Sharpening the Hadwiger-Finsler Inequality

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In memory of Alexandru Lupaş

1 Introduction and Preliminaries.

The Hadwiger–Finsler Inequality is known in the literature as a generalization of the following:

Theorem 1 In any triangle ABC with side lengths a, b, c, and area S, the following inequality holds:

$$a^2+b^2+c^2 \ \geq\ 4S\sqrt{3}\,.$$

This inequality is due to Weitzenböck [1] but also appeared in the International Mathematical Olympiad in 1961. In [5], one can find eleven proofs. In fact, in any triangle ABC the following sequence of inequalities is valid:

$$\begin{aligned} a^2+b^2+c^2 &\geq ab+bc+ca \ \geq\ a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab} \\ &\geq 3\sqrt[3]{a^2b^2c^2} \ \geq\ 4S\sqrt{3}\,. \end{aligned}$$

In 1937, Finsler and Hadwiger found a stronger version [2]:

Theorem 2 In any triangle ABC with side lengths a, b, c, and area S, the following inequality holds:

$$a^2+b^2+c^2 \ \geq\ 4S\sqrt{3}+(a-b)^2+(b-c)^2+(c-a)^2\,.$$

Now, we give an algebraic inequality due to Schur (See [3], for example), namely

Theorem 3 For non-negative x, y, z, and positive t, we have

$$x^t(x-y)(x-z)+y^t(y-x)(y-z)+z^t(z-y)(z-x) \ \geq\ 0\,.$$

When $t=1$ (the most common case), this is successively equivalent to:

$$\begin{aligned} x^3+y^3+z^3+3xyz &\geq xy(x+y)+yz(y+z)+zx(z+x)\,, \\ x^3+y^3+z^3+6xyz &\geq (x+y+z)(xy+yz+zx)\,. \end{aligned}$$

Since

$$x^3+y^3+z^3-3xyz \ =\ (x+y+z)(x^2+y^2+z^2-xy-yz-zx)\,,$$

one can easily deduce that

$$x^2+y^2+z^2+\frac{9xyz}{x+y+z} \ \geq\ 2(xy+yz+zx)\,.$$

We let $m = 1/x$, $n = 1/y$, and $p = 1/z$ to get the equivalent form below:

Theorem 4 For any positive reals m, n, and p, we have

$$\frac{mn}{p} + \frac{np}{m} + \frac{mp}{n} + \frac{9mnp}{mn+np+mp} \;\geq\; 2(m+n+p)\,.$$

2 Main result.

Our refinement of the Hadwiger–Finsler Inequality is as follows.

Theorem 5 In any triangle ABC with side lengths a, b, c, area S, inradius r, and circumradius R, the following inequality is valid:

$$a^2+b^2+c^2 \;\geq\; 4S\sqrt{3+\frac{4(R-2r)}{4R+r}} + (a-b)^2+(b-c)^2+(c-a)^2\,.$$

Proof: In Theorem 4, we set $m = \frac{1}{2}(b+c-a)$, $n = \frac{1}{2}(c+a-b)$, and $p = \frac{1}{2}(a+b-c)$. This yields

$$\sum_{\text{cyclic}} \frac{(b+c-a)(c+a-b)}{(a+b-c)} + \frac{9(b+c-a)(c+a-b)(a+b-c)}{\sum_{\text{cyclic}}(b+c-a)(c+a-b)} \;\geq\; 2(a+b+c)\,.$$

Let s be the semiperimeter. Since

$$ab+bc+ca \;=\; s^2+r^2+4Rr \tag{2}$$

$$\text{and}\quad a^2+b^2+c^2 \;=\; 2(s^2-r^2-4Rr)\,, \tag{3}$$

we deduce that

$$\sum_{\text{cyclic}}(b+c-a)(c+a-b) \;=\; 4r(4R+r)\,.$$

On the other hand, we have $(b+c-a)(c+a-b)(a+b-c) = 8sr^2$ by Heron's Formula; hence, our inequality is successively equivalent to

$$\sum_{\text{cyclic}} \frac{(b+c-a)(c+a-b)}{(a+b-c)} + \frac{18sr}{4R+r} \;\geq\; 4s\,;$$

$$\sum_{\text{cyclic}} \frac{(s-a)(s-b)}{(s-c)} + \frac{9sr}{4R+r} \;\geq\; 2s\,;$$

$$\sum_{\text{cyclic}} (s-a)^2(s-b)^2 + \frac{9s^2r^3}{4R+r} \;\geq\; 2s^2r^2\,.$$

Now, according to the identity

$$\sum_{\text{cyclic}} (s-a)^2(s-b)^2 \;=\; \Bigg(\sum_{\text{cyclic}}(s-a)(s-b)\Bigg)^2 - 2s^2r^2\,,$$

we have

$$\Big(\sum_{\text{cyclic}}(s-a)(s-b)\Big)^2 - 2s^2r^2 + \frac{9s^2r^3}{4R+r} \;\geq\; 2s^2r^2\,.$$

