SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

3214. [2007 : 110, 113] *Proposed by Mihály Bencze, Brasov, Romania.*

Let ABC be an acute-angled triangle.

(a) Prove that $\frac{\tan A}{A} + \frac{\tan B}{B} + \frac{\tan C}{C} > \left(\frac{6}{\pi}\right)^2$.

(b) Prove that $A\cot A + B\cot B + C\cot C < \left(\frac{\pi}{2}\right)^2$.

(c)★ Determine the best constants $c_1 \geq \left(\frac{6}{\pi}\right)^2$ and $0 < c_2 < c_3 \leq \left(\frac{\pi}{2}\right)^2$ such that

$$\sum_{\text{cyclic}} \frac{\tan A}{A} \geq c_1 \qquad \text{and} \qquad c_2 \leq \sum_{\text{cyclic}} A\cot A \leq c_3 .$$

Composite of solutions by Roy Barbara, Lebanese University, Fanar, Lebanon; Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; and Dragoljub Milošević, Pranjani, Serbia.

(a) and (c) Applying the AM–GM Inequality, we obtain (for any acute-angled triangle ABC)

$$\frac{\tan A}{A} + \frac{\tan B}{B} + \frac{\tan C}{C} \geq 3\left(\frac{\tan A\tan B\tan C}{ABC}\right)^{\frac{1}{3}} , \qquad (1)$$

and

$$\tan A + \tan B + \tan C \geq 3(\tan A\tan B\tan C)^{\frac{1}{3}} . \qquad (2)$$

Using (2) and the well-known identity

$$\tan A\tan B\tan C = \tan A + \tan B + \tan C ,$$

we get

$$\tan A\tan B\tan C \geq 3(\tan A\tan B\tan C)^{\frac{1}{3}} ;$$

whence,

$$\tan A\tan B\tan C \geq 3\sqrt{3} . \qquad (3)$$

Since $ABC \leq \left(\frac{1}{3}(A+B+C)\right)^3 = \left(\frac{\pi}{3}\right)^3$, inequalities (3) and (1) imply that

$$\frac{\tan A}{A} + \frac{\tan B}{B} + \frac{\tan C}{C} \geq \frac{9\sqrt{3}}{\pi} ,$$

with equality if and only if $A = B = C = \frac{\pi}{3}$. Thus, $c_1 = \frac{9\sqrt{3}}{\pi}$ in part (c). Since $\frac{9\sqrt{3}}{\pi} > \left(\frac{6}{\pi}\right)^2$, part (a) is also proved.

(b) and (c) Since $\lim\limits_{x\to 0} x\cot x = 1$, we may extend $h(x) = x\cot x$ to the continuous function $f(x)$ defined as

$$f(x) \;=\; \begin{cases} x\cot x & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

Then $f(x)$ is continuous on $\left[0, \frac{\pi}{2}\right]$. Next consider a lemma:

Lemma 1. The function $f(x) = x\cot x$ is concave on $\left[0, \frac{\pi}{2}\right]$.

Proof: The second derivative of $f(x)$ is

$$f''(x) \;=\; \frac{2(x\cos x - \sin x)}{\sin^3 x} \;<\; 0\,,$$

since $\tan x > x$ on $\left(0, \frac{\pi}{2}\right)$. Thus, $f(x)$ is concave on $\left[0, \frac{\pi}{2}\right]$. ■

Therefore,

$$f(A) + f(B) + f(C) \;\leq\; 3f\left(\frac{A+B+C}{3}\right)\;;$$

that is,

$$A\cot A + B\cot B + C\cot C \;\leq\; \frac{\pi\sqrt{3}}{3}\,,$$

with equality if and only if $A = B = C = \frac{\pi}{3}$. Thus, $c_3 = \frac{\pi\sqrt{3}}{3}$ in part (c). Since $\frac{\pi\sqrt{3}}{3} < \left(\frac{\pi}{2}\right)^2$, part (b) is also proved.

(c) It only remains to find c_2. For that purpose we introduce another lemma:

Lemma 2. Let f be a concave function on some real interval I, and let a, b, u, $v \in I$ be such that $a \leq u \leq v \leq b$ and $u + v = a + b$. Then $f(u) + f(v) \geq f(a) + f(b)$.

Proof: If $a = b$, the result is obvious. Now assume that $a < b$. Let ℓ be the line containing the points $(a, f(a))$ and $(b, f(b))$. Since ℓ is not vertical, it has finite slope. Let $y = Ax + B$ be the equation of ℓ. Since f is concave, we have $f(x) \geq Ax + B$ for every $x \in [a, b]$. In particular, we have

$$f(u) \;\geq\; Au + B \qquad \text{and} \qquad f(v) \;\geq\; Av + B\,.$$

Hence,

$$\begin{aligned} f(u) + f(v) \;&\geq\; (Au + B) + (Av + B) \;=\; A(u + v) + 2B \\ &=\; A(a + b) + 2B \;=\; (Aa + B) + (Ab + B) \\ &=\; f(a) + f(b)\,. \end{aligned}$$ ■

Without loss of generality, we may assume that $0 < A \le B \le C < \frac{\pi}{2}$. Note that $A + B > \frac{\pi}{2}$; hence, $0 < \frac{\pi}{2} - A < B$. Then $\frac{\pi}{2} - A < B \le C < \frac{\pi}{2}$ all belong to the interval $\left[0, \frac{\pi}{2}\right]$ and satisfy $B + C = \left(\frac{\pi}{2} - A\right) + \frac{\pi}{2}$. By Lemma 2 and the concavity of f, we have

$$f(B) + f(C) \ \ge\ f\left(\tfrac{\pi}{2} - A\right) + f\left(\tfrac{\pi}{2}\right).$$

Since $f\left(\frac{\pi}{2}\right) = 0$, we obtain $f(B) + f(C) = f\left(\frac{\pi}{2} - A\right)$. Then

$$\lambda \ =\ f(A) + f(B) + f(C) \ \ge\ f(A) + f\left(\tfrac{\pi}{2} - A\right).$$

Let $u = \min\left\{A,\ \frac{\pi}{2} - A\right\}$ and $v = \max\left\{A,\ \frac{\pi}{2} - A\right\}$. Then $0 < u \le v < \frac{\pi}{2}$ with $u + v = 0 + \frac{\pi}{2}$. By Lemma 2 again, we get

$$f(u) + f(v) \ \ge\ f(0) + f\left(\tfrac{\pi}{2}\right) \ =\ 1 + 0\,.$$

That is, $f(A) + f\left(\frac{\pi}{2} - A\right) \ge 1$. Therefore, $c_2 \ge 1$.

If we consider an acute isosceles triangle ABC with $AB = AC = k$ (a constant), then as $\angle A \to 0$ we see that $A\cot A \to 1$, $B\cot B \to 0$, and $C\cot C \to 0$. Hence, $\lambda \to 1$. Thus, $c_2 = 1$.

Also solved by CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; VEDULA N. MURTY, Dover, PA, USA (parts (a) *and* (b) *only); VO QUOC BA CAN, Can Tho University of Medicine and Pharmacy, Can Tho, Vietnam; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer (parts* (a) *and* (b) *only). Not all solvers of part* (c) *obtained a value for* c_2.

3215. [2007 : 110, 113] *Proposed by Shaun White, student, Vincent Massey Secondary School, Windsor, ON.*

Given any integers k, ℓ, m, greater than 2, an integer n is called *expressible* for (k, ℓ, m) if there exist positive real numbers a_1, a_2, $\dots$, a_k such that $\prod\limits_{i=1}^{k} a_i = 1$ and

$$\sum_{i=1}^{k}\left(\sum_{j=1}^{\ell} a_{i+j-1}\right)^{m} \ =\ n\,,$$

where the subscripts are taken modulo k.

Suppose that for some (k, ℓ, m) the integer 2005 is expressible while 1987 is not. Find the ordered triple (k, ℓ, m).

Solution by Joel Schlosberg, Bayside, NY, USA.

We will show that $(74, 3, 3)$ and $(16, 5, 3)$ are the only two possibilities for the triple (k, ℓ, m). This will be deduced from the fact that n is expressible for (k, ℓ, m) if and only if $n \ge k\ell^m$.

