THE OLYMPIAD CORNER

No. 268

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We begin this number of the *Corner* with the six problems of the Estonian IMO Team Selection Contest 2004/2005. My thanks go to Felix Recio, Canadian Team Leader to the IMO in Mexico, for collecting them for our use.

ESTONIAN IMO TEAM SELECTION CONTEST 2004–2005

First Day

1. In a plane, a line l and two circles c_1 and c_2 of different radii are given such that l touches both circles at point P. Point $M \neq P$ on l is chosen so that the angle Q_1MQ_2 is as large as possible, where Q_1 and Q_2 are the tangency points of the tangent lines drawn from M to c_1 and c_2, respectively, differing from l. Find $\angle PMQ_1 + \angle PMQ_2$.

2. The planet Automory has infinitely many inhabitants. Each Automorian loves exactly one Automorian and honours exactly one Automorian. Suppose that

(a) each Automorian is loved by some Automorian;

(b) if Automorian A loves Automorian B, then all Automorians honouring A love B;

(c) if Automorian A honours Automorian B, then all Automorians loving A honour B.

Does each Automorian then honour and love the same Automorian?

3. Find all pairs (x, y) of positive integers satisfying $(x + y)^x = x^y$.

Second Day

4. Find all pairs (a, b) of real numbers such that all roots of the polynomials $6x^2 - 24x - 4a$ and $x^3 + ax^2 + bx - 8$ are non-negative real numbers.

5. On a horizontal line, 2005 points are marked, each of which is either white or black. For each point, one finds the sum of the number of white points on the right of it and the number of black points on the left of it. Among the 2005 sums, exactly one number occurs an odd number of times. Find all possible values of this number.

6. In a plane, a line l and a circle c do not intersect, and the diameter AB of c is perpendicular to l, with B nearer to l than A. Let C be a point on c different from A and B. Line AC intersects l at point D, and E is the point of tangency of a line drawn from D to c such that E lies on the same side of AC as B. Line EB intersects l at point F, and line FA intersects c a second time at point G. Prove that the reflection of G in AB lies on FC.

Next, we give the four problems of the Trentième Olympiad Mathématique Belge Maxi Finale 2005. Thanks again to to Felix Recio, Canadian Team Leader to the IMO in Mexico, for collecting them for our use.

TRENTIÈME OLYMPIAD MATHÉMATIQUE BELGE
Maxi Finale
Mercredi 20 Avril 2005

1. Dans l'expression

$$(x_1 + x_2 + \cdots + x_n)^2 = x_1^2 + x_2^2 + \cdots + x_n^2 + 2x_1x_2 + 2x_1x_3 + \cdots + 2x_1x_n + 2x_2x_3 + \cdots + 2x_{n-1}x_n$$

les nombres réels non nuls x_1, x_2, x_3, $\ldots$, x_n ne sont pas tous positifs. Existe-t-il des valeurs de ces nombres réels qui rendent le nombre de doubles produits positifs égal au nombre de doubles produits négatifs

(a) si $n = 4$? (b) si $n = 2005$?

Donner une condition nécessaire et suffisante sur n pour que le nombre de doubles produits positifs soit égal au nombre de doubles produits négatifs.

2. Dans l'espace de dimension 3, existe-t-il deux points P et Q à coordonnées rationnelles tels que $|PQ| = \sqrt{7}$?

3. Dans le triangle ABC, les droites AE et CD sont les bissectrices intérieures des angles $\angle BAC$ et $\angle ACB$ respectivement; E appartient à BC et D appartient à AB. Pour quelles amplitudes de l'angle $\angle ABC$ a-t-on certainement

(a) $|AD| + |EC| = |AC|$? (b) $|AD| + |EC| > |AC|$?
(c) $|AD| + |EC| < |AC|$?

4. La suite infinie 1, 2, 3, 4, 0, 9, 6, 9, 4, 8, 7, 8, $\ldots$, ne comprend que des nombres appartenant à l'ensemble $\{0, 1, 2, \ldots, 9\}$ et est construite de la manière suivant: après le quatrième nombre, chaque nouveau nombre est formé du chiffre des unités de la somme des quatre nombres précédents.

(a) Les nombres 2, 0, 0, 5 apparaissent-ils de manière consécutive dans cette suite?

(b) Les nombres 1, 2, 3, 4 apparaissent-ils une deuxième fois de manière consécutive dans cette suite?

Next, we give the six problems of the 2005 Vietnam Mathematical Olympiad. Thanks again go to Felix Recio for collecting them for our use.

2005 VIETNAM MATHEMATICAL OLYMPIAD
Day 1 (Time: 3 hours)

1. Find the smallest and largest values of the expression $P = x + y$, where x and y are real numbers satisfying $x - 3\sqrt{x+1} = 3\sqrt{y+2} - y$.

2. In a plane, let Γ be a circle with centre O and radius R, and let A and B be points on Γ such that AB is not a diameter. Let C be a point on Γ distinct from A and B. Construct the circle Γ_1 through A and tangent to BC at C, and construct the circle Γ_2 through B and tangent to AC at C. Let the circles Γ_1 and Γ_2 intersect again at D, distinct from C.

Prove that $CD \leq R$, and that the line CD passes through a fixed point when C moves on Γ in such a way that C does not coincide with A and B.

