Solution by Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina.**

If \( x \) is an integer, then \( \{x\} = 0 \) and \( x = [x] \), and we have

\[
\sqrt{\frac{[x]}{x+[x]}} + \sqrt{\frac{\{x\}}{x+[x]}} = 1 + 0 = 1.
\]

Similarly, if \( x \) is from interval \((0, 1)\), then \([x] = 0\) and \( x = \{x\} \), and

\[
\sqrt{\frac{[x]}{x+[x]}} + \sqrt{\frac{\{x\}}{x+[x]}} = 0 + 1 = 1.
\]

(This means that in the problem "\(>\)" should be replaced by "\(\geq\)" because, as we see, equality can be achieved when \( x \) is an integer or \( x \in (0, 1) \).)

Next, let \([x]\) be denoted by \(a\) and \(\{x\}\) by \(b\). Then \( x = a + b \). Note that \(0 \leq b < 1\). We may assume that \(x\) is not an integer and that \(x \notin (0, 1)\). Thus, both \(a\) and \(b\) are greater than 0. Since \(a\) is an integer, we see that \(a \geq 1 > b\); thus, \(a \neq b\). We rewrite the given inequality as \( \frac{a}{a+2b} + \frac{b}{2a+b} > 1 \), which is equivalent to

\[
\sqrt{a(2a+b)} + \sqrt{b(a+2b)} > \sqrt{(a+2b)(2a+b)}.
\]

Since both sides are positive, after squaring this is equivalent to

\[
2a^2 + 2b^2 + 2ab + 2\sqrt{a(2a+b)}\sqrt{b(a+2b)} > 2a^2 + 2b^2 + 5ab,
\]

which is equivalent to

\[
2\sqrt{a(2a+b)}\sqrt{b(a+2b)} > 3ab.
\]

Since both sides are positive again, we can square this to obtain the equivalent inequality

\[
4(ab)(2a+b)(2b+a) > 9(ab)(ab),
\]
and, after dividing by \( ab > 0 \), we get \( 4(5ab + 2a^2 + 2b^2) > 9ab \), which is equivalent to \( 11ab + 8a^2 + 8b^2 > 0 \), which is true because \( 11ab \geq 0 \), \( 8a^2 \geq 8 \), and \( 8b^2 \geq 0 \). Therefore, the result follows.

Also solved by ARKADY ALT, San Jose, CA, USA; HASAN DENKER, Istanbul, Turkey; SAMUEL GÓMEZ MORENO, Universidad de Jaén, Jaén, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; J. SÜCK, Essen, Germany; and TITU ZVONARU, Comănești, Romania. There was also one incorrect solution submitted.

**M293.** Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John’s, NL.

Eight equal circles are mutually tangent in pairs and tangent externally to a unit circle. Determine the common radius of the eight smaller circles.

1. **Solution by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam.**

Let \( O_i (i = 1, 2, \ldots, 8) \) be the centre of the \( i \)-th small circle and let \( O \) be the centre of the circle of radius 1. Let \( r \) be the common radius of the eight smaller circles. We need to determine \( r \). Let \( \angle O_i O O_2 = \alpha \). By symmetry, \( \angle O_1 O O_2 = \angle O_2 O O_3 = \cdots = \angle O_8 O O_1 = \alpha \). We have \( 8 \alpha = 360^\circ \), whence, \( \alpha = 45^\circ \). Applying the Cosine Law to \( \triangle O_1 O O_2 \), and noting that \( O O_1 = O O_2 = 1 + r \) and \( O_1 O_2 = 2r \), we have

\[
\begin{align*}
O_1 O_2^2 &= OO_1^2 + OO_2^2 - 2OO_1 \cdot OO_2 \cos \angle O_1 O O_2, \\
(2r)^2 &= (1 + r)^2 + (1 + r)^2 - 2(1 + r)(1 + r) \frac{1}{2}, \\
4r^2 &= (2 - \sqrt{2})(1 + r)^2, \\
2r &= (1 + r)\sqrt{2} - \sqrt{2} \quad \text{(since } r > 0), \\
2r - r\sqrt{2} - \sqrt{2} &= \sqrt{2} - \sqrt{2}, \\
r &= \frac{\sqrt{2} - \sqrt{2}}{2 - \sqrt{2} - \sqrt{2}}.
\end{align*}
\]

II. **Solution by Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina.**

Let \( O \) denote the centre of the large circle and \( A \) the centre of one of the small circles. The line \( OA \) passes through \( C \), the point of tangency between the large and small circle. Draw the tangent \( OB \) to this small circle. Then \( OB \) is perpendicular to \( AB \). By symmetry, \( \angle BOA = \frac{1}{8} \times 360^\circ = 22.5^\circ \). Applying the Sine Law in \( \triangle OBA \) and using the fact that the radius of the
large circle is 1, we get

\[
\frac{AB}{\sin(22.5^\circ)} = \frac{OA}{\sin(90^\circ)}
\]

or

\[
\frac{r}{\sin(22.5^\circ)} = \frac{1 + r}{1}.
\]

Thus, \(\sin(22.5^\circ) = r(1 - \sin(22.5^\circ))\). Hence, \(r = \frac{\sin(22.5^\circ)}{1 - \sin(22.5^\circ)}\). Now, \(\frac{\sqrt{2}}{2} = \cos(45^\circ) = \cos(2(22.5^\circ)) = 1 - 2\sin^2(22.5^\circ)\), which implies that \(\sin^2(22.5^\circ) = \frac{2 - \sqrt{2}}{4}\). Therefore, \(\sin(22.5^\circ) = \sqrt{\frac{2 - \sqrt{2}}{4}}\), and we get

\[
r = \frac{\sqrt{\frac{2 - \sqrt{2}}{4}}}{1 - \sqrt{\frac{2 - \sqrt{2}}{4}}} = \frac{\sqrt{2 - \sqrt{2}}}{2 - \sqrt{2 - \sqrt{2}}}.
\]

Also solved by HASAN DENKER, Istanbul, Turkey; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; SAMUEL GÓMEZ MORENO, Universidad de Jaén, Jaén, Spain; KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India; OWEN REN, student, Magee Secondary School, Vancouver, BC; J. SUCK, Essen, Germany; NICK WILSON, student, Valley Catholic School, Beaverton, OR, USA, and TITU IVONARU, Comănești, Romania.