THE OLYMPIAD CORNER
No. 269
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In this number we begin with the Hungarian contests for 2004–2005. First we give the Hungarian National Olympiad, Competition for Specialized Classes. Thanks go to Felix Recio, Canadian Team Leader to the IMO in Mexico, for collecting them for our use.

HUNGARIAN NATIONAL OLYMPIAD 2004–2005
Specialized Mathematical Classes
First Round

1. The quadrilateral $ABCD$ is cyclic. Prove that

$$\frac{AC}{BD} = \frac{DA \cdot AB + BC \cdot CD}{AB \cdot BC + CD \cdot DA}.$$ 

2. How many real numbers $x$ are there in the interval $0 < x < 2004$ such that $x + [x^2] = x^2 + [x]$? (Here $[c]$ denotes the greatest integer $k$ such that $k \leq c$.)

3. Let $s(n)$ be the sum of those positive divisors of $n$ that are less than $n$. A triple of three integers, $(a, b, c)$, is a friendly triple if $1 < a \leq b \leq c$ and $s(a) + s(b) = c$, $s(b) + s(c) = a$, and $s(c) + s(a) = b$. Determine all friendly triples $(a, b, c)$ where $c$ is even.

4. The set $A$ of positive integers has $k$ elements. If the positive integers $x$ and $y$ are not in $A$, then $2x$, $2y$, and $x + y$ are also not in $A$. The sum of the elements in $A$ is $s$. Find the maximum possible value of $s$.

5. Let $ABCDE$ be a pyramid, where $ABCD$ is a cyclic quadrilateral. The perpendicular projection of $E$ onto the plane $ABCD$ is $F$. Prove that the perpendicular projections of $F$ onto $AE$, $BE$, $CE$, and $DE$ all lie on a circle.

Final Round

1. Let $ABCD$ be a trapezoid with parallel sides $AB$ and $CD$. Let $E$ be a point on the side $AB$ such that $EC$ and $AD$ are parallel. Further, let the area of the triangle determined by the lines $AC$, $BD$, and $DE$ be $t$, and the area of $ABC$ be $T$. Determine the ratio $AB : CD$, if $t : T$ is maximal.

2. Find the greatest integer $k$ which has the following property: For all integers $x$ and $y$, whenever $xy + 1$ is divisible by $k$, then $x + y$ is also divisible by $k$. 
3. Haydn and Beethoven celebrate the birthday of Mozart with a game. They take numbers alternately according to the following rules. First Haydn takes the number 2. The next player can take the sum or the product of any two numbers which were taken earlier (it is possible to choose just one number twice, thus taking the square of it). The numbers which are taken must be distinct and smaller than 1757. The winner is the player who takes the number 1756. Which player has a winning strategy?

Next we give the Hungarian National Olympiad for 2004–2005. Thanks again to Felix Recio for collecting them for our use.

**HUNGARIAN NATIONAL OLYMPIAD 2004–2005**

**Grades 11–12**

**Second Round**

1. Find all real solutions to the following system of equations:

\[
\begin{align*}
\sqrt{x + y} + \sqrt{x - y} &= 10, \\
x^2 - y^2 - z^2 &= 476, \\
2(\log |y| - \log z) &= 1.
\end{align*}
\]

2. In triangle \(ABC\), the points \(B_1\) and \(C_1\) are on \(BC\), point \(B_2\) is on \(AB\), and point \(C_2\) is on \(AC\) such that the segment \(B_1B_2\) is parallel to \(AC\) and the segment \(C_1C_2\) is parallel to \(AB\). Let the lines \(B_1B_2\) and \(C_1C_2\) meet at \(D\). Denote the areas of triangles \(BB_1B_2\) and \(CC_1C_2\) by \(b\) and \(c\), respectively.

(a) Prove that if \(b = c\), then the centroid of \(ABC\) is on the line \(AD\).

(b) Find the ratio \(b : c\) if \(D\) is the incentre of \(ABC\) and \(AB = 4, BC = 5,\) and \(CA = 6\).

3. At each vertex of a pentagon there is a real number. On each side and on each diagonal, the sum of the numbers at the end-points is written. Of these ten numbers, at least seven are integers. Prove that each of the ten numbers is an integer.

4. The divisors of \(n\) are \(d_1 < d_2 < \cdots < d_8\), where \(d_1 = 1\) and \(d_8 = n\). It is known that \(20 \leq d_6 \leq 25\). Find all possible values of \(n\).