3. Let A_i, $1 \leq i \leq 8$, be the vertices of an octagon in a plane, such that no three of its diagonals are concurrent. Any point of intersection of any two diagonals of the octagon is called a cross. A subquadrilateral of the octagon is any convex quadrilateral whose vertices are also vertices of the octagon.

Given a colouring of a subset of the crosses, let $s(i,k)$, $i \neq k$, be the number of subquadrilaterals having A_i and A_k as vertices and having a coloured cross as the point of intersection of their diagonals. Find the least positive integer n such that one can colour n crosses so that the values $s(i,k)$ are all equal.

Day 2 (Time: 3 hours)

4. Find all real-valued functions f defined on $\mathbb{R}$ that satisfy the identity $f(f(x-y)) = f(x)f(y) - f(x) + f(y) - xy$.

5. Find all triples of non-negative integers (x, y, m) such that $\dfrac{x! + y!}{n!} = 3^n$ (with the convention $0! = 1$).

6. Let the sequence x_1, x_2, x_3, $\ldots$, be defined by $x_1 = a$, where a is a real number, and the recursion $x_{n+1} = 3x_n^3 - 7x_n^2 + 5x_n$ for $n \geq 1$.

Find all values of a for which the sequence has a finite limit as n tends to infinity, and find this limit.

To round out your problem solving pleasure, here are the six problems of the German Mathematical Olympiad, Final Round, Grades 12–13. Thanks again go to Felix Recio for collecting these for our use.

2005 GERMAN MATHEMATICAL OLYMPIAD
Final Round, Grades 12–13
Saarbrücken, May 10–12, 2005

1. Determine all pairs (x, y) of reals, which satisfy the system of equations

$$\begin{aligned} x^3 + 1 - xy^2 - y^2 &= 0\,, \\ y^3 + 1 - x^2y - x^2 &= 0\,. \end{aligned}$$

2. Let A, B, and C be three distinct points on the circle k. Let the lines h and g each be perpendicular to BC with h passing through B and g passing through C. The perpendicular bisector of AB meets h in F and the perpendicular bisector of AC meets g in G. Prove that the product $|BF| \cdot |CG|$ is independent of the choice of A, whenever B and C are fixed.

3. A lamp is placed at each lattice point (x, y) in the plane (that is, x and y are both integers). At time $t = 0$ exactly one lamp is switched on. At any integer time $t \geq 1$, exactly those lamps are switched on which are at a distance of 2005 from some lamp which is already switched on. Prove that every lamp will be switched on at some time.

4. Let $Q(n)$ denote the sum of the digits of the positive integer n. Prove that $Q\Big(Q\big(Q(2005^{2005})\big)\Big) = 7$.

5. Let r be the radius of the inscribed sphere of a tetrahedron $ABCD$ and let r_1, r_2, r_3, and r_4 be the radii of the other four spheres each of which is tangent externally to one of the faces of $ABCD$ and also tangent to the planes containing the other three faces. Prove that

$$\frac{2}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4}\,.$$

6. A sequence x_0, x_1, x_2, …, of real numbers is periodic with period p, if $x_{n+p} = x_n$ for all non-negative integers n and $p > 0$.

(a) Prove that there exists a sequence with period 2, which satisfies

$$x_{n+1} = x_n - \frac{1}{x_n}\,, \qquad n = 0, 1, 2, \ldots\,.$$

(b) Prove that for any integer $p > 2$, there is a sequence satisfying the condition in part (a) and having p as smallest period.

Next, we turn to solutions from our readers to problems given in the April 2007 number of the *Corner*, starting with the XX Olimpiadi Italiane Della Matematica, Cesenatico, 7 May 2004, given at [2007 : 149–150].

1. Reading the temperatures in Cesenatico for the months of December and January, Stefano notices an odd feature: on each day in that period, except for the first and the last, the lowest temperature was the sum of the lowest temperatures on the day before and the day after.

The lowest temperature was 5°C on December 3 and 2°C on January 31. Find the lowest temperature on December 25.

Solved by Ioannis Katsikis, Athens, Greece; and Titu Zvonaru, Comăneşti, Romania. We give Zvonaru's write-up.

Let t_1, t_2, $\dots$, t_{62} be the lowest temperatures for the 62 days in the months of December and January.

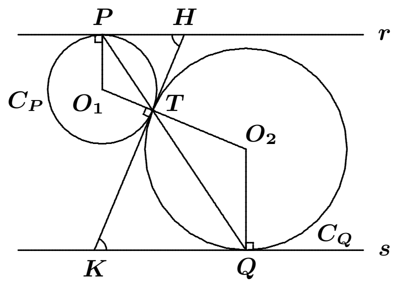
We know that $t_3 = 5$ and $t_{62} = 2$. For $i = 1, 2, \dots, 59$ we have $t_{i+1} = t_i + t_{i+2}$ and $t_{i+2} = t_{i+1} + t_{i+3}$, hence $t_i = -t_{i+3}$. It follows that for $i = 1, 2, \dots, 56$, we have $t_i = t_{i+6}$. Thus,

$$t_{26} = t_{32} = t_{38} = t_{44} = t_{50} = t_{56} = t_{62} = 2,$$
$$t_{24} = t_{18} = t_{12} = t_6 = -t_3 = -5.$$

Hence, the lowest temperature on December 25 is $t_{25} = t_{24} + t_{26} = -3$°C.

