MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

The Mayhem Editor is Ian VanderBurgh (University of Waterloo). The other staff members are Monika Khbeis (Ascension of Our Lord Secondary School, Mississauga), Eric Robert (Leo Hayes High School, Fredericton), and Larry Rice (University of Waterloo).

Mayhem Problems

Veuillez nous transmettre vos solutions aux problèmes du présent numéro avant le **premier juin 2008**. *Les solutions reçues après cette date ne seront prises en compte que s'il nous reste du temps avant la publication des solutions.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais.

La rédaction souhaite remercier Jean-Marc Terrier, de l'Université de Montréal, d'avoir traduit les problèmes.

M332. *Proposé par Dionne Bailey, Elsie Campbell, et Charles R. Diminnie, Angelo State University, San Angelo, TX, É-U.*

Le rayon et la longueur d'un cylindre circulaire droit fermé sont mesurés par des entiers. La valeur de son volume est quatre fois celle de sa surface totale (extrémités comprises). Trouver le plus petit volume possible pour ce cylindre.

M333. *Proposé par l'Équipe de Mayhem.*

Anne et Berthe jouent un jeu avec un tas de n allumettes. Elles jouent à tour de rôle et c'est Anne qui commence. Chacune doit enlever soit une, trois ou six allumettes. Celle qui enlève la dernière allumette a gagné. Pour quelles valeurs de n, entre 36 et 40, Berthe a-t-elle une stratégie gagnante ?

M334. *Proposé par l'Équipe de Mayhem.*

(a) Trouver tous les entiers x pour lesquels $\dfrac{x-3}{3x-2}$ est un entier.

(b) Trouver tous les entiers y pour lesquels $\dfrac{3y^3+3}{3y^2+y-2}$ est un entier.

M335. *Proposé par l'Équipe de Mayhem.*

Dans une suite de quatre nombres, le second est le double du premier. On a aussi que la somme du premier et du quatrième nombre est 9, la somme du deuxième et du troisième est 7, et la somme des carrés des quatre nombres est 78. Trouver toutes les suites ayant ces propriétés.

M336. *Proposé par l'Équipe de Mayhem.*

Un point réseau est un point (x, y) du plan dont les coordonnées x et y sont des entiers. Soit n un entier positif. Trouver le nombre de points réseau à l'intérieur de la région $|x| + |y| \leq n$.

M337. *Proposé par l'Équipe de Mayhem.*

Sur les côtés AB et CD d'un rectangle $ABCD$ avec $AD < AB$, on choisit les points F et E de sorte que $AFCE$ soit un losange.

(a) Si $AB = 16$ et $BC = 12$, déterminer EF.

(b) Si $AB = x$ et $BC = y$, déterminer EF en fonction de x et y.

. . . .

M332. *Proposed by Dionne Bailey, Elsie Campbell, and Charles R. Diminnie, Angelo State University, San Angelo, TX, USA.*

A closed right circular cylinder has an integer radius and an integer height. The numerical value of the volume is four times the numerical value of its total surface area (including its top and bottom). Determine the smallest possible volume for the cylinder.

M333. *Proposed by the Mayhem Staff.*

Anne and Brenda play a game which begins with a pile of n toothpicks. They alternate turns with Anne going first. On each player's turn, she must remove 1, 3, or 6 toothpicks from the pile. The player who removes the last toothpick wins the game. For which of the values of n from 36 to 40 inclusive does Brenda have a winning strategy?

M334. *Proposed by the Mayhem Staff.*

(a) Determine all integers x for which $\dfrac{x-3}{3x-2}$ is an integer.

(b) Determine all integers y for which $\dfrac{3y^3+3}{3y^2+y-2}$ is an integer.

M335. *Proposed by the Mayhem staff.*

In a sequence of four numbers, the second number is twice the first number. Also, the sum of the first and fourth numbers is 9, the sum of the second and third is 7, and the sum of the squares of the four numbers is 78. Determine all such sequences.

M336. *Proposed by the Mayhem Staff.*

A lattice point is a point (x, y) in the coordinate plane with each of x and y an integer. Suppose that n is a positive integer. Determine the number of lattice points inside the region $|x| + |y| \leq n$.