Suppose that n is expressible for (k,ℓ,m). Then by repeated use of the AM–GM Inequality, we have

$$\begin{aligned} n &= \sum_{i=1}^{k}\left(\sum_{j=1}^{\ell} a_{i+j-1}\right)^{m} \;\geq\; \sum_{i=1}^{k}\left(\ell\left(\prod_{j=1}^{\ell} a_{i+j-1}\right)^{\frac{1}{\ell}}\right)^{m} \\ &= \ell^m \sum_{i=1}^{k}\left(\prod_{j=1}^{\ell} a_{i+j-1}\right)^{\frac{m}{\ell}} \;\geq\; k\ell^m\left(\prod_{i=1}^{k}\left(\prod_{j=1}^{\ell} a_{i+j-1}\right)^{\frac{m}{\ell}}\right)^{\frac{1}{k}} \\ &= k\ell^m\left(\prod_{j=1}^{\ell}\prod_{i=1}^{k} a_{i+j-1}\right)^{\frac{m}{k\ell}} \;=\; k\ell^m\left(\prod_{j=1}^{\ell} 1\right)^{\frac{m}{k\ell}} \;=\; k\ell^m\,. \end{aligned}$$

For $a \geq 1$, let $f(a)$ be the value of $\sum\limits_{i=1}^{k}\left(\sum\limits_{j=1}^{\ell} a_{i+j-1}\right)^{m}$ when we set $(a_1,a_2,\ldots,a_n) = (a,a^{-1},1,\ldots,1)$. Then f is continuous, and $f(a)$ is clearly expressible for (k,ℓ,m) for all $a \geq 1$. Now, $f(1) = k\ell^m$, and $f(a) > a^m \geq a$; hence, $f(a) \to \infty$ as $a \to \infty$. By the Intermediate Value Theorem, for any integer $n \geq k\ell^m$, there exists some $a \geq 1$ such that $f(a) = n$, and thus, n is expressible for (k,ℓ,m).

Therefore, 2005 is expressible for (k,ℓ,m), and 1987 is not expressible for (k,ℓ,m), precisely when $1987 < k\ell^m \leq 2005$. In this case, $3\ell^3 \leq k\ell^m \leq 2005$, so that $3 \leq \ell \leq \lfloor (2005/3)^{1/3}\rfloor = 8$. For fixed ℓ the inequality $1987 < k\ell^m \leq 2005$ can hold for only finitely many m, and when ℓ and m are both fixed the inequality can hold for only finitely many k. We need only consider $\ell = 3$, 4, 5, and 7, for if $\ell = ab$, $b \geq 3$ and n is (k,l,m) expressible, then n is also (ka^m,b,m) expressible. Sifting the values of k, ℓ, and m reveals that (k,ℓ,m) is either $(74,3,3)$ or $(16,5,3)$.

There were two incomplete solutions submitted, each of which found only one of the two answers.

3216. [2007 : 110, 114] *Proposed by Mihály Bencze, Brasov, Romania.*

If a, b, c, and d are positive integers, prove that

$$\begin{aligned} 45\left(\frac{1}{a+b+c+d+1} - \frac{1}{(a+1)(b+1)(c+1)(d+1)}\right) \\ \leq\; 4 + \sum_{\text{cyclic}}\left[\frac{1}{a+1} + \frac{1}{(a+1)(b+1)}\right]. \end{aligned}$$

Solution by Vo Quoc Ba Can, Can Tho University of Medicine and Pharmacy, Can Tho, Vietnam.

The solver proved this inequality for any non-negative real numbers a, b, c, and d. Setting $x = \frac{1}{4}(a+b+c+d)$, and applying the AM–GM Inequality, we have

$$\begin{aligned}
\frac{1}{(a+1)(b+1)(c+1)(d+1)} &\geq \frac{256}{(a+b+c+d+4)^4} = \frac{1}{(x+1)^4}\,, \\
\sum_{\text{cyclic}} \frac{1}{a+1} &\geq \frac{4}{\sqrt[4]{(a+1)(b+1)(c+1)(d+1)}} \\
&\geq \frac{16}{a+b+c+d+4} = \frac{4}{x+1}\,, \\
\text{and} \qquad \sum_{\text{cyclic}} \frac{1}{(a+1)(b+1)} &\geq \frac{4}{\sqrt{(a+1)(b+1)(c+1)(d+1)}} \\
&\geq \frac{64}{(a+b+c+d+4)^2} = \frac{4}{(x+1)^2}\,.
\end{aligned}$$

It therefore suffices to show that

$$45\left(\frac{1}{4x+1} - \frac{1}{(x+1)^4}\right) \leq 4 + \frac{4}{x+1} + \frac{4}{(x+1)^2}\,,$$

or

$$\frac{16x^5 + 39x^4 - 86x^2 + 84x + 12}{(4x+1)(x+1)^4} \geq 0\,.$$

If $x \leq 1$, then

$$\begin{aligned}
&16x^5 + 39x^4 - 86x^2 + 84x + 12 \\
&\qquad = 16x^5 + 39x^4 + 84x(1-x) + 2(1-x^2) + 10 > 0\,.
\end{aligned}$$

If $x > 1$, then

$$\begin{aligned}
&16x^5 + 39x^4 - 86x^2 + 84x + 12 \\
&\qquad > 55x^3 - 86x^2 + 84x + 12 \\
&\qquad = 43x(x-1)^2 + 12x^3 + 41x + 12 > 0\,.
\end{aligned}$$

Therefore the inequality holds and the equality does not.

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; and the proposer.

3217. [2007 : 110, 114] *Proposed by Michel Bataille, Rouen, France.*

Let $\{L_n\}$ be the Lucas sequence defined by $L_0 = 2$, $L_1 = 1$, and $L_{n+1} = L_n + L_{n-1}$ for $n \geq 1$. Prove that, for all non-negative integers n, we have

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n+1}{2k+1} \frac{L_{2k}}{2^n} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} \frac{L_k}{2^{2k}}.$$

Solution by the proposer, expanded by the editor.

It is well known that $L_k = \alpha_1^k + \alpha_2^k$ for $k = 0, 1, 2, \ldots$, where $\alpha_1 = \frac{1}{2}(1+\sqrt{5})$ and $\alpha_2 = \frac{1}{2}(1-\sqrt{5})$ are the roots of $x^2 - x - 1 = 0$. The right side of the proposed identity then suggests the introduction of the polynomials $P_n(x) = \sum\limits_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} \left(\frac{x}{4}\right)^k$, for $n = 0, 1, 2, \ldots$. It is readily seen that $P_0(x) = P_1(x) = 1$.

We claim that $P_n(x)$ satisfies the following recurrence relation:

$$P_{n+1}(x) = P_n(x) + \frac{x}{4} P_{n-1}(x), \tag{1}$$

for all $n \in \mathbb{N}$. To establish (1), note first that

$$\begin{aligned} \frac{x}{4} P_{n-1}(x) &= \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-1-k}{k} \left(\frac{x}{4}\right)^{k+1} \\ &= \sum_{k=1}^{\lfloor (n-1)/2 \rfloor + 1} \binom{n-k}{k-1} \left(\frac{x}{4}\right)^k. \end{aligned}$$

If n is even (say $n = 2\ell$), then

$$\lfloor (n-1)/2 \rfloor + 1 = \lfloor (n+1)/2 \rfloor = \lfloor n/2 \rfloor = \ell.$$

Hence,

$$\begin{aligned} P_n(x) + \frac{x}{4} P_{n-1}(x) &= \sum_{k=0}^{\ell} \binom{n-k}{k} \left(\frac{x}{4}\right)^k + \sum_{k=1}^{\ell} \binom{n-k}{k-1} \left(\frac{x}{4}\right)^k \\ &= \sum_{k=1}^{\ell} \left[\binom{n-k}{k} + \binom{n-k}{k-1} \right] \left(\frac{x}{4}\right)^k + 1 \\ &= \sum_{k=1}^{\ell} \binom{n+1-k}{k} \left(\frac{x}{4}\right)^k + 1 \\ &= \sum_{k=0}^{\lfloor (n+1)/2 \rfloor} \binom{n+1-k}{k} \left(\frac{x}{4}\right)^k = P_{n+1}(x). \end{aligned}$$

Using the same argument with minor modification regarding the upper limits of the summations, we can easily show that (1) is also true when n is odd.

