MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as a Mathematical Journal for and by High School and University Students. It continues, with the same emphasis, as an integral part of Crux Mathematicorum with Mathematical Mayhem.

The Mayhem Editor is Ian VanderBurgh (University of Waterloo). The other staff members are Monika Khbeis (Ascension of Our Lord Secondary School, Mississauga), Eric Robert (Leo Hayes High School, Fredericton), Larry Rice (University of Waterloo), and Ron Lancaster (University of Toronto).

Mayhem Problems

Please send your solutions to the problems in this edition by 15 June 2008. Solutions received after this date will only be considered if there is time before publication of the solutions.

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English.

The editor thanks Jean-Marc Terrier of the University of Montreal for translations of the problems.

M338. Proposed by the Mayhem Staff.

Two students miscopy the quadratic equation \( x^2 + bx + c = 0 \) that their teacher writes on the board. Jim copies \( b \) correctly but miscopies \( c \); his equation has roots 5 and 4. Vazz copies \( c \) correctly, but miscopies \( b \); his equation has roots 2 and 4. What are the roots of the original equation?

M339. Proposed by the Mayhem Staff.

(a) Determine the number of integers between 100 and 199, inclusive, which contain exactly two equal digits.

(b) An integer between 1 and 999 is chosen at random, with each integer being equally likely to be chosen. What is the probability that the integer has exactly two equal digits?

M340. Proposed by the Mayhem Staff.

Let \( ABC \) be an isosceles triangle with \( AB = AC \), and let \( M \) be the mid-point of \( BC \). Let \( P \) be any point on \( BM \). A perpendicular is drawn to \( BC \) at \( P \), meeting \( BA \) at \( K \) and \( CA \) extended at \( T \). Prove that \( PK + PT \) is independent of the position of \( P \) (that is, the value of \( PK + PT \) is always the same, no matter where \( P \) is placed).
M341. Proposed by the Mayhem Staff.

Let $ABC$ be a right triangle with right angle at $B$. Sides $BA$ and $BC$ are in the ratio $3 : 2$. Altitude $BD$ divides $CA$ into two parts that differ in length by 10. What is the length of $CA$?

M342. Proposed by the Mayhem Staff.

Quincy and Celine have to move 10 small boxes and 10 large boxes. The chart below indicates the time that each person takes to move each type of box.

<table>
<thead>
<tr>
<th>Type</th>
<th>Celine</th>
<th>Quincy</th>
</tr>
</thead>
<tbody>
<tr>
<td>small box</td>
<td>1 min</td>
<td>3 min</td>
</tr>
<tr>
<td>large box</td>
<td>6 min</td>
<td>5 min</td>
</tr>
</tbody>
</table>

They start moving the boxes at 9:00 am. What is the earliest time at which they can be finished moving all of the boxes?

M343. Proposed by the Mayhem Staff.

The Fibonacci numbers are defined by $f_1 = f_2 = 1$ and, for $n \geq 2$, by $f_{n+1} = f_n + f_{n-1}$. The first few Fibonacci numbers are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, .... Find the sum of the first 100 even Fibonacci numbers.

M338. Proposé par l’Équipe de Mayhem.

Deux étudiants font une erreur en recopiant l’équation quadratique $x^2 + bx + c = 0$ que leur professeur écrit au tableau. Jean copie $b$ correctement, mais pas $c$ ; son équation possède alors les racines 5 et 4 ; Victor copie $c$ correctement, mais pas $b$ ; son équation possède les racines 2 et 4. Quelles sont les racines de l’équation originale ?

M339. Proposé par l’Équipe de Mayhem.

(a) Déterminer le nombre d’entiers entre 100 et 199, bornes comprises, contenant exactement deux chiffres égaux.

(b) Un entier entre 1 et 999 est choisi au hasard, chaque entier ayant la même chance d’être choisi. Quelle est la probabilité pour que cet entier ait exactement deux chiffres égaux?

M340. Proposé par l’Équipe de Mayhem.