M337. *Proposed by the Mayhem Staff.*

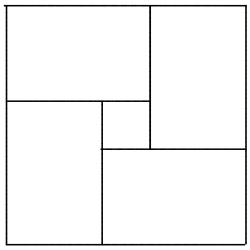
On sides AB and CD of rectangle $ABCD$ with $AD < AB$, points F and E are chosen so that $AFCE$ is a rhombus.

(a) If $AB = 16$ and $BC = 12$, determine EF.

(b) If $AB = x$ and $BC = y$, determine EF in terms of x and y.

Mayhem Solutions

M282. *Proposed by J. Walter Lynch, Athens, GA, USA.*



Four rectangles are arranged in a square pattern so that they enclose a smaller square. Let S be the area of the outer square and Q the area of the inner square. If $S/Q = 9 + 4\sqrt{5}$, determine the ratio of the sides of the rectangles.

Combination of solutions by Mihály Bencze, Brasov, Romania; Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam; Denise Cornwell, student, Angelo State University, San Angelo, TX, USA; Hasan Denker, Istanbul, Turkey; Richard I. Hess, Rancho Palos Verdes, CA, USA; John G. Heuver, Grande Prairie, AB; Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; Dragoljub Milošević, Pranjani, Serbia; Billy Suandito, Palembang, Indonesia; Nick Wilson, student, Valley Catholic School, Beaverton, OR, USA; and Titu Zvonaru, Comănești, Romania.

Let x and y represent the sides of one of the rectangles such that $x > y$. Then the outer square has side length $x + y$ and the inner square has side length $x - y$. The given ratio $S/Q = 9 + 4\sqrt{5}$ can then be represented as

$$\frac{(x+y)^2}{(x-y)^2} = \left(2 + \sqrt{5}\,\right)^2 .$$

Since $x > y$, we successively obtain the equivalent equations

$$\begin{aligned} \frac{x+y}{x-y} &= 2 + \sqrt{5}\,, \\ x + y &= \left(2 + \sqrt{5}\,\right)x - \left(2 + \sqrt{5}\,\right)y\,, \end{aligned}$$

and $x+\sqrt{5}x = 3y+\sqrt{5}y$. The ratio of the sides of the rectangle then yields

$$\frac{x}{y} = \frac{(3+\sqrt{5})(1-\sqrt{5})}{(1+\sqrt{5})(1-\sqrt{5})} = \frac{\sqrt{5}+1}{2},$$

which is the Golden Ratio! We can also compute $\frac{y}{x} = \dfrac{\sqrt{5}-1}{2}$.

There was one incorrect solution submitted.

M283. *Proposed by Neven Jurič, Zagreb, Croatia.*

Determine the relationship between x and y if

$$x^2 + y\cos^2\alpha = x\sin\alpha\cos\alpha \qquad \text{and} \qquad x\cos 2\alpha + y\sin 2\alpha = 0\,.$$

(Assume that both x and y are non-zero.)

Essentially the same solution by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam; and Titu Zvonaru, Cománeşti, Romania.

Using the identities $2\cos^2\alpha = 1+\cos 2\alpha$ and $2\sin\alpha\cos\alpha = \sin 2\alpha$, the first of the two given equations is successively equivalent to

$$\begin{aligned} x^2 + \tfrac{1}{2}y(1+\cos 2\alpha) &= \tfrac{1}{2}x\sin 2\alpha\,, \\ \text{and} \qquad x\sin 2\alpha - y\cos 2\alpha &= 2x^2 + y\,. \end{aligned}$$

Thus, the two given equations yield the following equivalent system of equations:

$$\begin{aligned} x\sin 2\alpha - y\cos 2\alpha &= 2x^2 + y\,, & (1) \\ x\cos 2\alpha + y\sin 2\alpha &= 0\,. & (2) \end{aligned}$$

Solving this system of equations for $\sin 2\alpha$ and $\cos 2\alpha$, we obtain

$$\sin 2\alpha = \frac{x(2x^2+y)}{x^2+y^2} \quad \text{and} \quad \cos 2\alpha = -\frac{y(2x^2+y)}{x^2+y^2}\,,$$

since x and y are non-zero. Squaring both equations and applying the Pythagorean Identity, $\sin^2\theta + \cos^2\theta = 1$, leads to

$$\frac{x^2(2x^2+y)^2}{(x^2+y^2)^2} + \frac{y^2(2x^2+y)^2}{(x^2+y^2)^2} = 1\,,$$

which simplifies to

$$\frac{(2x^2+y)^2}{x^2+y^2} = 1\,.$$

Hence, $4x^2+4y-1=0$, where we have again used the fact that $x \neq 0$.