Now, suppose that $x > -1$. Since the characteristic equation of the recurrence relation given in (1) is $4w^2 - 4w - x = 0$, the characteristic roots are $q_1 = \frac{1}{2}\left(1 + \sqrt{1+x}\,\right)$ and $q_2 = \frac{1}{2}\left(1 - \sqrt{1+x}\,\right)$.

From known theory, the general solution to the recurrence relation in (1) is given by $P_n(x) = c_1 q_1^{n+1} + c_2 q_2^{n+1}$, where c_1 and c_2 are to be determined. Solving $c_1q_1 + c_2q_2 = P_0(x) = 1$ and $c_1q_1^2 + c_2q_2^2 = P_1(x) = 1$, we find that $c_1 = 1/\sqrt{1+x}$ and $c_2 = -1/\sqrt{1+x}$. Let $z = \sqrt{1+x}$. We then have

$$\begin{aligned} P_n(x) &= \frac{1}{z}\left[\left(\frac{1+z}{2}\right)^{n+1} - \left(\frac{1-z}{2}\right)^{n+1}\right] \\ &= \frac{1}{2^{n+1}z}\left[\sum_{k=0}^{n+1}\binom{n+1}{k}z^k - \sum_{k=0}^{n+1}\binom{n+1}{k}(-z)^k\right] \\ &= \frac{1}{2^n z}\sum_{j=0}^{\lfloor n/2\rfloor}\binom{n+1}{2j+1}z^{2j+1} = \frac{1}{2^n}\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n+1}{2k+1}(1+x)^k\,. \end{aligned}$$

Therefore, we have established the identity

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n-k}{k}\left(\frac{x}{4}\right)^k = \frac{1}{2^n}\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n+1}{2k+1}(1+x)^k\,. \qquad (2)$$

Since $1 + \alpha_i = \alpha_i^2$ for $i = 1$ and $i = 2$, by substituting α_i for x in (2), we see that

$$\frac{1}{2^n}\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n+1}{2k+1}\alpha_1^{2k} = \sum_{k=0}^{\lfloor n/2\rfloor}\binom{n-k}{k}\frac{\alpha_1^k}{4^k} \qquad (3)$$

and
$$\frac{1}{2^n}\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n+1}{2k+1}\alpha_2^{2k} = \sum_{k=0}^{\lfloor n/2\rfloor}\binom{n-k}{k}\frac{\alpha_2^k}{4^k}\,. \qquad (4)$$

The proposed identity follows by adding (3) and (4).

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; and EDMUND SWYLAN, Riga, Latvia.

The proposer remarked that from the proof given above, we readily see that $\{L_n\}$ can be replaced by any sequence $\{U_n\}$ satisfying the recursion $U_{n+1} = U_n + U_{n-1}$ and in particular, by the Fibonacci sequence.

3218. [2007 : 111, 114] *Proposed by Ovidiu Furdui, University of Toledo, Toledo, OH, USA.*

Let n be an integer with $n \geq 2$. In $\mathbb{R}^n$, let E be the set of points $(x_1, x_2, \ldots, x_n)$ such that $x_i \geq 0$ for all i and $0 < x_1 + x_2 + \cdots + x_n \leq 1$. Calculate the integral over E of the fractional part of $\dfrac{1}{x_1 + x_2 + \cdots + x_n}$.

Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA, expanded by the editor.

Let $\{x\} = x - \lfloor x \rfloor$ be the fractional part of the real number x, let I be integral in question, and let $0 = z_0 < z_1 < \cdots < z_s = 1$ be a partition of $[0,1]$ with mesh size $\Delta = \max\{\Delta z_1,\ \Delta z_2,\ \ldots,\ \Delta z_s\}$, where $\Delta z_k = z_k - z_{k-1}$ for $k = 1,\ 2,\ \ldots,\ s$. By straightforward multiple integration, the volume of the simplex determined by $0 < x_1 + x_2 + \cdots + x_n \le z$ with $x_i \ge 0$ (for all i) is $V(z) = z^n/n!$. The k^{th} slice of the simplex E is determined by $z_{k-1} < x_1 + x_2 + \cdots + x_n \le z_k$, which has volume $V(z_k) - V(z_{k-1})$; this is equal to $V'(z_k^*)\Delta z_k$ for some $z_k^* \in (z_{k-1}, z_k)$, by the Mean-Value Theorem. The point $(z_k^*, 0, 0, \ldots, 0)$ is in the k^{th} slice, and the integrand evaluates to $\{1/z_k^*\}$ at this point. Thus, the limit of the sum

$$S = \sum_{k=1}^{s} \left\{\frac{1}{z_k^*}\right\} \big(V(z_k) - V(z_{k-1})\big)$$

is I as $\Delta \to 0$. On the other hand, $S = \sum\limits_{k=1}^{s} \left\{\frac{1}{z_k^*}\right\} V'(z_k^*)\Delta z_k$, and, when viewed this way, the limit of S is

$$\int_0^1 \left\{\frac{1}{z}\right\} V'(z)\,dz$$

as $\Delta \to 0$. Therefore,

$$\begin{aligned}
I &= \int_0^1 \left\{\frac{1}{z}\right\} V'(z)\,dz = \int_0^1 \left\{\frac{1}{z}\right\} \frac{z^{n-1}}{(n-1)!}\,dz \\
&= \frac{1}{(n-1)!} \sum_{j=1}^{\infty} \int_{\frac{1}{j+1}}^{\frac{1}{j}} \left(\frac{1}{z} - j\right) z^{n-1}\,dz \\
&= \frac{1}{(n-1)!} \left[\int_0^1 z^{n-2}\,dz - \sum_{j=1}^{\infty} \int_{\frac{1}{j+1}}^{\frac{1}{j}} j z^{n-1}\,dz\right] \\
&= \frac{1}{(n-1)!} \left[\frac{1}{n-1} - \frac{1}{n}\left(\frac{1}{1^n} - \frac{1}{2^n} + \frac{2}{2^n} - \frac{2}{3^n} + \frac{3}{3^n} - \frac{3}{4^n} + \cdots\right)\right] \\
&= \frac{1}{(n-1)!} \left[\frac{1}{n-1} - \frac{1}{n}\left(\frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \cdots\right)\right] \\
&= \frac{1}{(n-1)!} \left[\frac{1}{n-1} - \frac{\zeta(n)}{n}\right],
\end{aligned}$$

where ζ is the Riemann zeta function.

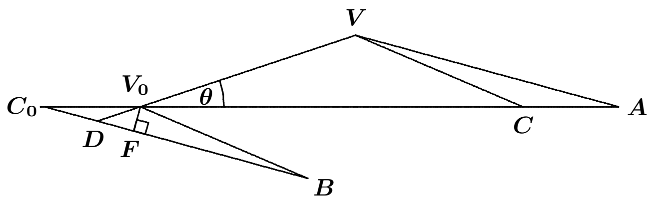
Also solved by JOEL SCHLOSBERG, Bayside, NY, USA; and the proposer.

Schlosberg obtains $1/(n-1)(n-1)!$ and $\zeta(n)/n!$ for the values of the integrals over E of $1/(x_1 + \cdots + x_n)$ and its integer part, respectively. The proposer passed from many variables to a single variable by using Liouville's result, that for suitable ϕ, the integral of $\phi(x_1 + \cdots + x_n)x_1^{p_1-1} \cdots x_n^{p_n-1}$ over E equals the integral of $\phi(z)z^{p_1+\cdots+p_n-1}$ over $[0,1]$ multiplied by $\Gamma(p_1)\cdots\Gamma(p_n)/\Gamma(p_1 + \cdots + p_n)$.