2. Let r and s be two parallel lines in the plane, and P and Q two points such that $P \in r$ and $Q \in s$. Consider circles C_P and C_Q such that C_P is tangent to r at P, C_Q is tangent to s at Q, and C_P and C_Q are tangent externally to each other at some point, say T. Find the locus of T when (C_P, C_Q) varies over all pairs of circles with the given properties.

Solution by Ioannis Katsikis, Athens, Greece.



Let KH be the common tangent of the circles C_P and C_Q. It is obvious that quadrilaterals PO_1TH and QO_2TK are cyclic, and the fact that $\angle K = \angle H$ means that $\angle PO_1T = \angle TO_2Q$.

Thus, $\triangle PO_1T \sim \triangle TO_2Q$ (similar triangles), and the fact that O_1TO_2 is a straight line tells us that PTQ is also a straight line.

Consequently, T belongs to the constant line PQ. Also, we get from the above similarity of triangles that $PT = PQ \cdot \dfrac{R}{R+r}$.

3. (a) Determine whether the number 2005^{2004} can be written as the sum of the squares of two positive integers.

(b) Determine whether the number 2004^{2005} can be written as the sum of the squares of two positive integers.

Combined solution to (a) *and* (b) *by R. Laumen, Deurne, Belgium.*

A natural number n is the sum of two squares if and only if the prime factorization of n does not contain any prime of the form $4k+3$ to an odd power [see W. Sierpiński, *Elementary Theory of Numbers*, Hafner, NY 1964, p. 351].

We have the prime power factorizations $2005^{2004} = 5^{2004} \cdot 401^{2004}$ and $2004^{2005} = 2^{4010} \cdot 3^{2005} \cdot 167^{2005}$, so we conclude that 2005^{2004} is the sum of two squares, and that 2004^{2005} is not the sum of two squares.

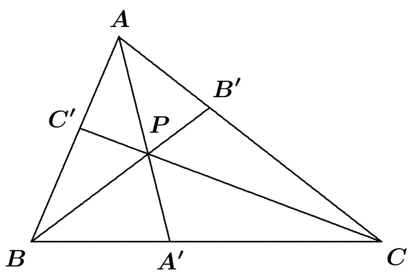
6. Let P be a point inside the triangle ABC. Say that the lines AP, BP, and CP meet the sides of ABC at A', B', and C', respectively. Let

$$x = \frac{AP}{PA'}, \quad y = \frac{BP}{PB'}, \quad z = \frac{CP}{PC'}.$$

Prove that $xyz = x+y+z+2$.

Solved by Jean-Claude Andrieux, Beaune, France; Michel Bataille, Rouen, France; Geoffrey A. Kandall, Hamden, CT, USA; Ioannis Katsikis, Athens, Greece; D.J. Smeenk, Zaltbommel, the Netherlands; and Titu Zvonaru, Comăneşti, Romania. We first give the solution of Andrieux.

Dans la suite, on notera $[UVW]$ l'aire d'un triangle UVW. On a



$$\begin{aligned} \frac{PA}{PA'} &= \frac{[PAB]}{[PA'B]} = \frac{[PAC]}{[PA'C]} \\ &= \frac{[PAB]+[PAC]}{[PA'B]+[PA'C]} \\ &= \frac{[PAB]+[PAC]}{[PBC]}. \end{aligned}$$

D'où, en posant $S_A = [PBC]$, $S_b = [PCA]$ et $S_c = [PAB]$:

$$x = \frac{PA}{PA'} = \frac{S_C+S_B}{S_A}, \qquad y = \frac{PB}{PB'} = \frac{S_A+S_C}{S_B},$$
$$\text{et} \qquad z = \frac{PC}{PC'} = \frac{S_B+S_A}{S_C}.$$

On a alors :

$$\begin{aligned} xyz &= \frac{S_C+S_B}{S_A} \times \frac{S_A+S_C}{S_B} \times \frac{S_B+S_A}{S_C} \\ &= \frac{2S_AS_BS_C + S_A^2S_C + S_C^2S_A + S_C^2S_B + S_B^2S_C + S_B^2S_A + S_A^2S_B}{S_AS_BS_C} \\ &= 2 + \frac{S_A+S_C}{S_B} + \frac{S_C+S_B}{S_A} + \frac{S_B+S_A}{S_C} = x+y+z+2. \end{aligned}$$

Next, we give the argument of Bataille.

Define positive real numbers α, β, and γ by $\alpha + \beta + \gamma = 1$ and $P = \alpha A + \beta B + \gamma C$. Then $P - \alpha A = \beta B + \gamma C = (1-\alpha)A'$; hence, $\alpha\overrightarrow{PA} + (1-\alpha)\overrightarrow{PA'} = \overrightarrow{0}$. It follows that

$$\frac{AP}{PA'} = x = \frac{1-\alpha}{\alpha}.$$

Similarly, $y = \frac{1-\beta}{\beta}$ and $z = \frac{1-\gamma}{\gamma}$; hence,

$$xyz = \frac{(1-\alpha)(1-\beta)(1-\gamma)}{\alpha\beta\gamma}$$

and

$$x + y + z + 2 = \frac{1-\alpha}{\alpha} + \frac{1-\beta}{\beta} + \frac{1-\gamma}{\gamma} + 2.$$

Thus, it is sufficient to show that

$$(1-\alpha)(1-\beta)(1-\gamma) = \beta\gamma(1-\alpha) + \gamma\alpha(1-\beta) + \alpha\beta(1-\gamma) + 2\alpha\beta\gamma.$$

Recalling that $\alpha + \beta + \gamma = 1$, this identity is easily checked.