**Final Round**

1. A positive integer \(n\) is **charming** if there are integers \(a_1, a_2, \ldots, a_n\) (not necessarily distinct) such that \(a_1 + a_2 + \cdots + a_n = a_1a_2\cdots a_n = n\). Find all charming integers.
2. Let \( a, b, \) and \( c \) be positive real numbers.

(a) Prove that
\[
\sqrt{\frac{a^2 + b^2}{2}} + \frac{2}{\frac{1}{a} + \frac{1}{b}} \geq \frac{a + b}{2} + \sqrt{ab}.
\]

(b) Is it true always that
\[
\sqrt{\frac{a^2 + b^2 + c^2}{3}} + \frac{3}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}} \geq \frac{a + b + c}{3} + \sqrt[3]{abc}?
\]

3. Triangle \( ABC \) is acute angled, \( \angle BAC = 60^\circ, AB = c, \) and \( AC = b \) with \( b > c \). The orthocentre and the circumcentre of \( ABC \) are \( M \) and \( O \), respectively. The line \( OM \) intersects \( AB \) and \( CA \) at \( X \) and \( Y \), respectively.

(a) Prove that the perimeter of triangle \( AXY \) is \( b + c \).

(b) Prove that \( OM = b - c \).

The next group of problems for your puzzling pleasure are those used to select the Indian Team to the IMO 2002. Thanks again go to Felix Recio, Canadian Team Leader to the IMO in Mexico, for collecting them for us.

**INDIAN TEAM SELECTION TEST TO IMO 2002**

1. Let \( A, B, \) and \( C \) be three points on a line with \( B \) between \( A \) and \( C \). Let \( \Gamma_1, \Gamma_2, \) and \( \Gamma_3 \) be semicircles, all on the same side of \( AC \), and with \( AC, AB, \) and \( BC \) as diameters, respectively. Let \( l \) be the line perpendicular to \( AC \) through \( B \). Let \( \Gamma \) be the circle which is tangent to the line \( l \), tangent to \( \Gamma_1 \) internally, and tangent to \( \Gamma_3 \) externally. Let \( D \) be the point of contact of \( \Gamma \) and \( \Gamma_3 \). The diameter of \( \Gamma \) through \( D \) meets \( l \) in \( E \). Show that \( AB = DE \).

2. Show that there is a set of 2002 consecutive positive integers containing exactly 150 primes. (You may use the fact that there are 168 primes less than 1000.)

3. Let \( X = \{2^m3^n \mid 0 \leq m, n \leq 9\} \). How many quadratics are there of the form \( ax^2 + 2bx + c \), with equal roots, and such that \( a, b, \) and \( c \) are distinct elements of \( X \)?

4. Let \( ABC \) be an acute triangle with orthocentre \( H \) and circumcentre \( O \). Show that there are points \( D, E, \) and \( F \) on \( BC, CA, \) and \( AB \), respectively, such that \( AD, BE, \) and \( CF \) are concurrent and
\[
DO + DH = EO + EH = FO + FH.
\]
5. Let \(a, b, \text{ and } c\) be positive real numbers such that \(a^2 + b^2 + c^2 = 3abc\). Prove that
\[
\frac{a}{b^2c^2} + \frac{b}{c^2a^2} + \frac{c}{a^2b^2} \geq \frac{9}{a + b + c}.
\]

6. Determine the number of \(n\)-tuples of integers \((x_1, x_2, \ldots, x_n)\) such that \(|x_i| \leq 10\) for each \(1 \leq i \leq n\) and \(|x_i - x_j| \leq 10\) for \(1 \leq i, j \leq n\).

7. Given two distinct circles touching each other internally, show how to construct a triangle with the inner circle as its incircle and the outer circle as its nine-point circle.

8. Let \(\sigma(n) = \sum_{d|n} d\), the sum of the positive divisors of an integer \(n > 0\).

   (a) Show that \(\sigma(mn) = \sigma(m)\sigma(n)\) for positive integers \(m\) and \(n\) with \(\gcd(m, n) = 1\).

   (b) Find all positive integers \(n\) such that \(\sigma(n)\) is a power of 2.

9. On each day of their tour of the West Indies, Sourav and Srinath have either an apple or an orange for breakfast. Sourav has oranges for the first \(m\) days, apples for the next \(m\) days, followed by oranges for the next \(m\) days, and so on. Srinath has oranges for the first \(n\) days, apples for the next \(n\) days, followed by oranges for the next \(n\) days, and so on.

   If \(\gcd(m, n) = 1\) and the tour lasted for \(mn\) days, on how many days did they eat the same kind of fruit?