[*Ed:* We could have squared equations (1) and (2) to get

$$\begin{aligned} x^2\sin^2 2\alpha - 2xy\sin 2\alpha\cos 2\alpha + y^2\cos^2 2\alpha &= (2x^2+y)^2\,, \\ x^2\cos^2 2\alpha + 2xy\cos 2\alpha\sin 2\alpha + y^2\sin^2 2\alpha &= 0\,, \end{aligned}$$

and then add to get

$$x^2(\sin^2 2\alpha + \cos^2 2\alpha) + y^2(\sin^2 2\alpha + \cos^2 2\alpha) \;=\; (2x^2 + y)^2\,,$$

or

$$x^2 + y^2 \;=\; (2x^2 + y)^2\,,$$

which gives $4x^2 + 4y - 1 = 0$, as above.]

Also solved by ARKADY ALT, San Jose, CA, USA; HASAN DENKER, Istanbul, Turkey; SAMUEL GÓMEZ MORENO, Universidad de Jaén, Jaén, Spain; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; and BILLY SUANDITO, Palembang, Indonesia. There were four incorrect solutions submitted.

M284. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Prove that

$$\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{1}{13}\right) \;=\; \frac{\pi}{4}\,.$$

Essentially the same solution by Mihály Bencze, Brasov, Romania; Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam; Hasan Denker, Istanbul, Turkey; José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain; Samuel Gómez Moreno, Universidad de Jaén, Jaén, Spain; Richard I. Hess, Rancho Palos Verdes, CA, USA; John G. Heuver, Grande Prairie, AB; Taichi Maekawa, Takatsuki City, Osaka, Japan; Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; Dragoljub Milošević, Pranjani, Serbia; Billy Suandito, Palembang, Indonesia; Daniel Tsai, student, Taipei American School, Taipei, Taiwan; Nick Wilson, student, Valley Catholic School, Beaverton, OR, USA; and Titu Zvonaru, Comănești, Romania.

Setting $a = \tan^{-1}\left(\frac{1}{2}\right)$, $b = \tan^{-1}\left(\frac{1}{4}\right)$, and $c = \tan^{-1}\left(\frac{1}{13}\right)$, we obtain $\tan a = \frac{1}{2}$, $\tan b = \frac{1}{4}$, and $\tan c = \frac{1}{13}$ with a, b, $c \in \left(0, \frac{\pi}{4}\right)$. Applying the identity $\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$ twice, we obtain

$$\begin{aligned}
\tan\big((a+b)+c\big) \;&=\; \frac{\tan(a+b) + \tan c}{1 - \tan(a+b)\tan c} \;=\; \frac{\dfrac{\tan a + \tan b}{1 - \tan a \tan b} + \tan c}{1 - \dfrac{\tan a + \tan b}{1 - \tan a \tan b}\tan c} \\
&=\; \frac{\tan a + \tan b + \tan c - \tan a \tan b \tan c}{1 - \tan a \tan b - \tan b \tan c - \tan a \tan c} \\
&=\; \frac{\frac{1}{2} + \frac{1}{4} + \frac{1}{13} - \frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{13}}{1 - \frac{1}{2}\cdot\frac{1}{4} - \frac{1}{4}\cdot\frac{1}{13} - \frac{1}{2}\cdot\frac{1}{13}} \;=\; 1\,.
\end{aligned}$$

Hence, $\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{1}{13}\right) = a + b + c = \tan^{-1}(1) = \frac{\pi}{4}$.

Also solved by MIHÁLY BENCZE, Brasov, Romania (second solution); JOSÉ HERNÁNDEZ SANTIAGO, student, Universidad Tecnológica de la Mixteca, Oaxaca, Mexico; and TITU ZVONARU, Comănești, Romania (second solution).

M285. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let a, b, and c be strictly positive numbers such that $a+b+c \geq 3abc$. Prove that $a^2+b^2+c^2 \geq 2abc$.