Next, we turn to solutions to problems of the 17th Irish Mathematical Olympiad, First Paper, given at [2007 : 150].

1. (a) For which positive integers n does $2n$ divide the sum of the first n positive integers?

(b) Determine, with proof, those positive integers n (if any) which have the property that $2n+1$ divides the sum of the first n positive integers.

Solved by Geoffrey A. Kandall, Hamden, CT, USA; Ioannis Katsikis, Athens, Greece; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Titu Zvonaru, Comăneşti, Romania. We give Kandall's solution.

Let $S_n = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$, and let P be the set of positive integers.

(a) We have $2n \mid S_n$ if and only if $2n \mid \frac{n(n+1)}{2}$, which holds if and only if $n(n+1) = 4np$ for some $p \in P$, and this is true if and only if $n+1 = 4p$ for some $p \in P$.

(b) We claim that $(2n+1) \nmid S_n$ for any $n \in P$. To see this, suppose (to the contrary) that $(2n+1) \mid S_n$; that is, suppose that $\frac{n(n+1)}{2} = (2n+1)p$ for some $p \in P$. Then $n(n+1) = 4np + 2p$; whence, $2p = np_1$ where $p_1 = n+1-4p \in P$, since $4p = 2n(n+1)/(2n+1) < n+1$. Consequently, $n+1 = 4p + p_1 = 2np_1 + p_1 \geq 2n+1$, a contradiction.

2. Each of the players in a tennis tournament played one match against each of the others. If every player won at least one match, show that there is a group A, B, C of three players for which A beat B, B beat C, and C beat A.

Solution by Titu Zvonaru, Comăneşti, Romania.

Let P_1, P_2, $\ldots$, P_n be the players, and let b_i be the number of players P_i beat. We know that $b_i \geq 1$. We can assume (by relabelling if necessary) that b_1 is minimum and that P_1 beat P_2, P_3, $\ldots$, P_{b_1+1}. Since b_1 is minimum, there is a player P_k such that P_2 beat P_k and $k > b_1+1$ (otherwise $b_2 < b_1$). Then P_k beat P_1, and we can take $A = P_1$, $B = P_2$, and $C = P_k$.

3. Let AB be a chord of length 6 of a circle of radius 5 centred at O. Let $PQRS$ denote the square inscribed in the sector OAB such that P is on the radius OA, S is on the radius OB, and Q and R are points on the arc of the circle between A and B. Find the area of $PQRS$.

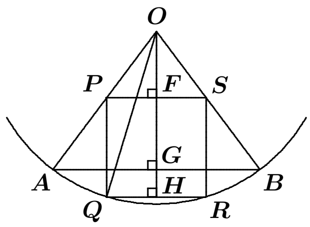
Solution by Geoffrey A. Kandall, Hamden, CT, USA.

Construct points F, G, and H, as in the diagram. Since $OA = 5$ and $AG = 3$, we see that $OG = 4$, from the Theorem of Pythagoras. Since $\triangle OPF \sim \triangle OAG$, it follows that $PF = 3t$ and $OF = 4t$ for some t. Then

$$OH \;=\; OF + FH \;=\; 4t + 6t \;=\; 10t$$

and $QH = 3t$. It now follows from the Theorem of Pythagoras (in $\triangle OQH$) that $(10t)^2 + (3t)^2 = 25$. Hence, $t^2 = \frac{25}{109}$ and

$$[PQRS] \;=\; (6t)^2 = \frac{900}{109}\,.$$



4. Prove that there are only two real numbers x such that

$$(x-1)(x-2)(x-3)(x-4)(x-5)(x-6) \;=\; 720\,.$$

Solved by Arkady Alt, San Jose, CA, USA; Michel Bataille, Rouen, France; José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain; Geoffrey A. Kandall, Hamden, CT, USA; Ioannis Katsikis, Athens, Greece; D.J. Smeenk, Zaltbommel, the Netherlands; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Titu Zvonaru, Comăneşti, Romania. We give the solution of Smeenk.

Setting $y = x^2 - 7x$, the given equation is successively equivalent to

$$\begin{aligned}
(x^2-7x+6)(x^2-7x+10)(x^2-7x+12) &= 720,\\
(y+6)(y+10)(y+12) &= 720,\\
y^3 + 28y^2 + 252y &= 0,\\
y\left((y+14)^2 + 56\right) &= 0.
\end{aligned}$$

Thus, $y = 0$ and $x \in \{0, 7\}$.

5. Let a, $b \geq 0$. Prove that

$$\sqrt{2}\left(\sqrt{a(a+b)^3}+b\sqrt{a^2+b^2}\right) \;\leq\; 3\left(a^2+b^2\right),$$

with equality if and only if $a = b$.

Solved by Arkady Alt, San Jose, CA, USA; Michel Bataille, Rouen, France; Ioannis Katsikis, Athens, Greece; and Titu Zvonaru, Comăneşti, Romania. We present the approach of Bataille.