Soit $ABC$ un triangle isocèle avec $AB = AC$, et soit $M$ le point milieu de $BC$. Soit $P$ un point quelconque sur $BM$. Par $P$, on dessine une perpendiculaire à $BC$, coupant $BA$ en $K$ et la droite $CA$ en $T$. Montrer que $PK + PT$ est indépendant de la position de $P$ (c’est-à-dire, la valeur de $PK + PT$ est toujours la même, peu importe la position de $P$).
M341. Proposé par l'Équipe de Mayhem.

Soit $ABC$ un triangle rectangle, d'angle droit en $B$. Ses côtés $BA$ et $BC$ sont dans le rapport $3 : 2$. La hauteur $BD$ divise $CA$ en deux parties dont la différence des longueurs est 10. Quelle est la longueur de $CA$?

M342. Proposé par l'Équipe de Mayhem.

Sophie et Céline doivent déplacer des boîtes, 10 grandes et 10 petites. Le tableau ci-dessous indique les temps requis pour ce faire, dans chaque cas et pour chaque personne.

<table>
<thead>
<tr>
<th></th>
<th>Céline</th>
<th>Sophie</th>
</tr>
</thead>
<tbody>
<tr>
<td>petite boîte</td>
<td>1 min.</td>
<td>3 min.</td>
</tr>
<tr>
<td>grande boîte</td>
<td>6 min.</td>
<td>5 min.</td>
</tr>
</tbody>
</table>

Leur travail commence à 9 heures du matin. Trouver à quelle heure, au plus tôt, elles pourraient finir leur déménagement?

M343. Proposé par l'Équipe de Mayhem.

Les nombres de Fibonacci sont définis par $f_1 = f_2 = 1$ et, pour $n \geq 2$, par $f_{n+1} = f_n + f_{n-1}$. Voici donc le début de la liste des nombres de Fibonacci : $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$. Trouver la somme des 100 premiers nombres pairs de cette liste.

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Mayhem Solutions

M288. Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.

The following figure can be cut into two pieces and reassembled into a square, by simply cutting off the ‘tab’ and placing it in the cutaway at the top, as shown in the second image.

![Image of the figure]

Determine a method to cut the given figure into three pieces which can be reassembled to form a square. (Find a method which is essentially different from cutting it into two pieces; for example, cutting the tab into two pieces would not be considered different from the two-piece dissection.)
Solution by Richard I. Hess, Rancho Palos Verdes, CA, USA.

Also solved by PETER HURTHIG, Columbia College, Vancouver, BC; OWEN REN, student, Magee Secondary School, Vancouver, BC; KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India; MRIDUL SINGH, student, Kendriya Vidyalaya School, Shillong, India; and JUSTIN YANG, student, Lord Byng Secondary School, Vancouver, BC. There was also a correct solution that was not essentially different than the two-piece dissection.

No two solutions to this problem were the same.

M289. Proposed by K.R.S. Sastry, Bangalore, India.

Solve the following equation for real $x$:

$$\log \left( x + \sqrt{5x - \frac{13}{4}} \right) = - \log \left( x - \sqrt{5x - \frac{13}{4}} \right).$$

Solution by Denise Cornwell, student, Angelo State University, San Angelo, TX, USA.

First we begin by bringing everything to one side to get

$$\log \left( x + \sqrt{5x - \frac{13}{4}} \right) + \log \left( x - \sqrt{5x - \frac{13}{4}} \right) = 0. $$

By the properties of logarithms, we can rewrite the equation as

$$\log \left[ \left( x + \sqrt{5x - \frac{13}{4}} \right) \left( x - \sqrt{5x - \frac{13}{4}} \right) \right] = 0. $$

We can now see that the product inside the logarithm must be 1, because $\log y = 0$ if and only if $y = 1$. Therefore, we successively obtain

$$\left( x + \sqrt{5x - \frac{13}{4}} \right) \left( x - \sqrt{5x - \frac{13}{4}} \right) = 1,$$

$$x^2 - 5x + \frac{13}{4} = 1,$$

$$x^2 - 5x + \frac{9}{4} = 0,$$

$$(x - \frac{1}{2})(x - \frac{9}{2}) = 0.$$ 

Thus, $x = \frac{1}{2}$ or $x = \frac{9}{2}$.