10. Let \(T\) be the set of all ordered triples \((p, q, r)\) of non-negative integers. Determine all functions \(f : T \to \mathbb{R}\) such that if \(pqr = 0\), then \(f(p, q, r) = 0\), and if \(pqr \neq 0\), then
\[
f(p, q, r) = 1 + \frac{1}{6} [f(p + 1, q - 1, r) + f(p - 1, q + 1, r) + f(p - 1, q, r + 1) + f(p + 1, q, r - 1) + f(p, q + 1, r - 1) + f(p, q - 1, r + 1)].
\]

11. Let \(ABC\) be a triangle and let \(P\) be an exterior point in the plane of the triangle. Let \(AP, BP, \text{ and } CP\) meet the (possibly extended) sides \(BC, CA, \text{ and } AB\) in \(D, E, \text{ and } F\), respectively. If the areas of the triangles \(PBD, PCE, \text{ and } PAF\) are all equal, prove that their common area is equal to the area of the triangle \(ABC\).

12. Let \(a\) and \(b\) be integers with \(0 < a < b\). A set \(\{x, y, z\}\) of non-negative integers is \textit{olympic} if \(x < y < z\) and if \(\{z - y, y - x\} = \{a, b\}\). Show that the set of all non-negative integers is the union of pairwise disjoint olympic sets.
13. Let $ABC$ and $PQR$ be two triangles such that
(a) $P$ is the mid-point of $BC$ and $A$ is the mid-point of $QR$, and
(b) $QR$ bisects $\angle BAC$ and $BC$ bisects $\angle QPR$.
Prove that $AB + AC = PQ + PR$.

14. Let $p$ be an odd prime and let $a$ be an integer not divisible by $p$. Show that there are $p^2 + 1$ triples of integers $(x, y, z)$ with $0 \leq x, y, z < p$ and such that $(x + y + z)^2 \equiv axyz \pmod{p}$.

15. Let $x_1, x_2, \ldots, x_n$ be real numbers. Prove that
\[
\frac{x_1}{1 + x_1^2} + \frac{x_2}{1 + x_2^2 + x_3^2} + \cdots + \frac{x_n}{1 + x_1^2 + x_2^2 + \cdots + x_n^2} < \sqrt{n}.
\]

16. Is there a set of positive integers, \{a_1, a_2, \ldots, a_{100} : a_i \leq 25000\}, with the property that the sums $a_i + a_j$, $1 \leq i < j \leq 100$, are all distinct?

17. Let $n$ be a positive integer, and let $(1 + iT)^n = f(T) + ig(T)$, where $i$ is the square root of $-1$, and $f$ and $g$ are polynomials with real coefficients. Show that for any real number $k$ the equation $f(T) + kg(T) = 0$ has only real roots.

18. Consider the square grid with $A = (0, 0)$ and $C = (n, n)$ at its diagonal ends. Paths from $A$ to $C$ are composed of moves one unit to the right or one unit up. Let $C_n$ be the number of paths from $A$ to $C$ which stay on or below the diagonal $AC$ ($C_n$ is the $n$th Catalan Number). Show that the number of paths from $A$ to $C$ which cross $AC$ from below at most twice is equal to $C_{n+2} - 2C_{n+1} + C_n$.

19. Let $PQR$ be an acute triangle. Let $SRP$, $TPQ$, and $UQR$ be isosceles triangles exterior to $PQR$, with $SP = SR$, $TP = TQ$, and $UQ = UR$, such that $\angle PSR = 2\angle QPR$, $\angle QTP = 2\angle RQP$, and $\angle RUQ = 2\angle PQR$. Let $S'$, $T'$, and $U'$ be the points of intersection of $SQ$ and $TU$, $TR$ and $US$, and $UP$ and $ST$, respectively. Determine the value of
\[
\frac{SQ}{SS'} + \frac{TR}{TT'} + \frac{UP}{UU'}.
\]

20. Let $a$, $b$, and $c$ be positive real numbers. Prove that
\[
\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{c}{b} + \frac{a}{c} + \frac{b}{a}.
\]

21. Given a prime $p$, show that there is a positive integer $n$ such that the decimal representation of $p^n$ has a block of 2002 consecutive zeros.
The next set of problems are from the 2004 Kőrnschák Competition, also collected for us by Felix Recio, Canadian Team Leader to the IMO in Mexico.

2004 KŐRNSCHÁK COMPETITION

1. The circle $k$ and the circumcircle of the triangle $ABC$ are touching externally. The circle $k$ also touches the rays $AB$ and $AC$ at the points $P$ and $Q$, respectively. Prove that the mid-point of the segment $PQ$ is the centre of the excircle touching the side $BC$ of the triangle $ABC$.