Solution by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam.

From the Arithmetic Mean–Geometric Mean Inequality, we have $\frac{1}{3}(a+b+c) \geq \sqrt[3]{abc}$, or $(a+b+c)^3 \geq 27abc$; hence,

$$(a+b+c)^4 \;=\; (a+b+c)^3(a+b+c) \;\geq\; (27abc)(3abc) \;=\; 81a^2b^2c^2\,.$$

Thus, taking square roots, we get $(a+b+c)^2 \geq 9abc$, since a, b, and c are all positive. Next,

$$\begin{aligned} a^2+b^2+c^2-\tfrac{1}{3}(a+b+c)^2 & \\ &= a^2+b^2+c^2-\tfrac{1}{3}\big(a^2+b^2+c^2+2ab+2bc+2ac\big) \\ &= \tfrac{2}{3}\big(a^2+b^2+c^2-ab-bc-ac\big) \\ &= \tfrac{2}{3}\Big(\tfrac{1}{2}\big((a-b)^2+(b-c)^2+(a-c)^2\big)\Big) \;\geq\; 0\,. \end{aligned}$$

Therefore, $a^2+b^2+c^2 \geq \frac{1}{3}(a+b+c)^2 \geq \frac{1}{3}(9abc) = 3abc > 2abc$.

Also solved by ARKADY ALT, San Jose, CA, USA; HASAN DENKER, Istanbul, Turkey; SAMUEL GÓMEZ MORENO, Universidad de Jaén, Jaén, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; and VEDULA N. MURTY, Dover, PA, USA. Three incomplete solutions were also submitted.

Three of the solvers actually proved the stronger inequality attained in the solution above.

M286. *Proposed by K.R.S. Sastry, Bangalore, India.*

If $xy+yz+zx=1$, show that

(a) $$\left|\frac{x}{1+x^2}+\frac{y}{1+y^2}+\frac{z}{1+z^2}\right| \;=\; \frac{2}{\sqrt{(1+x^2)(1+y^2)(1+z^2)}};$$

(b) $$\frac{x}{1+x^2}+\frac{y}{1+y^2}+\frac{z}{1+z^2} \;=\; \frac{2}{x+y+z-xyz}.$$

Solution by Vedula N. Murty, Dover, PA, USA, modified by the editor.

If $xy+yz+zx=1$, then $1+x^2 = xy+yz+zx+x^2 = (x+y)(x+z)$. Similarly, $1+y^2=(y+x)(y+z)$ and $1+z^2=(z+x)(z+y)$. This yields $\sqrt{(1+x^2)(1+y^2)(1+z^2)} = \pm(x+y)(y+z)(z+x)$. We also note that

$$\begin{aligned} (x+y)(y+z)(z+x) &= (1+x^2)(y+z) \\ &= y+z+x(xy+xz) \\ &= y+z+x(1-yz)\,; \end{aligned}$$

hence, $(x+y)(x+z)(y+z) = x+y+z-xyz$. Therefore,

$$\frac{x}{1+x^2}+\frac{y}{1+y^2}+\frac{z}{1+z^2} = \frac{x}{(x+y)(x+z)}+\frac{y}{(y+x)(y+z)}+\frac{z}{(z+x)(z+y)}\,.$$

The right side of the above equation is equal to

$$\frac{x(y+z)+y(x+z)+z(x+y)}{(x+y)(y+z)(z+x)} = \frac{2}{(x+y)(y+z)(z+x)}\,.$$

Since $(x+y)(y+z)(z+x) = x+y+z-xyz$ from above, we have proved part (b).

Since $\sqrt{(1+x^2)(1+y^2)(1+z^2)} = \pm(x+y)(y+z)(z+x)$, we see that part (a) also holds.