We now must check for extraneous solutions. When $x = \frac{1}{2}$, we get $x + \sqrt{5x - \frac{13}{4}} = \frac{1}{2} + \sqrt{5(\frac{1}{2}) - \frac{13}{4}} = \frac{1}{2} + \sqrt{\frac{3}{4}}$. Since we are looking for real values $x$, we can stop here and say that $x = \frac{1}{2}$ is not a solution.

When $x = \frac{9}{2}$, we have

$$x + \sqrt{5x - \frac{13}{4}} = \frac{9}{2} + \sqrt{\frac{45}{2} - \frac{13}{4}} = \frac{9 + \sqrt{77}}{2}$$
and 
\[ x - \sqrt{5x - \frac{13}{4}} = \frac{9}{2} - \sqrt{\frac{5}{2} - \frac{13}{4}} = \frac{9 - \sqrt{77}}{2} = \frac{(9 - \sqrt{77})(9 + \sqrt{77})}{2(9 + \sqrt{77})} = \frac{2}{9 + \sqrt{77}}. \]

When we substitute these back into the logarithm equation, we can see that \( x = \frac{3}{2} \) is a valid solution.

Also solved by HASAN DENKER, Istanbul, Turkey; JOSÉ HERNÁNDEZ SANTIAGO, student, Universidad Tecnológica de la Mixteca, Oaxaca, Mexico; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; SALEM MALIKIC, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; OWEN REN, student, Magee Secondary School, Vancouver, BC; NICK WILSON, student, Valley Catholic School, Beaverton, OR, USA; and the proposer. There were 3 incorrect solutions submitted.

**M290. Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.**

Give a purely geometric proof that \( \tan^{-1} \left( \frac{1}{2} \right) + \tan^{-1} \left( \frac{1}{3} \right) = \frac{\pi}{4} \).

**Solution by Nick Wilson, student, Valley Catholic School, Beaverton, OR, USA; and Titu Zvonaru, Comănești, Romania (independently).**

In the diagram, each of the six squares is a unit square. Note that \( \angle CAD = \tan^{-1} \left( \frac{1}{3} \right) \), since \( CD = 1 \) and \( AD = 3 \), and that \( \angle BAC = \tan^{-1} \left( \frac{1}{2} \right) \), since \( AB \) and \( BC \) are perpendicular, \( BC = \sqrt{2} \), and \( AB = 2\sqrt{2} \). Then \( \tan^{-1} \left( \frac{1}{3} \right) + \tan^{-1} \left( \frac{1}{2} \right) = \angle BAD = \frac{\pi}{4} \).

Also solved by DENISE CORNWELL, student, Angelo State University, San Angelo, TX, USA; HASAN DENKER, Istanbul, Turkey; SAÚL GÓMEZ MORENO, Universidad de Jaén, Jaén, Spain; JOSÉ HERNÁNDEZ SANTIAGO, student, Universidad Tecnológica de la Mixteca, Oaxaca, Mexico; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; SALEM MALIKIC, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; OWEN REN, student, Magee Secondary School, Vancouver, BC; KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India; D.J. SMEENK, Zaltbommel, the Netherlands; J. SUCK, Essen, Germany; and DANIEL TSAI, student, Taipei American School, Taipei, Taiwan.

**M291. Proposed by Robert Bilinski, Collège Montmorency, Laval, QC.**

The right triangle having sides 3, \( \sqrt{7} \), and 4, has the strange property that the two integer lengths sum to the value under the square root sign for the length of the third side.

1. Find another such triangle.
2. Prove that there are infinitely many such triangles, and show how to construct them.
3. Does the formula work only for integers?
Adapted from the solution by Denise Cornwell, student, Angelo State University, San Angelo, TX, USA.

(a) Another triangle with the same property is the right-angled triangle with legs 5 and $\sqrt{11}$ and hypotenuse 6.