2. Find the smallest positive integer $n$, different from 2004, with the property that there exists a polynomial $f(x)$ with integer coefficients such that the equation $f(x) = 2004$ has at least one integer solution and the equation $f(x) = n$ has at least 2004 distinct integer solutions.

3. Some red points and some blue points are on the circumference of a circle. The following operations can be performed:

(a) A new red point can be inserted somewhere and the colours of each of its two neighbours changed to the opposite colour.

(b) If there are at least three points, at least one of which is red, then a red point can be deleted and the colours of each of its two neighbours changed to the opposite colour.

At the start, there are exactly two points on the circle, both blue. Can these two blue points be changed into (exactly) two red points by a sequence of the two operations?

Next we turn to our file of solutions to problems given in the May number of the Corner. First is a solution to a problem from the final round of the 18th Korean Mathematical Olympiad given at [2007: 214–215].

2. Show that no pair of positive integers $x$ and $y$ satisfies $3y^2 = x^4 + x$.

Solved by Ioannis Katsikis, Athens, Greece; and Andrea Munaro, student, University of Trento, Trento, Italy. We give the solution of Munaro.

We have

$$3y^2 = x^4 + x = x(x + 1)(x^2 - x + 1).$$

Let $\gcd(a, b)$ be the greatest common divisor of the integers $a$ and $b$. Then

$$\gcd(x, x + 1) = 1,$$
$$\gcd(x^2 - x + 1, x) = \gcd(x^2 + 1, x) = 1,$$
$$\gcd(x^2 - x + 1, x + 1) = \gcd((x + 1)^2 - 3x, x + 1) = \gcd(3x, x + 1),$$
where the very last greatest common divisor is either 1 or 3. This leads to two cases.

**Case 1.** \( \gcd(x^2 - x + 1, x + 1) = 1 \).

The three factors on the right side of (1) are coprime in pairs; thus, one factor is of the form \( 3a^2 \) and the other two factors are perfect squares.

If \( x^2 - x + 1 = 3a^2 \) for some integer \( a \), then both \( x \) and \( x + 1 \) are perfect squares, a contradiction.

If \( x + 1 = 3a^2 \) or \( x = 3a^2 \) for some integer \( a \), then \( x^2 - x + 1 \) must be a perfect square. Noting that \( x^2 - 2x + 1 = (x - 1)^2 \) is a perfect square and that \( (x - 1)^2 < x^2 - x + 1 \leq x^2 \), we must have \( x^2 - x + 1 = x^2 \), yielding \( x = 1 \), which is not a solution to the original equation.

**Case 2.** \( \gcd(x^2 - x + 1, x + 1) = 3 \).

Then \( x + 1 = 3a \) and \( x^2 - x + 1 = 3b \) with \( \gcd(a, b) = 1 \), and \( x \) is a perfect square. Equation (1) becomes \( 3y^2 = x \cdot 3a \cdot 3b \) or \( y^2 = x \cdot a \cdot b \), with \( \gcd(x, a) = \gcd(x, b) = \gcd(a, b) = 1 \). Hence, \( a \) or \( b \) is divisible by 3. Simple calculations show that \( x^2 - x + 1 \) is never divisible by 9; thus, \( 3 \nmid b \). This means that \( 3 \nmid a \), implying that \( 9 \mid (x + 1) \). However, this implies that \( x \) and \( \frac{1}{8}(x + 1) \) are perfect squares, hence \( x \) and \( x + 1 \) are perfect squares, a contradiction.

We now present a solution from our readers to a problem from the 21st Balkan Mathematical Olympiad 2004, given at [2007 : 215].

3. Let \( O \) be the circumcentre of the acute triangle \( ABC \). The circles centred at the mid-points of the triangle's sides and passing through \( O \) intersect one another at the points \( K, L, \) and \( M \). Prove that \( O \) is the incentre of triangle \( KLM \).

