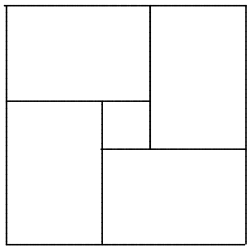
Mayhem Solutions



M282. *Proposed by J. Walter Lynch, Athens, GA, USA.*

Four rectangles are arranged in a square pattern so that they enclose a smaller square. Let S be the area of the outer square and Q the area of the inner square. If $S/Q = 9 + 4\sqrt{5}$, determine the ratio of the sides of the rectangles.

Combination of solutions by Mihály Bencze, Brasov, Romania; Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam; Denise Cornwell, student, Angelo State University, San Angelo, TX, USA; Hasan Denker, Istanbul, Turkey; Richard I. Hess, Rancho Palos Verdes, CA, USA; John G. Heuver, Grande Prairie, AB; Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; Dragoljub Milošević, Pranjani, Serbia; Billy Suandito, Palembang, Indonesia; Nick Wilson, student, Valley Catholic School, Beaverton, OR, USA; and Titu Zvonaru, Comăneşti, Romania.

Let x and y represent the sides of one of the rectangles such that $x > y$. Then the outer square has side length $x + y$ and the inner square has side length $x - y$. The given ratio $S/Q = 9 + 4\sqrt{5}$ can then be represented as

$$\frac{(x+y)^2}{(x-y)^2} = \left(2+\sqrt{5}\right)^2 .$$

Since $x > y$, we successively obtain the equivalent equations

$$\begin{aligned} \frac{x+y}{x-y} &= 2+\sqrt{5}, \\ x+y &= \left(2+\sqrt{5}\right)x - \left(2+\sqrt{5}\right)y, \end{aligned}$$

and $x + \sqrt{5}x = 3y + \sqrt{5}y$. The ratio of the sides of the rectangle then yields

$$\frac{x}{y} = \frac{\left(3+\sqrt{5}\right)\left(1-\sqrt{5}\right)}{\left(1+\sqrt{5}\right)\left(1-\sqrt{5}\right)} = \frac{\sqrt{5}+1}{2},$$

which is the Golden Ratio! We can also compute $\frac{y}{x} = \frac{\sqrt{5}-1}{2}$.

There was one incorrect solution submitted.

M283. *Proposed by Neven Jurič, Zagreb, Croatia.*

Determine the relationship between x and y if

$$x^2 + y\cos^2\alpha = x\sin\alpha\cos\alpha \qquad \text{and} \qquad x\cos 2\alpha + y\sin 2\alpha = 0 .$$

(Assume that both x and y are non-zero.)

Essentially the same solution by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam; and Titu Zvonaru, Comănești, Romania.

Using the identities $2\cos^2\alpha = 1 + \cos 2\alpha$ and $2\sin\alpha\cos\alpha = \sin 2\alpha$, the first of the two given equations is successively equivalent to

$$\begin{aligned} x^2 + \tfrac{1}{2}y(1+\cos 2\alpha) &= \tfrac{1}{2}x\sin 2\alpha\,, \\ \text{and} \qquad x\sin 2\alpha - y\cos 2\alpha &= 2x^2 + y\,. \end{aligned}$$

Thus, the two given equations yield the following equivalent system of equations:

$$\begin{aligned} x\sin 2\alpha - y\cos 2\alpha &= 2x^2 + y\,, & (1) \\ x\cos 2\alpha + y\sin 2\alpha &= 0\,. & (2) \end{aligned}$$

Solving this system of equations for $\sin 2\alpha$ and $\cos 2\alpha$, we obtain

$$\sin 2\alpha = \frac{x(2x^2+y)}{x^2+y^2} \quad \text{and} \quad \cos 2\alpha = -\frac{y(2x^2+y)}{x^2+y^2}\,,$$

since x and y are non-zero. Squaring both equations and applying the Pythagorean Identity, $\sin^2\theta + \cos^2\theta = 1$, leads to

$$\frac{x^2(2x^2+y)^2}{(x^2+y^2)^2} + \frac{y^2(2x^2+y)^2}{(x^2+y^2)^2} = 1\,,$$

which simplifies to

$$\frac{(2x^2+y)^2}{x^2+y^2} = 1\,.$$

Hence, $4x^2 + 4y - 1 = 0$, where we have again used the fact that $x \neq 0$.

[*Ed:* We could have squared equations (1) and (2) to get

$$\begin{aligned} x^2\sin^2 2\alpha - 2xy\sin 2\alpha\cos 2\alpha + y^2\cos^2 2\alpha &= (2x^2+y)^2\,, \\ x^2\cos^2 2\alpha + 2xy\cos 2\alpha\sin 2\alpha + y^2\sin^2 2\alpha &= 0\,, \end{aligned}$$

and then add to get

$$x^2(\sin^2 2\alpha + \cos^2 2\alpha) + y^2(\sin^2 2\alpha + \cos^2 2\alpha) = (2x^2+y)^2\,,$$

or

$$x^2 + y^2 = (2x^2+y)^2\,,$$

which gives $4x^2 + 4y - 1 = 0$, as above.]