3219. [2007 : 111, 114] *Proposed by Dan Vetter, Regina, SK.*

A vulture with a university education, when approached by a car while dining on the road, will always fly off in a direction chosen to maximize the distance of closest approach of the car. Show that the ratio of the speed of the car to the speed of the bird is $\sec\theta$, where θ is the angle that the vulture's flight path makes with the road.

Solution by Apostolis K. Demis, Varvakeio High School, Athens, Greece.



Let C and V be the positions of the car and vulture after t seconds, with C_0 and V_0 being their initial positions. The road then follows the line C_0C while the bird's flight path is along V_0V (with V_0 on C_0C). We denote $r = C_0V_0$, and the speeds of the car and the bird by v_c and v_b. With this notation, we have $\dfrac{C_0C}{V_0V} = \dfrac{v_c}{v_b}$. If A is the point on line C_0C for which $V_0A = C_0C$, we then have

$$\frac{V_0A}{V_0V} = \frac{v_c}{v_b}.$$

Next, let B be the point for which both $V_0B \parallel VC$ and $C_0B \parallel VA$, let D be the point where V_0V meets C_0B, and let F be the foot of the perpendicular from V_0 to C_0B. Since $C_0V_0 = CA$, the triangles VCA and BV_0C_0 are congruent. Consequently, $V_0B = CV$, the distance between the car and the vulture at time t. Because $\triangle V_0DC_0 \sim \triangle V_0VA$, we have $\dfrac{V_0D}{V_0V} = \dfrac{V_0C_0}{V_0A}$; that is, $V_0D = \dfrac{V_0V}{V_0A} \cdot C_0V_0$. Hence,

$$V_0D = \frac{v_b}{v_c} \cdot r.$$

In words, the length of V_0D is independent of the vulture's flight path. It follows that for a given flight path, the minimum distance between the car and the bird occurs when B is at F (so that V_0B and VC are perpendicular to C_0B). We therefore want to determine the flight path for which V_0F is as large as possible. But, for any flight path, we have $V_0D \geq V_0F$; that is, $\frac{v_b}{v_c} \cdot r \geq V_0F$. We interpret this to mean that $\frac{v_b}{v_c} \cdot r$ is the maximum value of the minimum distance between the car and the vulture. I believe that any well-educated vulture would choose the flight path that maximizes the distance V_0F—the direction in which V_0D coincides with V_0F (so that

$V_0D \perp C_0B$). Of course, we then have

$$\cos\theta \;=\; \cos VV_0A \;=\; \cos C_0V_0D \;=\; \frac{V_0D}{C_0V_0}\,,$$

or

$$\sec\theta \;=\; \frac{r}{\frac{v_b}{v_c}\cdot r} \;=\; \frac{v_c}{v_b}\,,$$

which is the desired relation.

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and the proposer.

All other solvers used coordinates, an approach that produces a function to be maximized. The problem is therefore suitable as an exercise in a beginning calculus course, although the maximum can easily be obtained without derivatives. Demis added that he doubts that the vulture would resort to calculus. Indeed, it is doubtful that many vultures would exhibit the behaviour called for in our problem; however, dogs do seem to use an optimal strategy when fetching tennis balls according to the recent debate in The College Mathematics Journal, *34:3 (May 2003) 178–192, 37:1 (January 2006) 16–23, and 38:5 (November 2007) 356–361. In our problem, coordinates make clear what happens when the bird moves as fast or faster than the car. The proposer deduces that if the speeds are equal, the vulture must fly directly away from the car to maintain the minimum distance. When the bird is faster, the critical point of the distance function is irrelevant: the required minimum distance occurs when the bird takes flight, as long as it flies in a direction that has a large enough component away from the car.*

3220. [2007 : 111, 114] *Proposed by Marian Tetiva, Bîrlad, Romania.*

Let n be a positive integer. Prove that the set $\{1^2,\, 2^2,\, \dots,\, n^2\}$ of the first n perfect squares can be partitioned into four subsets each having the same sum of elements if and only if $n = 8k$ or $n = 8k-1$ for some integer $k \geq 2$.

Composite of similar solutions by Brian D. Beasley, Presbyterian College, Clinton, SC, USA; Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; and the proposer.

Let $P(n)$ denote the statement that the set $\{1^2,\, 2^2,\, \dots,\, n^2\}$ can be partitioned into four subsets each having the same sum of elements.

Suppose first that $P(n)$ is true. Then $4 \,\Big|\, \sum\limits_{m=1}^{n} m^2$, which implies that $24 \,\Big|\, n(n+1)(2n+1)$. Since $2n+1$ is odd and $(n, n+1) = 1$, we must have $8 \mid n$ or $8 \mid n+1$. Hence, $n = 8k$ or $n = 8k-1$ for some $k \geq 1$. However, since $\frac{1}{4}\sum\limits_{m=1}^{7} m^2 = 35 < 7^2$ and $\frac{1}{4}\sum\limits_{m=1}^{8} m^2 = 51 < 8^2$, we may eliminate $n = 7$ and $n = 8$ as possibilities. Therefore, $n = 8k$ or $n = 8k-1$ for some $k \geq 2$.

To prove that the condition is sufficient, we first show that if $P(n)$ is true, then so is $P(n+32)$. To see this, it suffices to show that for non-negative integers x, the set $S_x = \{x^2,\, (x+1)^2,\, \dots,\, (x+31)^2\}$ can be partitioned into four subsets each having the same sum of elements. Define

$$
\begin{aligned}
B_1 &= \{0,\ 7,\ 11,\ 12,\ 18,\ 21,\ 25,\ 30\}\,, \\
B_2 &= \{1,\ 6,\ \ 8,\ 15,\ 19,\ 20,\ 26,\ 29\}\,, \\
B_3 &= \{2,\ 5,\ \ 9,\ 14,\ 16,\ 23,\ 27,\ 28\}\,, \\
B_4 &= \{3,\ 4,\ 10,\ 13,\ 17,\ 22,\ 24,\ 31\}\,;
\end{aligned}
$$

and then define $A_i = \{(x+a)^2 \mid a \in B_i\}$ for $i = 1, 2, 3, 4$.

Then by tedious but straightforward calculations, we find that the sum of the elements in A_k equals $8x^2 + 248x + 2604$ for $k = 1, 2, 3, 4$. Since $S_x = A_1 \cup A_2 \cup A_3 \cup A_4$, our claim is established.

To complete the proof, it now suffices to show that the statements $P(n)$ is true for $n \in \{15, 16, 23, 24, 31, 32, 39, 40\}$.

Setting $x = 0$ and $x = 1$ in the partition given above, we see that $P(31)$ and $P(32)$ are true. For the six remaining values of n, squaring each element below gives the required partitions.

$$
\begin{aligned}
&\{1, 2, \ldots, 15\} \\
&\quad = \{1, 7, 8, 14\} \cup \{2, 9, 15\} \cup \{3, 6, 11, 12\} \cup \{4, 5, 10, 13\}\,; \\
&\{1, 2, \ldots, 16\} \\
&\quad = \{1, 6, 9, 16\} \cup \{2, 4, 8, 11, 13\} \cup \{3, 5, 12, 14\} \cup \{7, 10, 15\}\,; \\
&\{1, 2, \ldots, 23\} \\
&\quad = \{1, 3, 10, 13, 19, 21\} \cup \{2, 8, 22, 23\} \cup \{4, 5, 9, 11, 15, 17, 18\} \\
&\qquad\quad \cup \{6, 7, 12, 14, 16, 20\}\,; \\
&\{1, 2, \ldots, 24\} \\
&\quad = \{1, 5, 7, 10, 13, 15, 16, 20\} \cup \{4, 14, 22, 23\} \\
&\qquad\quad \cup \{2, 3, 11, 17, 19, 21\} \cup \{6, 8, 9, 12, 18, 24\}\,; \\
&\{1, 2, \ldots, 39\} \\
&\quad = \{1, 8, 11, 15, 16, 19, 27, 28, 35, 37\} \cup \{2, 5, 20, 29, 30, 38, 39\} \\
&\qquad\quad \cup \{3, 6, 7, 12, 13, 14, 17, 21, 22, 23, 26, 32, 33\} \\
&\qquad\quad \cup \{4, 9, 10, 18, 24, 25, 31, 34, 36\}\,; \\
&\{1, 2, \ldots, 40\} \\
&\quad = \{1, 6, 18, 20, 32, 34, 35, 37\} \cup \{2, 7, 14, 15, 16, 28, 30, 39, 40\} \\
&\qquad\quad \cup \{3, 4, 5, 9, 13, 21, 22, 27, 29, 36, 38\} \\
&\qquad\quad \cup \{8, 10, 11, 12, 17, 19, 23, 24, 25, 26, 31, 33\}.
\end{aligned}
$$

This completes the proof.