(b) To find the formula to construct infinitely many such triangles, we let the integer sides be called $a$ and $c$. Then the side under the square root is $a + c$. We try to find an infinite family of triangles which are right-angled with legs of lengths $a$ and $\sqrt{a + c}$ and hypotenuse of length $c$. For this to happen, we must have

\[
\begin{align*}
  a^2 + (\sqrt{a + c})^2 &= c^2, \\
  a^2 + (a + c) &= c^2, \\
  a^2 + a &= c^2 - c, \\
  a^2 + a + \frac{1}{4} &= c^2 - c + \frac{1}{4}, \\
  (a + \frac{1}{2})^2 &= (c - \frac{1}{2})^2, \\
  a + \frac{1}{2} &= c - \frac{1}{2} \quad \text{(since $a$ and $c$ are positive)}, \\
  c &= a + 1.
\end{align*}
\]

Therefore, we may construct infinitely many of these right-angled triangles by letting the legs of triangle be $a$ and $\sqrt{a + (a + 1)} = \sqrt{2a + 1}$ and letting the hypotenuse be $a + 1$.

(c) The formula will work for any real number $a > 0$ (to ensure the triangle does not have a negative side).

Also solved by HASAN DENKER, Istanbul, Turkey; SAMUEL GÓMEZ MORENO, Universidad de Jaén, Jaén, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India; and TITU ZVONARU, Comănești, Romania.

There was some confusion about whether the word “construct” meant to demonstrate explicitly the infinite family or show how these triangles can be created using compass and straightedge.


Let $x$ be a positive number. Prove that $\sqrt{\frac{[x]}{x + [x]}} + \sqrt{\frac{\{x\}}{x + [x]}} > 1$, where $[x]$ and $\{x\}$ represent the integer part and the fractional part of $x$, respectively.

Solution by Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina.

If $x$ is an integer, then $\{x\} = 0$ and $x = [x]$, and we have

\[
\sqrt{\frac{[x]}{x + [x]}} + \sqrt{\frac{\{x\}}{x + [x]}} = 1 + 0 = 1.
\]
Similarly, if \( x \) is from interval \((0, 1)\), then \([x] = 0\) and \(x = \{x\}\), and
\[
\sqrt{\frac{[x]}{x + \{x\}}} + \sqrt{\frac{\{x\}}{x + [x]}} = 0 + 1 = 1.
\]
(This means that in the problem "\(\geq\)" should be replaced by "\(\geq\)" because, as we see, equality can be achieved when \(x\) is an integer or \(x \in (0, 1)\).)

Next, let \([x]\) be denoted by \(a\) and \(\{x\}\) by \(b\). Then \(x = a + b\). Note that \(0 \leq b < 1\). We may assume that \(x\) is not an integer and that \(x \notin (0, 1)\). Thus, both \(a\) and \(b\) are greater than 0. Since \(a\) is an integer, we see that \(a \geq 1 > b\); thus, \(a \neq b\). We rewrite the given inequality as
\[
\sqrt{\frac{a}{a + 2b}} + \sqrt{\frac{b}{2a + b}} > 1,
\]
which is equivalent to
\[
\sqrt{a(2a + b)} + \sqrt{b(a + 2b)} > \sqrt{(a + 2b)(2a + b)}.
\]
Since both sides are positive, after squaring this is equivalent to
\[
2a^2 + 2b^2 + 2ab + 2\sqrt{a(2a + b)}\sqrt{b(a + 2b)} > 2a^2 + 2b^2 + 5ab,
\]
which is equivalent to
\[
2\sqrt{a(2a + b)}\sqrt{b(a + 2b)} > 3ab.
\]
Since both sides are positive again, we can square this to obtain the equivalent inequality
\[
4(ab)(2a + b)(2b + a) > 9(ab)(ab),
\]
and, after dividing by \(ab > 0\), we get
\[
4(5ab + 2a^2 + 2b^2) > 9ab,
\]
which is equivalent to \(11ab + 8a^2 + 8b^2 > 0\), which is true because \(11ab \geq 0, 8a^2 \geq 8,\) and \(8b^2 \geq 0\). Therefore, the result follows.