Using the Cauchy-Schwarz Inequality

$$\begin{aligned}\sqrt{a(a+b)^3}+b\sqrt{a^2+b^2} &= (a+b)\cdot\sqrt{a^2+ab}+\sqrt{a^2+b^2}\cdot b \\ &\leq \sqrt{2a^2+2b^2+2ab}\cdot\sqrt{a^2+ab+b^2}\,;\end{aligned}$$

hence, since $2ab \leq a^2+b^2$,

$$\begin{aligned}\sqrt{2}\left(\sqrt{a(a+b)^3}+b\sqrt{a^2+b^2}\right) &\leq 2(a^2+ab+b^2) \\ &= 2(a^2+b^2)+2ab \;\leq\; 3(a^2+b^2)\,.\end{aligned}$$

Equality calls for $a = b$ (otherwise, $2ab < a^2+b^2$ and the inequality is strict) and conversely, equality holds if $a = b$, as is readily checked. The proof is complete.

Next, we turn to solutions from our readers to problems of the 17th Irish Mathematical Olympiad 2004, Second Paper, given at [2007 : 151].

1. Determine all pairs of prime numbers (p,q), with $2 \leq p$, $q < 100$, such that $p+6$, $p+10$, $q+4$, $q+10$, and $p+q+1$ are all prime numbers.

Solved by Ioannis Katsikis, Athens, Greece; and Titu Zvonaru, Comăneşti, Romania. We give Katsikis' write-up.

The only pairs are $(p,q) \in \{(7,3),\,(13,3),\,(37,3),\,(97,3)\}$.

Every prime different from 2 or 3 is of the form $6k+1$ or $6k+5$, where k is some positive integer. Obviously $p \neq 2$ and $p \neq 3$, since $p+6$ is prime. Since $p+10$ is a prime, $p \neq 6k+5$; thus, $p = 6k+1$ for some positive integer $k \geq 1$. If $q > 6$, the fact that $q+10$ is prime tells us that $q = 6\lambda+1$ for some positive integer $\lambda \geq 1$. However, we would then have

$$p+q+1 \;=\; 6k+1+6\lambda+1+1 \;=\; 3(2k+2\lambda+1)\,,$$

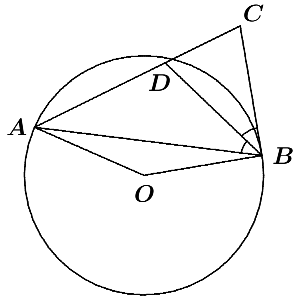
which is not prime. Thus, $q < 6$, and obviously $q \neq 2$ (since $q+4$ is prime). Therefore, we must have $q = 3$. Now we must find prime numbers $p = 6k+1$ for some integer k, $1 \leq k \leq 16$, such that $p+6$, $p+10$, and $p+4$ are all primes.

By direct checking, $k \in \{1,\,2,\,6,\,16\}$; that is, $p \in \{7,\,13,\,37,\,97\}$.

2. Let A and B be distinct points on a circle T. Let C be a point distinct from B such that $|AB| = |AC|$ and such that BC is tangent to T at B. Suppose that the bisector of $\angle ABC$ meets AC at a point D inside T. Show that $\angle ABC > 72°$.

Solution by Titu Zvonaru, Comăneşti, Romania.

Let O be the centre of T, let $AB = AC = b$ and $BC = a$, and let $\angle ABC = \alpha$. We then have $\cos\alpha = \frac{a}{2b}$, $\angle ABO = 90° - \alpha$, and $\angle OBD = 90° - \frac{\alpha}{2}$. In the isosceles triangle AOB, we have



$$OB = \frac{AB}{2\cos\angle ABO} = \frac{b}{2\sin\alpha}.$$

It is known that $BD = \frac{2ab\cos\frac{\alpha}{2}}{a+b}$ and that $\sin 18° = \frac{-1+\sqrt{5}}{4}$. The Law of Cosines gives

$$OD^2 = OB^2 + BD^2 - 2\,OB\cdot BD\cdot\cos\left(90° - \frac{\alpha}{2}\right).$$

Thus, $OD^2 < OB^2$ if and only if $BD^2 < 2\cdot OB\cdot BD\cdot\sin\frac{\alpha}{2}$; that is, D is inside T if and only if

$$\frac{2ab\cos\frac{\alpha}{2}}{a+b} < 2\cdot\frac{b}{2\sin\alpha}\cdot\sin\frac{\alpha}{2}.$$

If we cancel $2b$, write $\sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$, and $a = 2b\cos\alpha$, we get

$$\frac{2\cos\alpha\cos\frac{\alpha}{2}}{2\cos\alpha+1} < \frac{1}{4\cos\frac{\alpha}{2}}.$$

This gives $4\cos\alpha\cdot 2\cos^2\frac{\alpha}{2} - (2\cos\alpha+1) < 0$, which, by employing the identity $2\cos^2\frac{\alpha}{2} = 1+\cos\alpha$, is equivalent to $4\cos^2\alpha + 2\cos\alpha - 1 < 0$. Therefore, $\cos\alpha \in \left(\frac{-1-\sqrt{5}}{4}, \frac{-1+\sqrt{5}}{4}\right)$; thus,

$$\cos\alpha < \frac{-1+\sqrt{5}}{4} = \sin 18°.$$

However, $\sin 18° = \cos 72°$; hence, $\alpha > 72°$ since $\alpha < 90°$.