Also solved by PETER Y. WOO, Biola University, La Mirada, CA, USA. There was one incomplete solution submitted.

It should be pointed out that the partitions of the sets S_x and $\{1^2, 2^2, \ldots, n^2\}$ for $n \in \{15, 16, 23, 24, 31, 32, 39, 40\}$ are not unique in general; for example,

$$
\begin{aligned}
\{1^2, 2^2, \ldots, 23^2\} &= \{1^2, 5^2, 8^2, 9^2, 15^2, 18^2, 19^2\} \cup \{2^2, 7^2, 12^2, 20^2, 22^2\} \\
&\qquad \cup \{3^2, 4^2, 6^2, 11^2, 13^2, 17^2, 21^2\} \cup \{10^2, 14^2, 16^2, 23^2\}
\end{aligned}
$$

is an admissible partition different from the one given above.

3221. Correction. [2007 : 111, 114] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let ABC be a triangle with sides $a \geq b \geq c$ opposite the angles A, B, C, respectively. Let AH be perpendicular to the side BC with H on BC. Set $m = BH$ and $n = CH$. Prove that $a(bm + cn) - bc(b + c)$ is positive, negative, or zero according as $\angle A$ is obtuse, acute, or right-angled.

Essentially the same solution by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Charles R. Diminnie, Angelo State University, San Angelo, TX, USA; Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain; Francisco Javier García Capitán, IES Álvarez Cubero, Priego de Córdoba, Spain; Richard I. Hess, Rancho Palos Verdes, CA, USA; Geoffrey A. Kandall, Hamden, CT, USA; Vedula N. Murty, Dover, PA, USA; Xavier Ros, student, Universitat Politècnica de Catalunya, Barcelona, Spain; D.J. Smeenk, Zaltbommel, the Netherlands; and Peter Y. Woo, Biola University, La Mirada, CA, USA.

Since $m = c\cos B$ and $n = b\cos C$, we have

$$a(bm + cn) \;=\; abc(\cos B + \cos C)\,.$$

Using the Law of Cosines repeatedly, we obtain

$$\begin{aligned} a(bm+cn) - bc(b+c) &= abc(\cos B + \cos C) - bc(b+c) \\ &= b(ac\cos B) + c(ab\cos C) - bc(b+c) \\ &= b\frac{a^2+c^2-b^2}{2} + c\frac{a^2+b^2-c^2}{2} - \frac{2bc(b+c)}{2} \\ &= \frac{a^2b - bc^2 - b^3 + a^2c - b^2c - c^3}{2} \\ &= \frac{b+c}{2}(a^2 - b^2 - c^2) \\ &= -bc(b+c)\cos A\,. \end{aligned}$$

Therefore, the expression $a(bm + cn) - bc(b + c)$ is positive, negative, or zero if and only if $\cos A$ is negative, positive, or zero, respectively.

Also solved by ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (2 solutions); ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; KATRINA BRICKER and NATALIE KALMINK, students, California State University, Fresno, CA, USA; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; RICHARD I. HESS, Rancho Palos Verdes, CA, USA (second solution); JOE HOWARD, Portales, NM, USA; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; DRAGOLJUB MILOŠEVIĆ and G. MILANOVAC, Serbia; VEDULA N. MURTY, Dover, PA, USA; JOSÉ H. NIETO, Universidad del Zulia, Maracaibo, Venezuela; JOEL SCHLOSBERG, Bayside, NY, USA; SKIDMORE COLLEGE PROBLEM SOLVING GROUP, Skidmore College Saratoga Springs, NY, USA; D.J. SMEENK, Zaltbommel, the Netherlands (second solution); EDMUND SWYLAN, Riga, Latvia; TITU ZVONARU, Comănești, Romania; and the proposer.

3222. [2007 : 111, 115] *Proposed by Pham Van Thuan, Hanoi University of Science, Hanoi, Vietnam.*

Given positive real numbers a, b, c such that $a+b+c=1$, prove that

$$\frac{(1-a)(1-b)(1-c)}{\left(1-a^2\right)^2+\left(1-b^2\right)^2+\left(1-c^2\right)^2} \leq \frac{1}{8}\,.$$

Solution by Michel Bataille, Rouen, France.

We prove that $L \geq 8$, where

$$L = \frac{(1-a^2)^2+(1-b^2)^2+(1-c^2)^2}{(1-a)(1-b)(1-c)} = \sum_{\text{cyclic}} \frac{(1-a)(1+a)^2}{(1-b)(1-c)}\,.$$

From

$$(1+a)^2 = (1+1-b-c)^2 = \big((1-b)+(1-c)\big)^2 \geq 4(1-b)(1-c)\,,$$

and similar inequalities for $(1+b)^2$ and $(1+c)^2$, it follows that

$$L \geq 4(1-a)+4(1-b)+4(1-c) = 8\,.$$

Also solved by ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; ROY BARBARA, Lebanese University, Fanar, Lebanon; PAUL BRACKEN, University of Texas, Edinburg, TX, USA; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; HASAN DENKER, Istanbul, Turkey; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; KEE-WAI LAU, Hong Kong, China; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; D.M. MILOŠEVIĆ, Pranjani, Serbia; DRAGOLJUB MILOŠEVIĆ and G. MILANOVAC, Serbia; VEDULA N. MURTY, Dover, PA, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; GEORGE TSAPAKIDIS, Agrinio, Greece; VO QUOC BA CAN, Can Tho University of Medicine and Pharmacy, Can Tho, Vietnam; STAN WAGON, Macalester College, St. Paul, MN, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

3223. [2007 : 111, 115] *Proposed by Achilleas Pavlos Porfyriadis, student, American College of Thessaloniki "Anatolia", Thessaloniki, Greece.*

Let a, b, c be positive real numbers which satisfy

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} = \frac{1}{abc}\,.$$

Prove that

$$\frac{a}{a^2+1}+\frac{b}{b^2+1}+\frac{c}{c^2+1} \leq \frac{3\sqrt{3}}{4}\,.$$

Solution by Vo Quoc Ba Can, Can Tho University of Medicine and Pharmacy, Can Tho, Vietnam.

The given condition is equivalent to $ab + bc + ca = 1$. Thus,

$$\begin{aligned}\sum_{\text{cyclic}} \frac{a}{a^2+1} &= \sum_{\text{cyclic}} \frac{a}{a^2+ab+bc+ca} \\ &= \sum_{\text{cyclic}} \frac{a}{(a+b)(a+c)} = \frac{2}{(a+b)(b+c)(c+a)}\,.\end{aligned}$$

On the other hand, by the AM–GM Inequality, we have

$$\begin{aligned}(a+b)(b+c)(c+a) &= (a+b+c)(ab+bc+ca) - abc \\ &\geq \tfrac{8}{9}(a+b+c)(ab+bc+ca) = \tfrac{8}{9}(a+b+c) \\ &\geq \tfrac{8}{9}\sqrt{3(ab+bc+ca)} = \tfrac{8}{9}\sqrt{3}\,.\end{aligned}$$

Therefore,

$$\sum_{\text{cyclic}} \frac{a}{a^2+1} \leq \frac{3\sqrt{3}}{4}\,.$$

Equality holds if and only if $a = b = c = \frac{1}{\sqrt{3}}$.