*Solved by Michel Bataille, Rouen, France; and D.J. Smeenk, Zaltbommel, the Netherlands. We give Bataille's write-up.*

Let \( X, Y, \) and \( Z \) be the mid-points of \( BC, CA, \) and \( AB \), respectively. Since \( OK \) is perpendicular to the line through the centres \( Y \) and \( Z \), and \( YZ \parallel BC \), we see that \( KO \) is perpendicular to \( BC \) and it follows that \( O \) lies on the line \( KX \). Similarly, \( O \) is on \( LY \) and \( MZ \). Note that \( O \) is the orthocentre of \( \triangle XYZ \) and that \( K, L, \) and \( M \) are
the reflections of \( O \) in the sides \( YZ, ZX, \) and \( XY, \) respectively. It follows that \( K, L, \) and \( M \) are on the circumcircle \( \Gamma \) of \( \triangle XYZ. \) Since \( O \) is interior to \( \triangle XYZ \) (an acute-angled triangle, as it is similar to \( \triangle ABC), \) \( K \) is on the arc \( YZ \) of \( \Gamma \) which does not contain \( X. \) Analogous observations can be made for \( L \) and \( M, \) so that \( X, M, Y, K, Z, L, \) and \( X \) occur in this order on \( \Gamma. \) In addition, \( ZX \) is clearly the internal bisector of \( \angle LZO, \) so that \( X \) is the mid-point of the arc \( LM \) of \( \Gamma \) (recall that \( Z, O, \) and \( M \) are collinear). Thus, \( KX \) is the internal bisector of \( \angle MKL. \) The result follows.

Next we examine the solutions in our files to problems posed in the 14th Japanese Mathematical Olympiad given at [2007 : 215–216].

2. Find all functions \( f : \mathbb{R} \to \mathbb{R} \) such that, for all real numbers \( x \) and \( y, \)
\[
f(xf(x) + f(y)) = (f(x))^2 + y.
\]

Solved by Michel Bataille, Rouen, France; and Li Zhou, Polk Community College, Winter Haven, FL, USA. We give Zhou’s write-up.

Clearly \( f(x) = x \) and \( f(x) = -x \) are solutions. We show that they are the only solutions.

We obtain \( f(f(-f(0)^2)) = 0 \) by taking \( x = 0 \) and \( y = -f(0)^2 \) in the identity. Then setting \( x = f(-f(0)^2), \) we obtain \( f(f(y)) = y, \) for all \( y. \) Putting \( x = f(z) \) in the identity we now obtain
\[
f(f(z)z + f(y)) = z^2 + y,
\]
for all \( y \) and \( z, \) and hence \( f(z)^2 = z^2 \) for all \( z, \) as the left side of the above is also equal to \( f(z)^2 + y. \)

If \( f(1) = 1, \) then
\[
1 + 2x + x^2 = f(1 + x)^2 = f(1 \cdot f(1) + f(f(x)))^2 = (f(1)^2 + f(x))^2 = 1 + 2f(x) + x^2,
\]
and thus \( f(x) = x \) for all \( x. \) Similarly, if \( f(1) = -1, \) then
\[
1 - 2x + x^2 = f(-1 + x)^2 = f(1 \cdot f(1) + f(f(x)))^2 = (f(1)^2 + f(x))^2 = 1 + 2f(x) + x^2,
\]
and thus \( f(x) = -x \) for all \( x. \)

4. For positive real numbers \( a, b, \) and \( c \) with \( a + b + c = 1, \) show that
\[
\frac{1+a}{1-a} + \frac{1+b}{1-b} + \frac{1+c}{1-c} \leq 2 \left( \frac{b}{a} + \frac{c}{b} + \frac{a}{c} \right).
\]

You need not state when equality holds.
Solved by Arkady Alt, San Jose, CA, USA; Michel Bataille, Rouen, France; and Babis Stergiou, Chalkida, Greece. We give Stergiou’s account.

Since \(a + b + c = 1\), we can write

\[
\frac{1 + a}{1 - a} = \frac{a + b + c + a}{a + b + c - a} = \frac{2a + b + c}{b + c} = 2a + 1.
\]

Thus, the given inequality is successively equivalent to

\[
\left( \frac{2a}{b + c} + 1 \right) + \left( \frac{2b}{c + a} + 1 \right) + \left( \frac{2c}{a + b} + 1 \right) \leq 2 \left( \frac{b + c + a}{c} \right);
\]

\[
\left( \frac{a}{c} - \frac{a}{b + c} \right) + \left( \frac{b}{a} - \frac{b}{c + a} \right) + \left( \frac{c}{b} - \frac{c}{a + b} \right) \geq \frac{3}{2};
\]

\[
\frac{ab}{c(b + c)} + \frac{bc}{a(c + a)} + \frac{ca}{b(a + b)} \geq \frac{3}{2}.
\]

