Problem of the Month

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I hope that you've been doing some strengthening exercises recently—we're in for some heavy (and not-so-heavy) lifting this month.

Problem (2006 Cayley Contest)

Quincy and Celine have to move 16 small boxes and 10 large boxes. The chart below indicates the time that each person takes to move each type of box.

	Celine	Quincy
small box	2 min.	3 min.
large box	6 min.	5 min.

They start moving the boxes at 9:00 a.m. What is the earliest time at which they can be finished moving all of the boxes?

(A) 9:41 a.m. (B) 9:42 a.m. (C) 9:43 a.m. (D) 9:44 a.m. (E) 9:45 a.m.

First, some ground rules. No throwing boxes. No stacking boxes. You are not allowed to put one box inside another. (For some reason, when I do this type of problem with a group of students, they always want to try these sneaky things!)

While it makes sense to get out a piece of scrap paper and do a bit of investigation, let's look before we leap. Since Celine is faster on smaller boxes and Quincy is faster on large boxes, let's try having Celine move all of the small boxes (taking $16 \times 2 = 32$ minutes) and having Quincy move all of the large boxes (taking $10 \times 5 = 50$ minutes). Thus, they would finish at 9:50 a.m. Can you see why this minimizes the *total working time*? Interestingly, this does not necessarily mean that it gives the earliest end time using the given rules.

While this division of labour is fast in one sense, this does not seem optimal in others. Celine moves boxes for 32 minutes and then sits around for 18 more minutes while Quincy keeps moving boxes. (Does this remind you of what happens sometimes when you're doing chores at home?)

How could we improve on this time? What if Celine pitches in by also moving 1 large box? Quincy's time is reduced to $9 \times 5 = 45$ minutes and Celine's time increases to $16 \times 2 + 6 = 38$ minutes. We are now at 9:45 a.m. as the finishing time. This is definitely better. But is it the best possible?

Take a few minutes to see if you can do better than 9:45 a.m. While you're doing this, I'll try to distract you a bit by pointing out that there might be some unrealistic things in this problem. At the same time, though, this problem does seem to be more of a "real life" problem than some of the ones that we encounter. (I will say more about this at the end.)

Did you do better than 9:45 a.m.? In fact, 9:43 a.m. is possible. Suppose that Celine moves 15 small boxes and 2 large boxes. This takes her $15 \times 2 + 2 \times 6 = 42$ minutes. Quincy will thus move 1 small box and 8 large boxes, which will take $1 \times 3 + 8 \times 5 = 43$ minutes. Overall, they will finish at 9:43 a.m. (and Celine won't even have time to get much of a break in her 1 minute of spare time at the end).

After some more trial and error, you will probably get frustrated like I did when you can't get a shorter time than 43 minutes. So it seems that 9:43 a.m. is the earliest possible finish time. Let's prove this.

Solution: If Celine moves 15 small boxes and 2 large boxes, it takes her 42 minutes. If Quincy moves 8 large boxes and 1 small box, it takes him 43 minutes. In this configuration, the boxes are fully moved at 9:43 a.m.

Suppose in general that Celine moves x small and y large boxes. Thus, Quincy moves $16 - x$ and $10 - y$ small boxes. In this case, Celine takes $2x + 6y$ minutes and Quincy takes $3(16 - x) + 5(10 - y) = 98 - 3x - 5y$ minutes.

Suppose now that they finish the job at 9:42 a.m. or earlier. (We'll show that this cannot actually happen.) In this case, each works for at most 42 minutes, so the total time that they work would have to be at most 84 minutes. From the information above, their total time is

$$(2x + 6y) + (98 - 3x - 5y) \;=\; 98 + y - x\,.$$

If their total time is at most 84 minutes, then $98 + y - x \leq 84$; that is, $x - y \geq 14$. But x and y are non-negative integers with $0 \leq x \leq 16$ and $0 \leq y \leq 10$. Therefore, $x - y \leq 16$ (because x is at most 16). Hence, we have $14 \leq x - y \leq 16$.

The possible pairs (x, y) that satisfy these restrictions are $(16, 0)$, $(16, 1)$, $(16, 2)$, $(15, 0)$, $(15, 1)$, and $(14, 0)$. Let's make a table of the length of time that each of Celine and Quincy takes for these values of x and y:

x	y	$2x + 6y$	$98 - 3x - 5y$	Finish at
16	0	32	50	9:50 a.m.
16	1	38	45	9:45 a.m.
16	2	44	40	9:44 a.m.
15	0	30	53	9:53 a.m.
15	1	36	48	9:48 a.m.
14	0	28	56	9:56 a.m.

Since these are all of the possible configurations in which Celine and Quincy could possibly finish at or before 9:42 a.m., and in none of them do they actually finish at 9:42 a.m. or before, we conclude that they cannot finish at or before 9:42 a.m.

Summarizing what we have seen, Celine and Quincy can finish at 9:43 a.m., and cannot finish any earlier. Hence, 9:43 a.m. is the earliest possible finishing time.

Looking back, we solved this problem by looking at the combined time and showing that we couldn't make the pair of eager workers finish earlier than 9:43 a.m.

We could have tried looking at the conditions $2x + 6y \leq 42$ and $98 - 3x - 5y \leq 42$ (or $3x + 5y \geq 56$) to show that they cannot be satisfied at the same time. We could approach this algebraically or even graphically (showing that there isn't a point in this region with non-negative integer coordinates).

This problem can actually be solved in "easier" ways using more advanced techniques from "linear programming". However, it is important from time to time to try to solve these problems using more elementary techniques.

I want to return briefly to the problem and the "real life" aspect of it. Suppose that we change the problem to the following:

> The Dunkley Piano Company makes small pianos and large pianos in their factories in Caracas and Quito. The chart below indicates the time that each factory takes to make each type of piano.
>
> | | Caracas | Quito |
> |---|---|---|
> | small piano | 2 days | 3 days |
> | large piano | 6 days | 5 days |
>
> They start making the pianos on March 1. What is the earliest date on which they can be finished making all of the pianos?

This is exactly the same problem, and so may not appear to be that interesting. But you can quite easily see how applicable this type of problem could be in business and industry. So math (and even math contest problems) can be useful!