Also solved by ARKADY ALT. San Jose, CA, USA; HASAN DENKER, Istanbul, Turkey; SAMUEL GOMEZ MORENO, Universidad de Jaen, Jaen, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; J. SUCK, Essen, Germany; and TITU IVONARU, Comănești, Romania. There was also one incorrect solution submitted.

**M293.** Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John’s, NL.

Eight equal circles are mutually tangent in pairs and tangent externally to a unit circle. Determine the common radius of the eight smaller circles.

Let $O_i$ ($i = 1, 2, \ldots, 8$) be the centre of the $i$th small circle and let $O$ be the centre of the circle of radius 1. Let $r$ be the common radius of the eight smaller circles. We need to determine $r$. Let $\angle O_1 O O_2 = \alpha$. By symmetry, $\angle O_1 O O_2 = \angle O_2 O O_3 = \ldots = \angle O_8 O O_1 = \alpha$. We have $8 \alpha = 360^\circ$; whence, $\alpha = 45^\circ$. Applying the Cosine Law to $\triangle O_1 O O_2$, and noting that $OO_1 = OO_2 = 1 + r$ and $O_1 O_2 = 2r$, we have

\[
O_1 O_2^2 = OO_1^2 + OO_2^2 - 2OO_1 \cdot OO_2 \cos \angle O_1 O O_2, \\
(2r)^2 = (1 + r)^2 + (1 + r)^2 - 2(1 + r)(1 + r) \frac{1}{\sqrt{2}}, \\
4r^2 = (2 - \sqrt{2})(1 + r)^2, \\
2r = (1 + r)\sqrt{2} - \sqrt{2} \quad \text{(since } r > 0), \\
2r - r\sqrt{2} - \sqrt{2} = \sqrt{2} - \sqrt{2}, \\
r = \frac{\sqrt{2} - \sqrt{2}}{2 - \sqrt{2} - \sqrt{2}}.
\]

II. Solution by Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina.

Let $O$ denote the centre of the large circle and $A$ the centre of one of the small circles. The line $OA$ passes through $C$, the point of tangency between the large and small circle. Draw the tangent $OB$ to this small circle. Then $OB$ is perpendicular to $AB$. By symmetry, $\angle BOA = \frac{1}{8} \times \frac{1}{8} \times 360^\circ = 22.5^\circ$. Applying the Sine Law in $\triangle OBA$ and using the fact that the radius of the large circle is 1, we get

\[
\frac{AB}{\sin(22.5^\circ)} = \frac{OA}{\sin(90^\circ)}.
\]

or

\[
\frac{r}{\sin(22.5^\circ)} = \frac{1 + r}{1}.
\]

Thus, $\sin(22.5^\circ) = r(1 - \sin(22.5^\circ))$. Hence, $r = \frac{\sin(22.5^\circ)}{1 - \sin(22.5^\circ)}$. Now, $\frac{\sqrt{2}}{2} = \cos(45^\circ) = \cos(2(22.5^\circ)) = 1 - 2\sin^2(22.5^\circ)$, which implies that $\sin^2(22.5^\circ) = \frac{2 - \sqrt{2}}{4}$. Therefore, $\sin(22.5^\circ) = \sqrt{\frac{2 - \sqrt{2}}{4}}$, and we get

\[
r = \frac{\sqrt{\frac{2 - \sqrt{2}}{4}}}{1 - \sqrt{\frac{2 - \sqrt{2}}{4}}} = \frac{\sqrt{2 - \sqrt{2}}}{2 - \sqrt{2} - \sqrt{2}}.
\]

Also solved by HASAN DENKER, Istanbul, Turkey; RICHARD J. HESS, Rancho Palos Verdes, CA, USA; SAMUEL GOMEZ MORENO, Universidad de Jaen, Jaen, Spain; KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India; OWEN REN, student, Magee Secondary School, Vancouver, BC. J. SUCK, Essen, Germany; NICK WILSON, student, Valley Catholic School, Beaverton, OR, USA; and TITU ZVONARU, Comanesti, Romania.
Problem of the Month

Ian VanderBurgh

This month, we look at two similar problems involving exponents and numbers of digits.