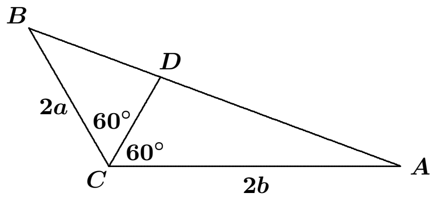
Also solved by ARKADY ALT, San Jose, CA, USA; MIHÁLY BENCZE, Brasov, Romania; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; HASAN DENKER, Istanbul, Turkey; SAMUEL GÓMEZ MORENO, Universidad de Jaén, Jaén, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie, AB; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; D.M. MILOŠEVIĆ, Pranjani, Serbia; BILLY SUANDITO, Palembang, Indonesia (part (b) *only*)*; TITU ZVONARU, Comănești, Romania; and the proposer.*

Part (a) *originally appeared without the absolute value signs on the left side. Malikić and Gómez Moreno both provided a counterexample to the equation as it originally appeared.*

M287. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Given two positive real numbers a and b, construct their harmonic mean with straightedge and compass.

Solution by Taichi Maekawa, Takatsuki City, Osaka, Japan.



Construction: As shown in the diagram, draw a triangle whose sides AB and AC have length $2a$ and $2b$, respectively, in such a way that $\angle ACB = 120^\circ$ [*Ed.:* this is well known to be constructible with straightedge and compass]. Let D be the point of intersection of AB and the internal angle bisector of $\angle ACB$. Then the length of CD is the harmonic mean of a and b.

Proof: Since the area of $\triangle ACD$ plus the area of $\triangle BCD$ is equal to the area of $\triangle ABC$, we see that

$$\tfrac{1}{2}2a\cdot CD\cdot\sin 60^\circ + \tfrac{1}{2}2b\cdot CD\cdot\sin 60^\circ = \tfrac{1}{2}2a\cdot 2b\cdot\sin 120^\circ\,.$$

Therefore, $CD = \dfrac{2ab}{a+b}$; hence, CD is the harmonic mean of a and b.

Also solved by MIHÁLY BENCZE, Brasov, Romania; HASAN DENKER, Istanbul, Turkey; SAMUEL GÓMEZ MORENO, Universidad de Jaén, Jaén, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie, AB; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; DANIEL TSAI, student, Taipei American School, Taipei, Taiwan; and TITU ZVONARU, Comănești, Romania. One incorrect solution was also submitted.

Interestingly enough, the proposer had an article in the issue previous to the one in which this proposal appeared [2007 : 17–18] *which showed a way to construct harmonic means. The method indicated there was similar to the above solution.*

Problem of the Month

Ian VanderBurgh

I hope that you've been doing some strengthening exercises recently—we're in for some heavy (and not-so-heavy) lifting this month.

Problem (2006 Cayley Contest)

Quincy and Celine have to move 16 small boxes and 10 large boxes. The chart below indicates the time that each person takes to move each type of box.

	Celine	Quincy
small box	2 min.	3 min.
large box	6 min.	5 min.

They start moving the boxes at 9:00 a.m. What is the earliest time at which they can be finished moving all of the boxes?

(A) 9:41 a.m. (B) 9:42 a.m. (C) 9:43 a.m. (D) 9:44 a.m. (E) 9:45 a.m.

First, some ground rules. No throwing boxes. No stacking boxes. You are not allowed to put one box inside another. (For some reason, when I do this type of problem with a group of students, they always want to try these sneaky things!)

While it makes sense to get out a piece of scrap paper and do a bit of investigation, let's look before we leap. Since Celine is faster on smaller boxes and Quincy is faster on large boxes, let's try having Celine move all of the small boxes (taking $16 \times 2 = 32$ minutes) and having Quincy move all of the large boxes (taking $10 \times 5 = 50$ minutes). Thus, they would finish at 9:50 a.m. Can you see why this minimizes the *total working time*? Interestingly, this does not necessarily mean that it gives the earliest end time using the given rules.

While this division of labour is fast in one sense, this does not seem optimal in others. Celine moves boxes for 32 minutes and then sits around for 18 more minutes while Quincy keeps moving boxes. (Does this remind you of what happens sometimes when you're doing chores at home?)

How could we improve on this time? What if Celine pitches in by also moving 1 large box? Quincy's time is reduced to $9 \times 5 = 45$ minutes and Celine's time increases to $16 \times 2 + 6 = 38$ minutes. We are now at 9:45 a.m. as the finishing time. This is definitely better. But is it the best possible?