Comment. A proof that $\sin 18^\circ = \dfrac{-1+\sqrt{5}}{4}$ follows. We have

$$\begin{aligned}\sin 18^\circ \cos 36^\circ &= \frac{2\sin 18^\circ \cos 18^\circ \cos 36^\circ}{2\cos 18^\circ} = \frac{\sin 36^\circ \cos 36^\circ}{2\cos 18^\circ} \\ &= \frac{\sin 72^\circ}{4\cos 18^\circ} = \frac{\cos 18^\circ}{4\cos 18^\circ} = \frac{1}{4}\,. \qquad (1)\end{aligned}$$

We also see that

$$\begin{aligned}\sin 18^\circ - \cos 36^\circ &= \cos 72^\circ - \cos 36^\circ \\ &= -2\sin\frac{72^\circ+36^\circ}{2}\sin\frac{72^\circ-36^\circ}{2} \\ &= -2\sin 54^\circ \sin 18^\circ = -2\sin 18^\circ\cos 36^\circ \\ &= -\frac{1}{2}\,. \qquad (2)\end{aligned}$$

From (2), we get $\cos 36^\circ = \sin 18^\circ + \frac{1}{2}$. Then, using (1), we see that

$$\sin 18^\circ\left(\sin 18^\circ + \frac{1}{2}\right) = \frac{1}{4}\,.$$

Since $\sin 18^\circ > 0$, we obtain $\sin 18^\circ = \dfrac{-1+\sqrt{5}}{4}$.

3. Suppose n is an integer ≥ 2. Determine the first digit after the decimal point in the decimal expansion of the number $\sqrt[3]{n^3+2n^2+n}$.

Solved by Ioannis Katsikis, Athens, Greece; and Titu Zvonaru, Comăneşti, Romania. We give Katsikis' solution.

We prove that the first digit after the decimal point is 6.

First of all, we observe that

$$n = \sqrt[3]{n^3} < \sqrt[3]{n^3+2n^2+n} < \sqrt[3]{n^3+3n^2+3n+1} = n+1\,.$$

It is sufficient to show that

$$n + \frac{6}{10} < \sqrt[3]{n^3+2n^2+n} < n + \frac{7}{10}\,. \qquad (1)$$

The first inequality of (1) holds for $n \geq 2$, since

$$(5n+3)^3 < 125(n^3+2n^2+n) \iff 5n(5n-2) > 27\,.$$

The second inequality of (1) holds for $n \geq 2$, since

$$(10n+7)^3 > 1000(n^3+2n^2+n) \iff 100n^2+470n+343 > 0\,.$$

Thus, (1) holds for $n \geq 2$, and the first decimal digit is 6.

4. Define the function m of the three real variables x, y, and z by

$$m(x,y,z) = \max\{x^2,\, y^2,\, z^2\}\,.$$

Determine, with proof, the minimum value of m if x, y, and z vary in $\mathbb{R}$ subject to the restrictions $x+y+z=0$ and $x^2+y^2+z^2=1$.

Solved by Arkady Alt, San Jose, CA, USA; and Michel Bataille, Rouen, France. We give Bataille's write-up.

We show that the required minimum of m is $\frac{1}{2}$.

Let x, y, z be such that $x + y + z = 0$ and $x^2 + y^2 + z^2 = 1$. Clearly, x, y, and z are neither all positive nor all negative; hence, up to a change of order or sign, we may suppose that $x \leq 0$ and y, $z \geq 0$. Then

$$2x^2 \;=\; x^2 + (-y - z)^2 \;=\; 1 + 2yz \;\geq\; 1\,.$$

Thus, $m(x^2, y^2, z^2) \geq \frac{1}{2}$.

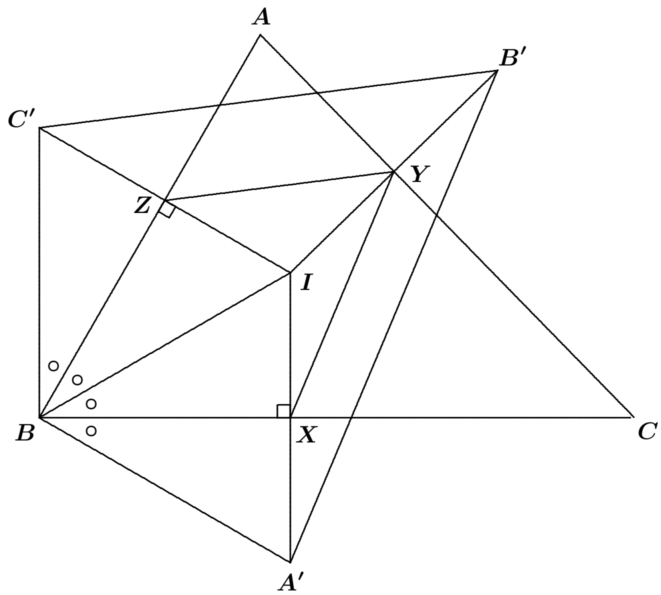
To complete the proof, we just observe that for $x = -\frac{\sqrt{2}}{2}$, $y = 0$, $z = \frac{\sqrt{2}}{2}$, we have $x + y + z = 0$, $x^2 + y^2 + z^2 = 1$ and $m(x^2, y^2, z^2) = \frac{1}{2}$.

Next, we give solutions to problems of the New Zealand Mathematical Olympiad, IMO Squad Selection Problems 2004 given at [2007 : 151–152].

1. Let I be the incentre of triangle ABC, and let A', B', and C' be the reflections of I in BC, CA, and AB, respectively. The circle through A', B', and C' passes also through B. Find the angle $\angle ABC$.