Also solved by ARKADY ALT, San Jose, CA, USA; HASAN DENKER, Istanbul, Turkey; SAMUEL GÓMEZ MORENO, Universidad de Jaén, Jaén, Spain; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; and BILLY SUANDITO, Palembang, Indonesia. There were four incorrect solutions submitted.

M284. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Prove that

$$\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{4}\right)+\tan^{-1}\left(\frac{1}{13}\right) \;=\; \frac{\pi}{4}\,.$$

Essentially the same solution by Mihály Bencze, Brasov, Romania; Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam; Hasan Denker, Istanbul, Turkey; José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain; Samuel Gómez Moreno, Universidad de Jaén, Jaén, Spain; Richard I. Hess, Rancho Palos Verdes, CA, USA; John G. Heuver, Grande Prairie, AB; Taichi Maekawa, Takatsuki City, Osaka, Japan; Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; Dragoljub Milošević, Pranjani, Serbia; Billy Suandito, Palembang, Indonesia; Daniel Tsai, student, Taipei American School, Taipei, Taiwan; Nick Wilson, student, Valley Catholic School, Beaverton, OR, USA; and Titu Zvonaru, Comăneşti, Romania.

Setting $a=\tan^{-1}\left(\frac{1}{2}\right)$, $b=\tan^{-1}\left(\frac{1}{4}\right)$, and $c=\tan^{-1}\left(\frac{1}{13}\right)$, we obtain $\tan a=\frac{1}{2}$, $\tan b=\frac{1}{4}$, and $\tan c=\frac{1}{13}$ with $a,\ b,\ c\in\left(0,\frac{\pi}{4}\right)$. Applying the identity $\tan(x+y)=\dfrac{\tan x+\tan y}{1-\tan x\tan y}$ twice, we obtain

$$\begin{aligned}
\tan\big((a+b)+c\big) &= \frac{\tan(a+b)+\tan c}{1-\tan(a+b)\tan c} = \frac{\dfrac{\tan a+\tan b}{1-\tan a\tan b}+\tan c}{1-\dfrac{\tan a+\tan b}{1-\tan a\tan b}\tan c} \\
&= \frac{\tan a+\tan b+\tan c-\tan a\tan b\tan c}{1-\tan a\tan b-\tan b\tan c-\tan a\tan c} \\
&= \frac{\frac{1}{2}+\frac{1}{4}+\frac{1}{13}-\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{13}}{1-\frac{1}{2}\cdot\frac{1}{4}-\frac{1}{4}\cdot\frac{1}{13}-\frac{1}{2}\cdot\frac{1}{13}} \;=\; 1\,.
\end{aligned}$$

Hence, $\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{4}\right)+\tan^{-1}\left(\frac{1}{13}\right)=a+b+c=\tan^{-1}(1)=\frac{\pi}{4}$.

Also solved by MIHÁLY BENCZE, Brasov, Romania (second solution); JOSÉ HERNÁNDEZ SANTIAGO, student, Universidad Tecnológica de la Mixteca, Oaxaca, Mexico; and TITU ZVONARU, Comăneşti, Romania (second solution).

M285. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let a, b, and c be strictly positive numbers such that $a+b+c\geq 3abc$. Prove that $a^2+b^2+c^2\geq 2abc$.

Solution by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam.

From the Arithmetic Mean–Geometric Mean Inequality, we have

$\frac{1}{3}(a+b+c) \geq \sqrt[3]{abc}$, or $(a+b+c)^3 \geq 27abc$; hence,

$$(a+b+c)^4 \;=\; (a+b+c)^3(a+b+c) \;\geq\; (27abc)(3abc) \;=\; 81a^2b^2c^2\,.$$

Thus, taking square roots, we get $(a+b+c)^2 \geq 9abc$, since a, b, and c are all positive. Next,

$$\begin{aligned} &a^2+b^2+c^2-\tfrac{1}{3}(a+b+c)^2 \\ &\qquad= a^2+b^2+c^2-\tfrac{1}{3}\big(a^2+b^2+c^2+2ab+2bc+2ac\big) \\ &\qquad= \tfrac{2}{3}\big(a^2+b^2+c^2-ab-bc-ac\big) \\ &\qquad= \tfrac{2}{3}\Big(\tfrac{1}{2}\big((a-b)^2+(b-c)^2+(a-c)^2\big)\Big) \;\geq\; 0\,. \end{aligned}$$

Therefore, $a^2+b^2+c^2 \geq \frac{1}{3}(a+b+c)^2 \geq \frac{1}{3}(9abc) = 3abc > 2abc$.

Also solved by ARKADY ALT, San Jose, CA, USA; HASAN DENKER, Istanbul, Turkey; SAMUEL GÓMEZ MORENO, Universidad de Jaén, Jaén, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; and VEDULA N. MURTY, Dover, PA, USA. Three incomplete solutions were also submitted.