Also solved by MOHAMMED AASSILA, Strasbourg, France; ARKADY ALT, San Jose, CA, USA; MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; MIHÁLY BENCZE, Brasov, Romania; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; HASAN DENKER, Istanbul, Turkey; JOE HOWARD, Portales, NM, USA; KEE-WAI LAU, Hong Kong, China; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; DRAGOLJUB MILOŠEVIĆ and G. MILANOVAC, Serbia; VEDULA N. MURTY, Dover, PA, USA; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; D.J. SMEENK, Zaltbommel, the Netherlands; PANOS E. TSAOUSSOGLOU, Athens, Greece; GEORGE TSAPAKIDIS, Agrinio, Greece; STAN WAGON, Macalester College, St. Paul, MN, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comăneşti, Romania; and the proposer.

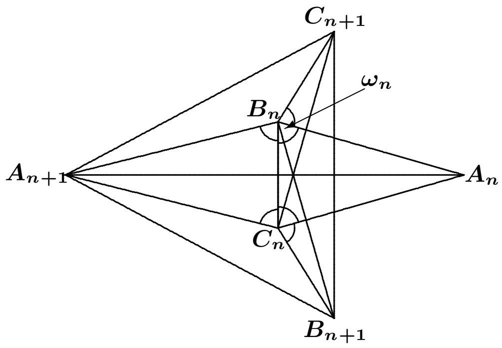
3224. [2007 : 112, 115] *Proposed by J. Chris Fisher and Harley Weston, University of Regina, Regina, SK.*

Let $A_0B_0C_0$ be an isosceles triangle whose apex angle A_0 is not 120°. We define a sequence of triangles $A_nB_nC_n$ in which $\triangle A_{i+1}B_{i+1}C_{i+1}$ is obtained from $\triangle A_iB_iC_i$ by reflecting each vertex in the opposite side (that is, B_iC_i is the perpendicular bisector of A_iA_{i+1}, and so forth). Prove that all three angles approach 60° as $n \to \infty$.

[*Ed.:* This problem is a special case of an open problem described by Judah Schwartz in "Can technology help us make the mathematics curriculum intellectually stimulating and socially responsible?", *International Journal of Computers for Mathematical Learning*, 4 (1999), pp. 99–119.]

Solution by Apostolis K. Demis, Varvakeio High School, Athens, Greece, modified by the editor.

We start with the observation that the problem, as stated, is incorrect. The condition $\angle A_0 \neq 2\pi/3$ does not guarantee that the angles of $\triangle A_nB_nC_n$ approach $\pi/3$ as $n \to \infty$. In what follows, we will assume that $\angle A_0$ is such that $\angle A_n \neq 2\pi/3$ *for all* n. [*Ed.:* This immediately evokes the important question of whether such angles A_0 exist. Perhaps, our readers will share some insights on the set of "prohibited values" for $\angle A_0$; we will also say a bit more on that later. At this point, we will just assume that this set is not the entire interval $(0, \pi)$].



Let ω_n denote the angle $C_nB_nA_n$, $j_n = \cos 2\omega_n + i\sin 2\omega_n$ (then $\overline{j_n} = \cos 2\omega_n - i\sin 2\omega_n$), $x_n = \dfrac{B_nC_n}{A_nB_n}$, and a_n, b_n, and c_n be the complex numbers representing the points A_n, B_n, and C_n, respectively. An easy induction shows that $\triangle A_nB_nC_n$ is isosceles and that $0 < \omega_n < \frac{\pi}{2}$. Then $\angle C_nB_nC_{n+1} = \angle A_{n+1}B_nA_n = \angle B_{n+1}C_nB_n = 2\omega_n$, and

$$(a_n - c_n)j_n = b_n - a_n\,, \tag{1}$$
$$(c_n - b_n)j_n = c_{n+1} - b_n\,, \tag{2}$$
$$(a_{n+1} - b_n)j_n = a_n - b_n\,, \tag{3}$$
and
$$(b_{n+1} - c_n)j_n = b_n - c_n\,. \tag{4}$$

From equations (1), (4), and (2), respectively, we obtain

$$a_n = \frac{1}{j_n+1}b_n + \frac{j_n}{j_n+1}c_n\,, \tag{5}$$
$$b_{n+1} = \frac{1}{j_n}b_n + \frac{j_n-1}{j_n}c_n\,, \tag{6}$$
$$c_{n+1} = (1-j_n)b_n + j_nc_n\,, \tag{7}$$

and from (3) and (5), we get

$$a_{n+1} = \frac{j_n}{j_n+1}b_n + \frac{1}{j_n+1}c_n\,. \tag{8}$$

We have $\cos\omega_n = \dfrac{B_nC_n}{2A_nB_n}$ or $x_n = 2\cos\omega_n$. Clearly, $0 < x_n < 2$.

Using $j_n\overline{j_n} = 1$ and equations (6), (7), and (8), we obtain a recurrence for the sequence $\{x_n\}$:

$$\begin{aligned} x_{n+1} &= \frac{B_{n+1}C_{n+1}}{A_{n+1}B_{n+1}} = \frac{|c_{n+1}-b_{n+1}|}{|b_{n+1}-a_{n+1}|} = \frac{\left|\dfrac{j_n^2-j_n+1}{j_n}(c_n-b_n)\right|}{\left|\dfrac{j_n^2-j_n-1}{j_n(j_n+1)}(c_n-b_n)\right|} \\ &= \frac{|(j_n+1)(j_n^2-j_n+1)|}{|j_n^2-j_n-1|} = \frac{|j_n+1|\cdot|(j_n^2-j_n+1)\overline{j_n}|}{|(j_n^2-j_n-1)\overline{j_n}|} \\ &= \frac{|j_n+1|\cdot|(j_n+\overline{j_n}-1)|}{|(j_n-\overline{j_n}-1)|} \\ &= \frac{|\cos 2\omega_n+1+i\sin 2\omega_n|\cdot|2\cos 2\omega_n-1|}{|-1+2i\sin 2\omega_n|} \\ &= \frac{\sqrt{2+2\cos 2\omega_n}\cdot|2\cos 2\omega_n-1|}{\sqrt{1+4\sin^2 2\omega_n}} \\ &= \frac{\sqrt{2(1+\cos 2\omega_n)}\cdot|2(2\cos^2\omega_n-1)-1|}{\sqrt{1+16\sin^2\omega_n\cos^2\omega_n}} \\ &= \frac{2\cos\omega_n\cdot|4\cos^2\omega_n-3|}{\sqrt{1+16\cos^2\omega_n-16\cos^4\omega_n}} = \frac{x_n\cdot|x_n^2-3|}{\sqrt{1+4x_n^2-x_n^4}}. \end{aligned}$$

Define the functions $f : [0,2] \to [0,2]$ as $f(x) = \dfrac{x\cdot|x^2-3|}{\sqrt{1+4x^2-x^4}}$ and $g : [0,2] \to [0,4]$ as $g(x) = \big(f(x)\big)^2 = \dfrac{x^2(x^2-3)^2}{1+4x^2-x^4}$. Then $x_{n+1} = f(x_n)$. Observe that if the sequence $\{x_n\}$ has a limit L, then $L = \dfrac{L\cdot|L^2-3|}{\sqrt{1+4L^2-L^4}}$, which gives 0, 1, and 2 as possible values for L.

We can now determine the "prohibited values" for x_0. Clearly, the equation $f(x) = \sqrt{3}$ has a solution x_0 in $(0,2)$ (by the Intermediate Value Theorem, since $f(0) = 0$, $f(2) = 2$, and f is continuous on $[0,2]$). Then $x_1 = f(x_0) = \sqrt{3}$, and the geometry problem does not make sense. Similarly, it does not make sense if x_0 is a solution of $f^n(x) = \sqrt{3}$, where $f^n(x)$ denotes $\underbrace{f(f(\ldots f}_{n \text{ times},\ n\geq 1}(x)))$. Hence, the set of "prohibited values" for x_0 is $\{x \mid f^n(x) = \sqrt{3}\} \cap (0,2)$.