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Geoffrey A. Kandall, Hamden, CT, USA; Ioannis Katsikis, Athens, Greece; D.J. Smeenk, Zaltbommel, the Netherlands; and Titu Zvonaru, Comănești, Romania. We give the write-up of Amengual Covas.

Let X, Y, and Z be the feet of the perpendiculars from I to the sides BC, CA, and AB , respectively.



Then X, Y, Z are the mid-points of segments IA', IB', IC' respectively. Therefore XY is parallel to $A'B'$ and YZ is parallel to $B'C'$, so we have

$$\begin{aligned}\angle A'B'C' = \angle XYZ &= \tfrac{1}{2}\angle XIZ = \tfrac{1}{2}(180^\circ - \angle ZBX)\\ &= \tfrac{1}{2}(180^\circ - \angle ABC) = 90^\circ - \tfrac{1}{2}\angle ABC\end{aligned}$$

On the other hand, since BI bisects $\angle ABC$, we have

$$\angle C'BZ = \angle ZBI = \tfrac{1}{2}\angle ABC = \angle IBX = \angle XBA';$$

whence,

$$\begin{aligned}\angle C'BA' &= \angle C'BZ + \angle ZBI + \angle IBX + \angle XBA'\\ &= 4\cdot\tfrac{1}{2}\angle ABC = 2\angle ABC\,.\end{aligned}$$

Since A', B', C', and B are concyclic, we have $\angle C'BA' + \angle A'B'C' = 180^\circ$, which, on substitution, gives $2\cdot\angle ABC + \left(90^\circ - \frac{1}{2}\angle ABC\right) = 180^\circ$. Therefore, $\angle ABC = 60^\circ$.

3. For positive x_1, x_2, y_1, y_2, prove the inequality

$$\frac{x_1^2}{y_1} + \frac{x_2^2}{y_2} \geq \frac{(x_1+x_2)^2}{y_1+y_2}\,.$$

Solved by Arkady Alt, San Jose, CA, USA; Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain; Xavier Ros, student, Universitat Politècnica de Catalunya, Barcelona, Spain; and Titu Zvonaru, Cománeşti, Romania. We give Alt's solutions and comment.

Solution 1. We have

$$\begin{aligned}(y_1+y_2)\left(\frac{x_1^2}{y_1}+\frac{x_2^2}{y_2}\right) &= x_1^2+x_2^2+\frac{x_1^2y_2}{y_1}+\frac{x_2^2y_1}{y_2}\\ &\geq x_1^2+x_2^2+2\sqrt{\frac{x_1^2y_2}{y_1}\cdot\frac{x_2^2y_1}{y_2}}\\ &= x_1^2+x_2^2+2x_1x_2 = (x_1+x_2)^2\,.\end{aligned}$$

Solution 2. Let $a = \dfrac{x_1}{x_1+x_2}$ and $b = \dfrac{y_1}{y_1+y_2}$. Then $1-a = \dfrac{x_2}{x_1+x_2}$ and $1-b = \dfrac{y_2}{y_1+y_2}$, and the original inequality can be rewritten in the form

$$\frac{a^2}{b} + \frac{(1-a)^2}{1-b} \geq 1\,,$$

where a, $b \in (0,1)$. This is successively equivalent to

$$\begin{aligned}a^2(1-b)+b(1-a)^2 &\geq b(1-b)\,,\\ a^2-a^2b+b-2ab+a^2b &\geq b-b^2\,,\end{aligned}$$

or $a^2+b^2 \geq 2ab$, which is true.

Solution 3. Since $\dfrac{x^2}{y} \geq 2x - y$ for $y > 0$, we apply this twice to obtain

$$\frac{a^2}{b} + \frac{(1-a)^2}{1-b} \;\geq\; (2a-b) + \big(2(1-a) - (1-b)\big) \;=\; 1\,,$$

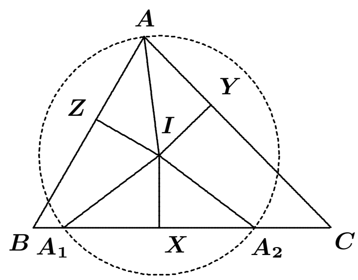
where a and b are defined as in Solution 2.

Comment. The original inequality is very simple relative to the high level of math olympiads, but it is a good occasion to perform different elementary techniques at an introductory level and, as well, for generalizations obtained by applying the Cauchy-Schwarz Inequality to $(\sqrt{y_1}, \sqrt{y_2}, \ldots, \sqrt{y_n})$ and $\left(\dfrac{x_1}{\sqrt{y_1}}, \dfrac{x_2}{\sqrt{y_2}}, \ldots, \dfrac{x_n}{\sqrt{y_n}}\right)$, where each x_i and y_i is a positive number:

$$\frac{x_1^2}{y_1} + \frac{x_2^2}{y_2} + \cdots + \frac{x_n^2}{y_n} \;\geq\; \frac{(x_1 + x_2 + \cdots + x_n)^2}{y_1 + y_2 + \cdots + y_n}\,.$$

5. Let I be the incentre of triangle ABC. Let points $A_1 \neq A_2$ lie on the line BC, points $B_1 \neq B_2$ lie on the line AC, and points $C_1 \neq C_2$ lie on the line AB so that $AI = A_1I = A_2I$, $BI = B_1I = B_2I$, $CI = C_1I = C_2I$. Prove that $A_1A_2 + B_1B_2 + C_1C_2 = P$, where P is the perimeter of $\triangle ABC$.