To prove the last inequality, we use a consequence of the Cauchy-Schwartz Inequality,

\[
a^2 + b^2 + c^2 \geq \frac{(a + b + c)^2}{x + y + z},
\]

where \(x, y,\) and \(z\) are positive integers. Using this inequality, we obtain

\[
\frac{ab}{c(b + c)} + \frac{bc}{a(c + a)} + \frac{ca}{b(a + b)} = \frac{(ab)^2}{abc(b + c)} + \frac{(bc)^2}{abc(c + a)} + \frac{(ca)^2}{abc(a + b)} \geq \frac{(ab + bc + ca)^2}{2abc(a + b + c)}.
\]

Thus, it suffices to prove that

\[
\frac{(ab + bc + ca)^2}{2abc(a + b + c)} \geq \frac{3}{2},
\]

which is the same as proving \((ab + bc + ca)^2 \geq 3abc(a + b + c)\), which follows from the basic inequality \((x + y + z)^2 \geq 3(xy + yz + zx)\). In the last inequality we have equality only if \(x = y = z\), so in the given inequality we have equality only if \(a = b = c = \frac{1}{3}\).

Next we turn to problems in the September 2007 Corner. We present solutions to selected problems of the Thai Mathematical Olympiad 2003, given at [2007: 277–278].

1. Triangle \(ABC\) has \(\angle A = 70^\circ\) and \(CA + AI = BC\), where \(I\) is the incentre of triangle \(ABC\). Find \(\angle B\).
Solved by Geoffrey A. Kandall, Hamden, CT, USA; Pavlos Maragoudakis, Pireas, Greece; and Titu Zvonaru, Comăneşti, Romania. We give Kandall’s version.

First note that $IA$ bisects $\angle BAC$, and $IB$ bisects $\angle ABC$. Extend $BA$ to a point $D$ such that $AD = AI = a - b$. Let $F$ be the foot of the perpendicular from $I$ onto $AB$; $F$ is the point of contact of the incircle with $AB$. It is well-known that $BF = s - b$ and $AF = s - a$. Then

$$DF = (a - b) + (s - a) = s - b = BF,$$

so $IB = ID$. Consequently, $\triangle IBF = \triangle IDA = \triangle AID = \frac{1}{2}\angle BAI$; hence, $\angle ABC = \angle BAI = \frac{1}{2}\angle BAC$.

We were given that $\angle BAC = 70^\circ$, so $\angle ABC = 35^\circ$.

2. Let $f : \mathbb{Q} \to \mathbb{Q}$, where $\mathbb{Q}$ is the set of all rational numbers, be such that

$$f(x + y) = f(x) + f(y) + 2547,$$

for all $x, y \in \mathbb{Q}$ and $f(2004) = 2547$. Find $f(2547)$.

Solved by Michel Bataille, Rouen, France; Pavlos Maragoudakis, Pireas, Greece; Daniel Tsai, student, Taipei American School, Taipei, Taiwan; and Titu Zvonaru, Comăneşti, Romania. We first give Bataille’s write-up.

We show that $f(2547) = \frac{2547 \cdot 515}{334}$. Let $g : \mathbb{Q} \to \mathbb{Q}$ be defined by

$$g(x) = f(x) + 2547.$$

Then,

$$g(x + y) = f(x + y) + 2547 = f(x) + 2547 + f(y) + 2547 = g(x) + g(y)$$

for all rational numbers $x$ and $y$. It follows that for some rational number $r$, we have $g(x) = rx$ for all $x \in \mathbb{Q}$ (this is a well-known result about additive functions on the rational numbers).

Since $f(2004) = 2547$, we get $g(2004) = 2 \cdot 2547$ and, observing that $g(2004) = 2004r$ as well, we obtain

$$r = \frac{2 \cdot 2547}{2004} = \frac{849}{334}.$$ 

Finally, $f(2547) = g(2547) - 2547 = 2547 - \frac{849}{334} - 2547 = 2547 \cdot \frac{515}{334}.$
Next we give Tsai’s write-up.

Let \( x \in \mathbb{Q} \) and let \( n \) be a positive integer. We prove by induction on \( n \) that

\[
f(nx) = nf(x) + 2547(n - 1) .
\]

For \( n = 1 \), this is trivial. Assume the above equation holds for some positive integer \( n \) and for all \( x \in \mathbb{Q} \). The induction step is completed by the calculation

\[
f((n + 1)x) = f(nx + x) = f(nx) + f(x) + 2547
\]

\[
= nf(x) + 2547(n - 1) + f(x) + 2547
\]

\[
= (n + 1)f(x) + 2547n .
\]