Problem 1 (2005 Senior Australian Mathematics Competition)

The number of digits in the decimal expansion of $2^{2005}$ is closest to

(A) 400  (B) 500  (C) 600  (D) 700  (E) 800

Problems involving exponents always present interesting challenges. Exponents and logarithms tend to mystify many students, who seem to enjoy creating their own exponent and logarithm rules to try to solve this sort of problem. I promise that we will not make up any rules while trying to solve this one.

We can deduce from the problem above that calculators are not likely allowed on the AMC; otherwise, I'm pretty sure that we could solve this in some snazzy way using a calculator. Unfortunately, I can't find mine right now, so we'll try to do this without one.

The general strategy to solve such a problem is to try to estimate the size of the given number ($2^{2005}$) compared to powers of 10.

Let's do some preliminary work before looking at the solution. We first look at how we might find an estimate for a large power of 10 that is less than $2^{2005}$. We will use the fact that if $a$ and $b$ are positive with $a > b$ and $n$ is a positive integer, then $a^n > b^n$.

Since $2^4 = 16 > 10^1$, then $2^{2005} = 2^1 2^{2004} = 2^1 (2^4)^{501} > 2 \cdot 10^{501}$.

Since $2^5 = 32 > 10^1$, then $2^{2005} = (2^5)^{401} > 10^{401}$.

Since $2^6 = 64 > 10^1$, then $2^{2005} = 2^1 2^{2004} = 2^1 (2^6)^{334} > 2 \cdot 10^{334}$.

Of these three attempts, the first inequality gives us the best estimate (that is, the largest lower bound). In general, it will be the first power of 2 larger than a given power of 10 (or the last power of 2 smaller than a given power of 10) that will give us the best estimates. Let's use this principle to solve the problem.

Let's also write out the first several powers of 2: 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096, 8192, 16384. We'll use this list to determine the largest power of 2 smaller than a given power of 10 and the smallest power of 2 larger than a given power of 10.

Solution to Problem 1: We look at the smallest powers of 2 larger than the first several powers of 10.

Since $2^4 = 16 > 10^1$, then $2^{2005} = 2^1 2^{2004} = 2^1 (2^4)^{501} > 2 \cdot 10^{501}$ (as above).

Since $2^7 = 128 > 10^2$, then $2^{2005} = 2^3 2^{2002} = 2^3 (2^7)^{286} > 8 \cdot 10^{572}$.

Since $2^{10} = 1024 > 10^3$, then $2^{2005} = 2^5 (2^{10})^{200} > 32 \cdot 10^{600}$. 

Since $2^{14} = 16384 > 10^4$, then $2^{2005} = 2^3(2^{14})^{143} > 8 \cdot 10^{572}$.

Okay! We got larger lower bounds for a little while, then it got worse. So let’s collect our thoughts and try to find an upper bound, noting that the best we know now is that $2^{2005} > 32 \cdot 10^{600}$.

We look at the largest powers of 2 which are smaller than the first few powers of 10.

Since $2^3 = 8 < 10^1$, then $2^{2005} = 2^1(2^3)^{668} < 2 \cdot 10^{668}$.

Thus, we definitely know that $32 \cdot 10^{600} < 2^{2005} < 2 \cdot 10^{668}$. Can you translate this into a range for the number of digits for $2^{2005}$? Try this before reading on.

From the last inequality, $2^{2005}$ has between 602 and 669 digits. Hence, the answer is either (C) or (D). In order to answer the question, we need to refine our estimate to determine if the number of digits is between 600 and 649 or between 650 and 699. Let’s keep going.

Since $2^6 = 64 < 10^2$, then $2^{2005} = 2^1(2^6)^{334} < 2 \cdot 10^{668}$. (That didn’t help much.)

Since $2^9 = 512 < 10^3$, then $2^{2005} = 2^7(2^9)^{222} < 128 \cdot 10^{666}$, which we can express as $1.28(10^{666})$. That’s a slightly better bound than $2 \cdot 10^{668}$, but it doesn’t actually reduce the number of digits!