Further, we have

$$g'(x) = \frac{-2x(x^2-3)(x^2+1)\left(x^2-(3-\sqrt{6})\right)\left(x^2-(3+\sqrt{6})\right)}{(1+4x^2-x^4)^2},$$

so that both functions g and f are strictly increasing on $\left[0,\sqrt{3-\sqrt{6}}\right]$ and $\left[\sqrt{3},2\right]$, and strictly decreasing on $\left[\sqrt{3-\sqrt{6}},\sqrt{3}\right]$.

Now consider the difference $g(x) - x^2 = \dfrac{2x^2\left(x^2-1\right)\left(x^2-4\right)}{1+4x^2-x^4}$. Since $1+4x^2-x^4 = 5-(x^2-2)^2 > 0$ for $x \in [0,2]$, we have $g(x) - x^2 > 0$ for every $x \in (0,1)$ and $g(x) - x^2 < 0$ for every $x \in (1,2)$. Hence, $f(x) > x$ for every $x \in (0,1)$, $f(x) < x$ for every $x \in (1,2)$, and $f(x) = x$ if and only if $x = 0$, $x = 1$, or $x = 2$.

Let $a = \sqrt{3-\sqrt{6}}$. Since $a < 1$, we have $a < f(a)$. Let $I = \left[a, f(a)\right]$. It is easy to check that $1 < f(a) < \sqrt{3}$. Thus, we have $f\big(f(a)\big) < f(a)$, since $f(a) > 1$. Since f is strictly decreasing and continuous on I and $I \subseteq \left[\sqrt{3-\sqrt{6}}, \sqrt{3}\right]$, we see that $f(I) = \left[f\big(f(a)\big), f(a)\right]$, which implies that $f(I) \subseteq I$, since $f\big(f(a)\big) = \sqrt{\frac{5346}{1345} - \frac{1674}{1345}\sqrt{6}} > a$. Therefore, $a < f\big(f(a)\big) < 1 < f(a) < \sqrt{3}$.

We are now going to show that, for each $x_0 \in (0,2)$ which is not "prohibited", there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} \in I$. Then it will follow that $x_n \in I$ for every $n \in \mathbb{N}$ with $n \geq n_0$, since $f(I) \subseteq I$.

Suppose, for the purpose of contradiction, that the sequence $\{x_n\}$ has the following property:

$x_n \notin I$ for every $n \in \mathbb{N}$.

Let us denote this property by $\mathcal{P}$. We consider several cases.

Case 1. $x_0 < a$.

Then (by induction) we obtain $x_n < a$ for every $n \in \mathbb{N}$, since f is strictly increasing on $[0,a]$. (If $x_k < a$ for some $k \in \mathbb{N}$, then $f(x_k) < f(a)$, or $x_{k+1} < f(a)$. By property $\mathcal{P}$, we get $x_{k+1} < a$.) But then we have $x_n < x_{n+1}$ for every $n \in \mathbb{N}$, since $x < f(x)$ for every $x \in (0,1)$. Thus, the sequence $\{x_n\}$ is increasing and bounded above, and therefore it has a limit, L. Recall that the possible values of L are 0, 1, and 2. Since $x_n \in (0,a)$ and $a < 1$, then L must be 0. This is a contradiction, because an increasing sequence with terms in $(0,a)$ cannot have limit 0.

Case 2. $f(a) < x_0 < \sqrt{3}$.

Then $f(x_0) < f\big(f(a)\big)$, or $x_1 < f\big(f(a)\big)$, since the function f is strictly decreasing on $\left[a, \sqrt{3}\right]$. By property $\mathcal{P}$, we get $x_1 < a$. But in this case, by induction, we get $x_n < a$ for every $n \in \mathbb{N}$ with $n \geq 1$, which leads us to a contradiction, similar to the one obtained in Case 1.

Case 3. $\sqrt{3} \leq x_0$ and there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} < \sqrt{3}$.

Then $x_n < a$ for every $n \in \mathbb{N}$ with $n \geq n_0 + 1$, and we again obtain a contradiction, similar to those obtained in Cases 1 and 2.

Case 4. $\sqrt{3} \leq x_n$ for every $n \in \mathbb{N}$.

Then we have $f(x_n) < x_n$, or $x_{n+1} < x_n$ for every $n \in \mathbb{N}$, since $f(x) < x$ for every $x \in (1,2)$. Hence, the sequence $\{x_n\}$ is strictly decreasing and therefore, there exists $\lim x_n = L$, with $L \in \left[\sqrt{3}, 2\right]$. The only possible value for L is 2, but this is a contradiction, because a decreasing sequence with terms in $[\sqrt{3}, 2)$ cannot have limit 2.

Consequently, the assumption that the sequence $\{x_n\}$ has property $\mathcal{P}$ leads to a contradiction. Therefore, it is true that there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} \in I$ and then $x_n \in I$ for every $n \in \mathbb{N}$ with $n \geq n_0$, since $f(I) \subseteq I$. Now define the sequence $\{y_n\}$ as $y_n = x_{n+n_0}$ for every $n \in \mathbb{N}$. Then $y_n \in I$ for every $n \in \mathbb{N}$. Define the function $h : I \to \mathbb{R}$ with $h(x) = f\big(f(x)\big)$. The function h is strictly increasing on I, since f is strictly decreasing on I. We have $y_{2n+2} = h(y_{2n})$ and $y_{2n+3} = h(y_{2n+1})$ for every $n \in \mathbb{N}$. The sequences $\{y_{2n}\}$ and $\{y_{2n+1}\}$ are monotone. (For example, if $y_0 \leq y_2$, then $y_2 = h(y_0) \leq h(y_2) = y_4$, etc.; an easy induction completes the proof that $\{y_{2n}\}$ is increasing.) Both sequences $\{y_{2n}\}$ and $\{y_{2n+1}\}$ are bounded, because $y_n \in I$. Therefore, there exist L_1, $L_2 \in I$ such that $\lim y_{2n} = L_1$ and $\lim y_{2n+1} = L_2$. Observe that

$$L_1 \;=\; \lim y_{2n+2} \;=\; \lim h(y_{2n}) \;=\; h(\lim y_{2n}) \;=\; h(L_1)\,.$$

Similarly, $L_2 = h(L_2)$. In fact, $L_1 = L_2 = 1$, as $x = 1$ is the only root in I of $h(x) = x$. To see this, let $b \in I \setminus \{1\}$ satisfy $h(b) = b$. By the Mean-Value Theorem, $b - f(b) = f(f(b)) - f(b) = f'(x)(f(b) - b)$ for some x between b and $f(b)$. Since $b \neq f(b)$ (as f has no fixed points in $I \setminus \{1\}$), we have $f'(x) = -1$ and $x \in I$. Then $f'(x) = -1$ reduces to $\ell(x) = r(x)$, where $\ell(x) = x^6 + 3 + \big(5 - (x^2-2)^2\big)^2$ and $r(x) = x^2(5x^2+3)$. Now $\ell(x)$ and $r(x)$ are strictly increasing on I, $\ell(a) > r(1)$ (a long calculation), and $\ell(1) > r(f(a))$, and therefore $\ell(x) > r(x)$ on I, a contradiction. Thus $x = 1$ is the only root in I of $h(x) = x$.

Finally, we have

$$\begin{aligned} \lim y_n = 1 \quad &\Longleftrightarrow \quad \lim x_n = 1 \\ &\Longleftrightarrow \quad \lim \cos\omega_n = \tfrac{1}{2} \\ &\Longleftrightarrow \quad \lim \omega_n = \tfrac{\pi}{3}\,, \end{aligned}$$

which completes the proof.

There were also two incomplete solutions submitted.

3225★. [2007 : 112, 115, 297] *Proposed by George Tsapakidis, Agrinio, Greece.*

The sides ℓ and m of an acute angle α with vertex A intersect the sides of a fixed acute angle β with vertex B in four distinct points P, Q, R, and S, labelled so that P lies between A and Q and also between B and S.