Now, we have \( f(2004 \cdot 2547) = 2004f(2547) + 2547 \cdot 2003 \) and also \( f(2547 \cdot 2004) = 2547f(2004) + 2547 \cdot 2546 \); thus,

\[
f(2547) = \frac{2547f(2004) + 2546 \cdot 2547 - 2547 \cdot 2003}{2004}
\]

\[
= \frac{2547^2 + 2546 \cdot 2547 - 2547 \cdot 2003}{2004}
\]

\[
= \frac{1311705}{334} .
\]

Remark. Let \( C \in \mathbb{Q} \) and let \( f : \mathbb{Q} \to \mathbb{Q} \). Then \( f(x + y) = f(x) + f(y) + C \) for all \( x, y \in \mathbb{Q} \) if and only if \( f(nx) = nf(x) + C(n - 1) \) for all \( x \in \mathbb{Q} \) and all \( n \in \mathbb{N} \).

3. Let \( a, b, \) and \( c \) be positive real numbers such that \( a + b + c \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \).

Prove that \( a^3 + b^3 + c^3 \geq a + b + c \).

Solved by Arkady Alt, San Jose, CA, USA; Michel Bataille, Rouen, France; Pavlos Maragoudakis, Piraeus, Greece; Vedula N. Murty, Dover, PA, USA; George Tsapakidis, Agrinoi, Greece; and Panos E. Tsouissoglou, Athens, Greece. We give Alt’s generalization.

By Jensen’s Inequality, we have

\[
\frac{a^3 + b^3 + c^3}{3} \geq \left( \frac{a + b + c}{3} \right)^3 .
\]

We also have

\[
(a + b + c)^2 \geq (a + b + c) \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \geq 9 .
\]

We conclude that

\[
a^3 + b^3 + c^3 \geq \frac{(a + b + c)(a + b + c)^2}{9} \geq a + b + c .
\]
**Generalization.** Let \( n \) be a non-negative integer. With the same hypotheses, we have
\[
a^{n+1} + b^{n+1} + c^{n+1} \geq a^{n-1} + b^{n-1} + c^{n-1}.
\]

**Proof.** For non-negative integers \( n \) and \( m \), we have
\[
a^{n+m} + b^{n+m} + c^{n+m} \geq \frac{(a^n + b^n + c^n)(a^m + b^m + c^m)}{3}.
\]
Indeed,
\[
3(a^{n+m} + b^{n+m} + c^{n+m}) - (a^n + b^n + c^n)(a^m + b^m + c^m)
\]
\[
= \sum_{cyclic}(a^{n+m} + b^{n+m} - a^n b^m - a^m b^n)
\]
\[
= \sum_{cyclic}(a^n - b^n)(a^m - b^m) \geq 0.
\]

Using this inequality and
\[
a^2 + b^2 + c^2 \geq \frac{(a + b + c)^2}{3} \geq \frac{(a + b + c)(\frac{1}{a} + \frac{1}{b} + \frac{1}{c})}{3} \geq \frac{9}{3} = 3,
\]
we immediately obtain
\[
a^{n+1} + b^{n+1} + c^{n+1} \geq (a^{n-1} + b^{n-1} + c^{n-1}) \left(\frac{a^2 + b^2 + c^2}{3}\right)
\]
\[
\geq a^{n-1} + b^{n-1} + c^{n-1}.
\]

6. Let \( ABCD \) be a convex quadrilateral. Prove that
\[
[ABCD] \leq \frac{1}{4} (AB^2 + BC^2 + CD^2 + DA^2).
\]

Solved by Michel Bataille, Rouen, France; Geoffrey A. Kandall, Hamden, CT, USA; Pavlos Maragoudakis, Pireas, Greece; George Tsapakis, Agrinio, Greece; and Titu Zvonaru, Comănești, Romania. We give the solution by Zvonaru.

We have
\[
[ABCD] = [ABC] + [CDA]
\]
\[
= \frac{1}{2} \cdot AB \cdot BC \cdot \sin \angle ABC + \frac{1}{2} CD \cdot DA \cdot \sin \angle CDA
\]
\[
\leq \frac{1}{2} AB \cdot BC + \frac{1}{2} CD \cdot DA
\]
\[
\leq \frac{1}{2} \left(\frac{1}{2}(AB^2 + BC^2) + \frac{1}{2}(CD^2 + DA^2)\right)
\]
\[
= \frac{1}{4}(AB^2 + BC^2 + CD^2 + DA^2).
\]
Equality holds if and only if \( ABCD \) is a square.
9. Given a right triangle $ABC$ with $\angle B = 90^\circ$, let $P$ be a point on the angle bisector of $\angle A$ inside $ABC$ and let $M$ be a point on the side $AB$ (with $A \neq M \neq B$). Lines $AP$, $CP$, and $MP$ intersect $BC$, $AB$, and $AC$ at $D$, $E$, and $N$, respectively. Suppose that $\angle MPB = \angle PCN$ and $\angle NPC = \angle MBP$. Find $[APC]/[ACDE]$.