Three of the solvers actually proved the stronger inequality attained in the solution above.

M286. *Proposed by K.R.S. Sastry, Bangalore, India.*

If $xy+yz+zx=1$, show that

(a) $$\left|\frac{x}{1+x^2}+\frac{y}{1+y^2}+\frac{z}{1+z^2}\right| \;=\; \frac{2}{\sqrt{(1+x^2)(1+y^2)(1+z^2)}};$$

(b) $$\frac{x}{1+x^2}+\frac{y}{1+y^2}+\frac{z}{1+z^2} \;=\; \frac{2}{x+y+z-xyz}.$$

Solution by Vedula N. Murty, Dover, PA, USA, modified by the editor.

If $xy+yz+zx=1$, then $1+x^2 = xy+yz+zx+x^2 = (x+y)(x+z)$. Similarly, $1+y^2 = (y+x)(y+z)$ and $1+z^2 = (z+x)(z+y)$. This yields $\sqrt{(1+x^2)(1+y^2)(1+z^2)} = \pm(x+y)(y+z)(z+x)$. We also note that

$$\begin{aligned} (x+y)(y+z)(z+x) &= (1+x^2)(y+z) \\ &= y+z+x(xy+xz) \\ &= y+z+x(1-yz)\,; \end{aligned}$$

hence, $(x+y)(x+z)(y+z) = x+y+z-xyz$. Therefore,

$$\begin{aligned} &\frac{x}{1+x^2}+\frac{y}{1+y^2}+\frac{z}{1+z^2} \\ &\qquad= \frac{x}{(x+y)(x+z)}+\frac{y}{(y+x)(y+z)}+\frac{z}{(z+x)(z+y)}\,. \end{aligned}$$

The right side of the above equation is equal to

$$\frac{x(y+z)+y(x+z)+z(x+y)}{(x+y)(y+z)(z+x)} = \frac{2}{(x+y)(y+z)(z+x)}\,.$$

Since $(x+y)(y+z)(z+x) = x+y+z-xyz$ from above, we have proved part (b).

Since $\sqrt{(1+x^2)(1+y^2)(1+z^2)} = \pm(x+y)(y+z)(z+x)$, we see that part (a) also holds.

Also solved by ARKADY ALT, San Jose, CA, USA; MIHÁLY BENCZE, Brasov, Romania; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; HASAN DENKER, Istanbul, Turkey; SAMUEL GÓMEZ MORENO, Universidad de Jaén, Jaén, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie, AB; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; D.M. MILOŠEVIĆ, Pranjani, Serbia; BILLY SUANDITO, Palembang, Indonesia (part (b) *only*)*; TITU ZVONARU, Comăneşti, Romania; and the proposer.*

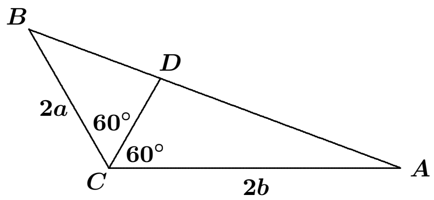
Part (a) *originally appeared without the absolute value signs on the left side. Malikić and Gómez Moreno both provided a counterexample to the equation as it originally appeared.*

M287. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Given two positive real numbers a and b, construct their harmonic mean with straightedge and compass.

Solution by Taichi Maekawa, Takatsuki City, Osaka, Japan.

Construction: As shown in the diagram, draw a triangle whose sides AB and AC have length $2a$ and $2b$, respectively, in such a way that $\angle ACB = 120°$ [*Ed.:* this is well known to be constructible with straightedge and compass]. Let D be the point of intersection of AB and the internal angle bisector of $\angle ACB$. Then the length of CD is the harmonic mean of a and b.



Proof: Since the area of $\triangle ACD$ plus the area of $\triangle BCD$ is equal to the area of $\triangle ABC$, we see that

$$\tfrac{1}{2}2a\cdot CD\cdot\sin 60° + \tfrac{1}{2}2b\cdot CD\cdot\sin 60° = \tfrac{1}{2}2a\cdot 2b\cdot\sin 120°\,.$$

Therefore, $CD = \dfrac{2ab}{a+b}$; hence, CD is the harmonic mean of a and b.

Also solved by MIHÁLY BENCZE, Brasov, Romania; HASAN DENKER, Istanbul, Turkey; SAMUEL GÓMEZ MORENO, Universidad de Jaén, Jaén, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie, AB; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; DANIEL TSAI, student, Taipei American School, Taipei, Taiwan; and TITU ZVONARU, Comăneşti, Romania. One incorrect solution was also submitted.

Interestingly enough, the proposer had an article in the issue previous to the one in which this proposal appeared [2007 : 17–18] *which showed a way to construct harmonic means. The method indicated there was similar to the above solution.*