Since $2^{13} = 8192 < 10^4$, then $2^{2005} = 2^4(2^{13})^{154} < 8 \cdot 10^{616}$.

Aha! We can combine this with our lower bound to conclude that $32 \cdot 10^{600} < 2^{2005} < 8 \cdot 10^{616}$. Therefore, $2^{2005}$ has between 602 and 617 digits, so (C) is the answer.

When you stop to think about it, we have actually done a pretty good job of narrowing down the range. Hold on a second! I found my calculator. Using the calculator, we get $\log_{10}(2^{2005}) \approx 603.57$, which tells us that in fact $10^{603} < 2^{2005} < 10^{604}$, so that $2^{2005}$ has exactly 604 digits.

In our second problem, we’ll try to determine the exact number of digits of a power of 2 using the method above.

**Problem 2** (1995 Special K Competition)

Determine the exact number of digits in the decimal expansion of $2^{100}$.

Let’s use our method above and try to narrow the range as much as possible.

**Solution to Problem 2:** We look at the smallest powers of 2 which are larger than the first several powers of 10.

Since $2^4 = 16 > 10^1$, then $2^{100} = (2^4)^{25} > 10^{25}$.

Since $2^7 = 128 > 10^2$, then $2^{100} = 2^2(2^7)^{14} > 2^2(10^2)^{14} = 4 \cdot 10^{28}$.

Since $2^{10} = 1024 > 10^3$, then $2^{100} = (2^{10})^{10} > (10^3)^{10} = 10^{30}$.

Since $2^{14} = 16384 > 10^4$, then $2^{100} = 2^2(2^{14})^7 > 2^2(10^4)^7 = 4 \cdot 10^{28}$.

Again, this has stopped getting better so let’s switch directions noting that our best estimate here is $2^{100} > 10^{30}$.

Next, we look at the largest powers of 2 which are smaller than the first few powers of 10.
Since \(2^3 = 8 < 10^1\), then \(2^{100} = 2^1(2^3)^{33} < 2^110^{33} = 2 \cdot 10^{33}\).

Since \(2^6 = 64 < 10^2\), then \(2^{100} = 2^4(2^6)^{16} < 2^4(10^2)^{16} = 16 \cdot 10^{32}\).

(That's only slightly better.)

Since \(2^9 = 512 < 10^3\), then \(2^{100} = 2^1(2^9)^{11} < 2^1(10^3)^{11} = 2 \cdot 10^{33}\).

(That's not any better.)

Since \(2^{13} = 8192 < 10^4\), then \(2^{100} = 2^9(2^{13})^7 < 2^9(10^4)^7\), which is \(512 \cdot 10^{28}\).

Does that help? This tells us that \(2^{100} < 5.12(10^{30})\). Aha! If we combine this with our earlier findings, we see that \(10^{30} < 2^{100} < 5.12(10^{30})\). This range is narrow enough to conclude that \(2^{100}\) has exactly 31 digits, since both the lower bound and the upper bound are integers with 31 digits.

So we used the same technique to, in the first case, bound the number of digits and, in the second case, determine the exact number of digits. In theory, we should be able to use this method to determine the exact number of digits of \(2^{2005}\), but we might need an enormous number of estimates to get this to work.

Next month, we'll have a problem that will literally and figuratively make you dizzy.

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**Note from the Mayhem Editor**

Greetings from your friendly neighbourhood Mathematical Mayhem Editor! I am very excited about joining the **CRUX with MAYHEM** team in a more significant way. You may have noticed already a slightly different flavour to the problems through the first few issues in 2008. We are going to make a real effort to keep the Mayhem problems at a more accessible level as we move forward. In addition, we are also going to move our timelines for submission up to try to get solutions published 6 issues after the problems are printed rather than 8 issues later. To facilitate this, you will notice that we have already moved up the submission deadline for the problems in this issue and you will see a plethora of Mayhem solutions in the first couple of issues this coming Fall. Happy problem solving!