Since $\sum\limits_{\text{cyclic}}(s-a)(s-b) = r(4R+r)$, it follows that

$$r^2(4R+r)^2 + \frac{9s^2r^3}{4R+r} \;\geq\; 4s^2r^2\,,$$

which can be rewritten as

$$\Big(\frac{4R+r}{s}\Big)^2 + \frac{9r}{4R+r} \;\geq\; 4\,.$$

From the identities (2) and (3) we deduce that

$$\frac{4R+r}{s} \;=\; \frac{2(ab+bc+ca)-(a^2+b^2+c^2)}{4S}\,,$$

so our final succession of inequalities is

$$\Big(\frac{2(ab+bc+ca)-(a^2+b^2+c^2)}{4S}\Big)^2 \;\geq\; 4 - \frac{9r}{4R+r}\,,$$

$$\Big(\frac{(a^2+b^2+c^2)-((a-b)^2+(b-c)^2+(c-a)^2)}{4S}\Big)^2 \;\geq\; 3 + \frac{4(R-2r)}{4R+r}\,,$$

$$a^2+b^2+c^2 \;\geq\; 4S\sqrt{3+\frac{4(R-2r)}{4R+r}} + (a-b)^2+(b-c)^2+(c-a)^2\,,$$

the last of which is the desired refinement. ∎

We remark that by using Euler's Inequality, $R \geq 2r$, we get Theorem 2.

3 Applications.

In this section, we give some basic applications of our refinement of the Hadwiger–Finsler Inequality. We begin with

Problem 1. In any triangle ABC with sides of lengths a, b, c and with exradii r_a, r_b, r_c, prove that

$$\frac{a}{r_a} + \frac{b}{r_b} + \frac{c}{r_c} \;\geq\; 2\sqrt{3+\frac{4(R-2r)}{4R+r}}\,.$$

Solution: From the well-known relation $r_a = S/(s-a)$ and its analogues, the inequality is equivalent to

$$\begin{aligned}\frac{a}{r_a} + \frac{b}{r_b} + \frac{c}{r_c} \;&=\; \frac{2(ab+bc+ca)-(a^2+b^2+c^2)}{2S}\\ &\geq\; 2\sqrt{3+\frac{4(R-2r)}{4R+r}}\,,\end{aligned}$$

where the last inequality follows from Theorem 5. ∎

Problem 2. In any triangle ABC with sides of lengths a, b, and c, prove that

$$\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2} \;\geq\; \sqrt{3+\frac{4(R-2r)}{4R+r}}\,.$$

Solution: From the cosine law we get $a^2 = b^2+c^2-2bc\cos A$. Since $S=\frac{1}{2}bc\sin A$, it follows that

$$a^2 \;=\; (b-c)^2+4S\cdot\frac{1-\cos A}{\sin A}\,.$$

On the other hand, by the trigonometric formulae $1-\cos A = 2\sin^2\frac{A}{2}$ and $\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}$, we get

$$a^2 \;=\; (b-c)^2+4S\tan\frac{A}{2}\,.$$

Doing the same for all sides of the triangle ABC and adding up we obtain

$$\begin{aligned} a^2+b^2+c^2 \;&=\; (a-b)^2+(b-c)^2+(c-a)^2 \\ &\qquad +4S\left(\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\right)\,. \end{aligned}$$

Now the inequality follows from Theorem 5. ∎

The following are left as exercises:

Problem 3. In any triangle ABC with sides of lengths a, b, c and with corresponding exradii and altitudes r_a, r_b, r_c and h_a, h_b, h_c, prove that

$$\frac{1}{h_ar_a}+\frac{1}{h_br_b}+\frac{1}{h_cr_c} \;\geq\; \frac{1}{S}\sqrt{3+\frac{4(R-2r)}{4R+r}}\,.$$

Problem 4. In any triangle ABC with sides of lengths a, b, and c, prove that

$$\frac{1}{b+c-a}+\frac{1}{c+a-b}+\frac{1}{c+a-b} \;\geq\; \frac{1}{2r}\sqrt{4-\frac{9r}{4R+r}}\,.$$

Problem 5. In any triangle ABC with sides of lengths a, b, and c, prove that

$$\frac{a^2}{b+c-a}+\frac{b^2}{c+a-b}+\frac{c^2}{a+b-c} \;\geq\; 3R\sqrt{4-\frac{9r}{4R+r}}\,.$$

[We note that by combining the above with Euler's Inequality, $R\geq 2r$, we obtain the weaker result

$$\frac{a^2}{b+c-a}+\frac{b^2}{c+a-b}+\frac{c^2}{a+b-c} \;\geq\; 3R\sqrt{3}\,,$$

which represents an old proposal of Laurenţiu Panaitopol at the Romanian IMO Team Selection Test, held in 1990.]

Problem 6. In any triangle ABC with sides of lengths a, b, and c, and with corresponding exradii r_a, r_b, and r_c, prove that

$$\frac{r_a}{a} + \frac{r_b}{b} + \frac{r_c}{c} \geq \frac{s(5R - r)}{R(4R + r)} .$$

Problem 7. In any triangle ABC with sides of lengths a, b, and c, prove that

$$\frac{1}{b + c - a} + \frac{1}{c + a - b} + \frac{1}{c + a - b} \geq \frac{1}{2r}\sqrt{4 - \frac{9r}{4R + r}} .$$

Problem 8. In any triangle ABC with sides of lengths a, b, and c, prove that

$$\frac{1}{a(b + c - a)} + \frac{1}{b(c + a - b)} + \frac{1}{c(a + b - c)} \geq \frac{r}{8R}\left(5 - \frac{9r}{4R + r}\right) .$$

Problem 9. In any triangle ABC with sides of lengths a, b, and c, prove that

$$\frac{1}{(b + c - a)^2} + \frac{1}{(c + a - b)^2} + \frac{1}{(a + b - c)^2} \geq \frac{1}{r^2}\left(\frac{1}{2} - \frac{9r}{4(4R + r)}\right) .$$

References

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