Take a few minutes to see if you can do better than 9:45 a.m. While you're doing this, I'll try to distract you a bit by pointing out that there might be some unrealistic things in this problem. At the same time, though, this problem does seem to be more of a "real life" problem than some of the ones that we encounter. (I will say more about this at the end.)

Did you do better than 9:45 a.m.? In fact, 9:43 a.m. is possible. Suppose that Celine moves 15 small boxes and 2 large boxes. This takes her $15 \times 2 + 2 \times 6 = 42$ minutes. Quincy will thus move 1 small box and 8 large boxes, which will take $1 \times 3 + 8 \times 5 = 43$ minutes. Overall, they will finish at 9:43 a.m. (and Celine won't even have time to get much of a break in her 1 minute of spare time at the end).

After some more trial and error, you will probably get frustrated like I did when you can't get a shorter time than 43 minutes. So it seems that 9:43 a.m. is the earliest possible finish time. Let's prove this.

Solution: If Celine moves 15 small boxes and 2 large boxes, it takes her 42 minutes. If Quincy moves 8 large boxes and 1 small box, it takes him 43 minutes. In this configuration, the boxes are fully moved at 9:43 a.m.

Suppose in general that Celine moves x small and y large boxes. Thus, Quincy moves $16 - x$ and $10 - y$ small boxes. In this case, Celine takes $2x + 6y$ minutes and Quincy takes $3(16 - x) + 5(10 - y) = 98 - 3x - 5y$ minutes.

Suppose now that they finish the job at 9:42 a.m. or earlier. (We'll show that this cannot actually happen.) In this case, each works for at most 42 minutes, so the total time that they work would have to be at most 84 minutes. From the information above, their total time is

$$(2x + 6y) + (98 - 3x - 5y) \;=\; 98 + y - x\,.$$

If their total time is at most 84 minutes, then $98 + y - x \leq 84$; that is, $x - y \geq 14$. But x and y are non-negative integers with $0 \leq x \leq 16$ and $0 \leq y \leq 10$. Therefore, $x - y \leq 16$ (because x is at most 16). Hence, we have $14 \leq x - y \leq 16$.

The possible pairs (x, y) that satisfy these restrictions are $(16, 0)$, $(16, 1)$, $(16, 2)$, $(15, 0)$, $(15, 1)$, and $(14, 0)$. Let's make a table of the length of time that each of Celine and Quincy takes for these values of x and y:

x	y	$2x + 6y$	$98 - 3x - 5y$	Finish at
16	0	32	50	9:50 a.m.
16	1	38	45	9:45 a.m.
16	2	44	40	9:44 a.m.
15	0	30	53	9:53 a.m.
15	1	36	48	9:48 a.m.
14	0	28	56	9:56 a.m.

Since these are all of the possible configurations in which Celine and Quincy could possibly finish at or before 9:42 a.m., and in none of them do they actually finish at 9:42 a.m. or before, we conclude that they cannot finish at or before 9:42 a.m.

Summarizing what we have seen, Celine and Quincy can finish at 9:43 a.m., and cannot finish any earlier. Hence, 9:43 a.m. is the earliest possible finishing time.

Looking back, we solved this problem by looking at the combined time and showing that we couldn't make the pair of eager workers finish earlier than 9:43 a.m.

We could have tried looking at the conditions $2x + 6y \leq 42$ and $98 - 3x - 5y \leq 42$ (or $3x + 5y \geq 56$) to show that they cannot be satisfied at the same time. We could approach this algebraically or even graphically (showing that there isn't a point in this region with non-negative integer coordinates).

This problem can actually be solved in "easier" ways using more advanced techniques from "linear programming". However, it is important from time to time to try to solve these problems using more elementary techniques.

I want to return briefly to the problem and the "real life" aspect of it. Suppose that we change the problem to the following:

> The Dunkley Piano Company makes small pianos and large pianos in their factories in Caracas and Quito. The chart below indicates the time that each factory takes to make each type of piano.
>
> | | Caracas | Quito |
> |---|---|---|
> | small piano | 2 days | 3 days |
> | large piano | 6 days | 5 days |
>
> They start making the pianos on March 1. What is the earliest date on which they can be finished making all of the pianos?

This is exactly the same problem, and so may not appear to be that interesting. But you can quite easily see how applicable this type of problem could be in business and industry. So math (and even math contest problems) can be useful!