Solved by Pavlos Maragoudakis, Pireas, Greece; and Titu Zvonaru, Comăneşti, Romania. We give the solution of Zvonaru.

Let $a = BC$, $b = CA$, and $c = AB$, and let $c_a = AD$, the bisector of $\angle A$. We have

$$c_a = \frac{2bc \cos \frac{A}{2}}{b + c}.$$  

By the Bisector Theorem, we deduce that

$$DC = \frac{ab}{b + c}, \quad BD = \frac{ac}{b + c}.$$  

We have $\angle BMP = \angle CNP$; thus, $\triangle AMN$ is isosceles. Hence, $AM = AN$ and $P$ is the mid-point of $MN$. Let $x$ be the common length of $AM$ and $AN$. It follows that

$$MP = NP = x \sin \frac{A}{2}, \quad AP = x \cos \frac{A}{2}.$$  

Since $\triangle MBP$ and $\triangle CPN$ are similar, we have $\frac{BM}{PN} = \frac{PM}{NC}$, which yields successively

$$(b - x)(c - x) = x^2 \sin^2 \frac{A}{2};$$

$$x^2 \left(1 - \sin^2 \frac{A}{2}\right) - (b + c)x + bc = 0;$$

$$\frac{x^2}{2} \left(1 + \cos A\right) - (b + c)x + bc = 0;$$

$$(b + c)x^2 - 2b(b + c)x + 2b^2c = 0.$$  

Solving this equation we obtain

$$x = \frac{b(b + c) \pm \sqrt{b^2(b + c)^2 - 2b^2c(b + c)}}{b + c}$$

$$= \frac{b(b + c) \pm b\sqrt{(b + c)(b - c)}}{b + c}$$

$$= \frac{b(b + c \pm a)}{b + c},$$

and since $x < b$, we take $x = \frac{b(b + c - a)}{b + c}$. 
We now have
\[
\frac{AP}{PD} = \frac{x \cos \frac{A}{2}}{c_a - x \cos \frac{A}{2}} = \frac{(b + c)x \cos \frac{A}{2}}{(b + c)c_a - (b + c)x \cos \frac{A}{2}}
\]
\[
= \frac{b(b + c - a) \cos \frac{A}{2}}{2bc \cos \frac{A}{2} - b(b + c - a) \cos \frac{A}{2}} = \frac{b + c - a}{a + c - b}.
\]

Menelaus’ Theorem applied to \(\triangle ABD\) gives \(\frac{CD}{CB} \cdot \frac{EB}{EA} \cdot \frac{PA}{PD} = 1\); hence, gives \(\frac{EB}{EA} = \frac{CB}{CD} \cdot \frac{PD}{PA}\). Substituting the previously obtained expressions
\[
\frac{EB}{EA} = \frac{(b + c)(a - b + c)}{b(b + c - a)}.
\]

If \(r\) is the ratio \(EB : EA\), then the ratio \(EB : c\) is equal to \(r : 1 + r\). Hence,
\[
EB = \frac{(b + c)(a - b + c)}{a + b + c}.
\]

Finally, we have
\[
\begin{align*}
[ACDE] &= [ABC] - [BDE] \\
&= \frac{ac}{2} - \frac{ac}{2} \cdot \frac{(b + c)(a - b + c)}{a + b + c} \\
&= \frac{ac}{2} \left(1 - \frac{a - b + c}{a + b + c}\right) = \frac{abc}{a + b + c},
\end{align*}
\]
and also
\[
\begin{align*}
[APC] &= \frac{AP \cdot AC \cdot \sin \frac{A}{2}}{2} = \frac{x \cos \frac{A}{2} \cdot b \sin \frac{A}{2}}{2} \\
&= \frac{bx \sin A}{4} = b \cdot \frac{b(b + c - a)}{4(b + c)} \cdot \frac{a}{b} = \frac{ab(b + c - a)}{4(b + c)};
\end{align*}
\]
hence,
\[
\frac{[APC]}{[ACDE]} = \frac{(b + c - a)(a + b + c)}{4c(b + c)}.
\]

That completes this number of the Corner. Send solutions soon!