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Michel Bataille, Rouen, France; Geoffrey A. Kandall, Hamden, CT, USA; D.J. Smeenk, Zaltbommel, the Netherlands; and Titu Zvonaru, Comăneşti, Romania. We give the write-up of Amengual Covas.



Let X, Y, Z be the feet of the perpendiculars from I to the sides BC, CA, AB respectively. Then $IX = IY = IZ$. Right triangles IAZ, IA_1X, IA_2X, IYA are congruent, and since X is the mid-point of segment A_1A_2, we have

$$A_1A_2 \;=\; A_1X + XA_2 \;=\; AZ + YA\,.$$

Similarly, we obtain

$$\begin{aligned} B_1B_2 \;&=\; B_1Y + YB_2 \;=\; BX + ZB \\ \text{and} \qquad C_1C_2 \;&=\; C_1Z + ZC_2 \;=\; CY + XC\,. \end{aligned}$$

Therefore,

$$\begin{aligned} A_1A_2 + B_1B_2 + C_1C_2 \;&=\; (AZ + YA) + (BX + ZB) + (CY + XC) \\ &=\; (AZ + ZB) + (BX + XC) + (CY + YA) \\ &=\; AB + BC + CA \;=\; P\,. \end{aligned}$$

7. A function $f(x)$ is defined on the interval $[0,1]$, so that $f(0)=f(1)=0$ and

$$f\left(\frac{a+b}{2}\right) \leq f(a)+f(b).$$

for all a and b from $[0,1]$.

(a) Show that the equation $f(x)=0$ has infinitely many solutions on $[0,1]$.

(b) Are there functions on $[0,1]$ which satisfy the above conditions but are not identically zero?

Solved by Ioannis Katsikis, Athens, Greece; Xavier Ros, student, Universitat Politècnica de Catalunya, Barcelona, Spain; and Li Zhou, Polk Community College, Winter Haven, FL, USA. We give the solution of Katsikis.

(a) For $a=b\in[0,1]$, we have that $f(a)\leq 2f(a)$, hence $f(a)\geq 0$, for all $a\in[0,1]$. Let $a_n=\left(\frac{1}{2}\right)^n$, $n\geq 1$. We will show by induction that $f(a_n)=0$ for each n.

Taking $a=0$ and $b=1$, we have $f\left(\frac{0+1}{2}\right)\leq f(0)+f(1)$, which gives $f\left(\frac{1}{2}\right)\leq 0$; whence, $f(a_1)=f\left(\frac{1}{2}\right)=0$. Next, we show that if $f(a_n)=0$, then $f(a_{n+1})=0$. Taking $a=0$ and $b=a_n$, the basic inequality yields $f\left(\frac{0+a_n}{2}\right)\leq f(0)+f(a_n)=0$. Since $f\left(\frac{a_n}{2}\right)\leq 0$, we have $f(a_{n+1})=0$.

(b) Let $c>0$. Define the function

$$f(x) = \begin{cases} 0, & \text{if } x\in[0,1]\cap\mathbb{Q}, \\ c, & \text{if } x\in[0,1]\cap(\mathbb{R}\setminus\mathbb{Q}). \end{cases}$$

We will prove that f satisfies the condition of the hypothesis.

Case 1. Let x_1 and x_2 be rational numbers.
Then we have

$$f\left(\frac{x_1+x_2}{2}\right) = 0 \leq 0+0 = f(x_1)+f(x_2).$$

Case 2. Let x_1 be a rational number and x_2 an irrational number.
Then we have

$$f\left(\frac{x_1+x_2}{2}\right) = c \leq 0+c = f(x_1)+f(x_2).$$

Case 3. Let x_1 and x_2 be irrational numbers.
Then we have

$$f\left(\frac{x_1+x_2}{2}\right) \leq c < c+c = f(x_1)+f(x_2).$$

Thus, there exist infinitely many functions on $[0,1]$ that are not identically zero, and such that the conditions of the hypothesis hold.

8. Prove that any prime number $2^{2^n}+1$ cannot be represented as a difference of two fifth powers of integers.

Solution by Ioannis Katsikis, Athens, Greece.

For $n=1$, we have $2^{2^n}+1=5$. For integers x and y such that

$$5 \;=\; x^5-y^5 \;=\; (x-y)(x^4+x^3y+x^2y^2+xy^3+y^4)\,,$$

we have $x-y=1$, or $x=y+1$. Generally, if $a \equiv U \pmod 5$, where $1 \le U \le 4$, then $a^5 \equiv U \pmod 5$. Thus, from $5=x^5-y^5$, we get $x \equiv y \pmod 5$, which contradicts the fact that $x=y+1$. Consequently, 5 is not the difference of two fifth powers of integers.

For $n>1$, the equation $2^{2^n}+1=x^5-y^5$ again implies $x=y+1$, since $2^{2^n}+1$ is prime. Thus,

$$\begin{aligned} 2^{2^n}+1 &= (y+1)^4+(y+1)^3y+(y+1)^2y^2+(y+1)y^3+y^4 \\ &= 5y^4+10y^3+10y^2+5y+1\,, \end{aligned}$$

which is a contradiction, since 2^{2^n} cannot be a multiple of 5.

That completes the material on file for this number of the *Corner*. Send me your nice solutions and generalizations.