(a) If the measure of $\angle\alpha$ is fixed, can A and ℓ be chosen so that

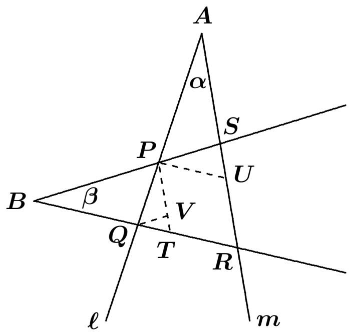
$$[PBQ] + [APS] \;=\; [PQRS]\,,$$

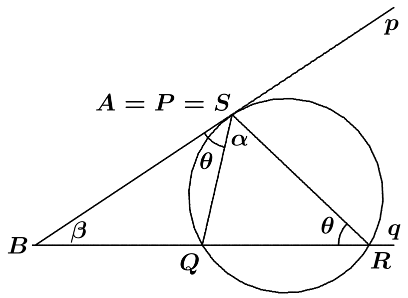
where $[XYZ]$ denotes the area of polygon XYZ?

(b) When are the lines ℓ and m constructible with Euclidean tools to satisfy the condition in part (a) for a given fixed value of α?

Solution to part (a) by Peter Y. Woo, Biola University, La Mirada, CA, USA, modified by the editor.

We shall investigate the degenerate case in which the points A, P, and S coincide on one side of the angle whose vertex is B. Since $[APS] = 0$ for the degenerate configuration, the area condition becomes $[ABQ] = [AQR]$, and Q is necessarily the mid-point of BR.





Claim 1. To any non-degenerate configuration that satisfies the area condition there corresponds a degenerate configuration that likewise satisfies the area condition.

Proof: Let the parallel to QR through P meet SR at U, and the parallel to SR at P meet QR at T; thus, $PTRU$ is a parallelogram inside $PQRS$. Moreover, let V be the point on PT where the parallel to PS through Q meets PT. Then,

$$\frac{AS}{SU} = \frac{PV}{VT} = \frac{BQ}{QT};$$

the first equality follows because the triangles APU and PQT (as well as their corresponding cevians PS and QV) are homothetic, and the second because VQ is parallel to the base BP of $\triangle BTP$. It follows that $TQ \leq BQ$ and $US \leq SA$; otherwise, should $TQ > BQ$ and $US > SA$, we would have $[PQRS] > [PQT] + [PUS] > [PBQ] + [APS]$, contrary to the area condition. Note that $\angle QPT = \alpha$. Rotate this angle about P in either direction (sliding Q and T along the line BR so that $\angle QPT$ remains equal to α): in one direction the distance between Q and B shrinks to zero, while in the other the distance between Q and T grows without bound. Either way there is a position in which Q is the mid-point of BT. In those positions, the points P, B, Q, and T play the roles of A, B, Q, and R in a degenerate configuration with angle α at A and β at B, in which the area condition $[ABQ] = [AQR]$ is satisfied. ■

Claim 2. There exists a degenerate configuration that satisfies the area condition if and only if

$$\alpha \leq 2\tan^{-1}\left[\left(3 - 2\sqrt{2}\right)\cot\frac{\beta}{2}\right].$$

It is constructible with Euclidean tools.

Proof: Denote the rays that bound the given angle β by p and q. The point R can be arbitrarily chosen on q, while Q is defined to be the mid-point of BR. Construct the circular arc QXR for which $\angle QXR = \alpha$ (on the same side of BR as p). This circle will intersect p in two, one, or zero points. In the first case, either of those points can be labeled A to obtain the desired configuration; in the last case, no such configuration can exist. The second case, where the circle is tangent to p, therefore provides the maximum permissible value of α for the given β. We define A to be the point where the circle QXR is tangent to p. It follows that $BA^2 = BQ \cdot BR = 2BQ^2$; whence,

$$\frac{BR}{BA} = \frac{BA}{BQ} = \sqrt{2}\,.$$

Moreover, $\angle BAQ = \angle ARQ$ equals some value θ, say. Then, in $\triangle BRA$, we have $2\theta = \pi - \alpha - \beta$. Also, the Sine Law applied to that triangle gives

$$\frac{\sin(\alpha+\theta)}{\sin\theta} = \frac{BR}{BA} = \sqrt{2}\,.$$

Replace θ in this last equation by $(\pi - \alpha - \beta)/2$ to get

$$\cos\frac{\alpha-\beta}{2} = \sqrt{2}\cos\frac{\alpha+\beta}{2}\,,$$

or

$$\cos\frac{\alpha}{2}\cos\frac{\beta}{2} + \sin\frac{\alpha}{2}\sin\frac{\beta}{2} = \sqrt{2}\left(\cos\frac{\alpha}{2}\cos\frac{\beta}{2} - \sin\frac{\alpha}{2}\sin\frac{\beta}{2}\right)\,.$$

Dividing both sides by the cosines yields

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2} = \frac{\sqrt{2}-1}{\sqrt{2}+1} = 3 - 2\sqrt{2}\,,$$

which provides the promised extreme value of α. ∎

[*Editor's comments.* The above argument provides a technically sound solution to the problem. We note that there exists a procedure, however, that turns a degenerate configuration into a non-degenerate one having the same α and β:

Claim 3. If $\alpha < 2\tan^{-1}\left[(3-2\sqrt{2})\cot\frac{\beta}{2}\right]$, then there is a non-degenerate configuration that satisfies the area condition.

Proof: Because of the strict inequality for α, the circle QXR in the proof of Claim 2 intersects the ray p in two points. Define P to be any point of p between those two points. Define a new position of Q on BR such that $\angle QPR = \alpha$. Since now we have $BQ > QR$, we also have $[PBQ] > [PQR]$. It remains to find a new position for R so that the line through it that makes an angle of α with PQ will intersect PQ in A, and p in S, in such a way that the area condition is satisfied. Such a position for R will exist because the area $[PQRS]$ grows faster than $[APS]$ as R moves away from B on the line BQ: at its initial position $[PBQ]+[APS]=[PBQ]>[PQR]=[PQRS]$, while the inequality is reversed when R is sufficiently removed from Q. ∎

Claim 1 tells us that the condition on α is necessary, so our construction provides a solution to part (a). It provides no information, however, about the constructability of the configuration; therefore, part (b) remains open in the non-degenerate case.]

No other solutions were received; in particular, there was no completely satisfactory solution to part (b) *submitted.*

3227. [2007 : 169, 172] *Proposed by Ovidiu Furdui, University of Toledo, Toledo, OH, USA.*

Let $\alpha \in [0,1]$ and define

$$x_n = \left(\frac{\zeta(2)+\cdots+\zeta(n+1)}{n}\right)^{n^\alpha},$$

where ζ is the Riemann Zeta Function, defined by $\zeta(k)=\sum\limits_{p=1}^{\infty}\frac{1}{p^k}$. Prove that

$$\lim_{n\to\infty} x_n = \begin{cases} 1, & \text{if } \alpha\in[0,1), \\ e, & \text{if } \alpha = 1. \end{cases}$$

Solution by Michel Bataille, Rouen, France.

Note first that

$$\begin{aligned}
&\zeta(2)+\zeta(3)\cdots+\zeta(n+1) \\
&\quad= \sum_{p=1}^{\infty}\left(\frac{1}{p^2}+\frac{1}{p^3}+\cdots+\frac{1}{p^{n+1}}\right) = \sum_{p=1}^{\infty}\frac{1}{p^2}\left(1+\frac{1}{p}+\cdots+\frac{1}{p^{n-1}}\right) \\
&\quad= n+\sum_{p=2}^{\infty}\frac{1}{p^2}\cdot\frac{1-\frac{1}{p^n}}{1-\frac{1}{p}} = n+\sum_{p=2}^{\infty}\frac{p^n-1}{p^{n+1}(p-1)}\,. \qquad (1)
\end{aligned}$$

Since $\dfrac{p^n-1}{p^{n+1}(p-1)}=\dfrac{1}{p(p-1)}-\dfrac{1}{p^{n+1}(p-1)}$ and

$$\sum_{p=2}^{\infty}\frac{1}{p(p-1)} = \sum_{p=2}^{\infty}\left(\frac{1}{p-1}-\frac{1}{p}\right) = 1\,,$$

we obtain

$$\sum_{p=2}^{\infty}\frac{p^n-1}{p^{n+1}(p-1)} = 1-\sum_{p=1}^{\infty}\frac{1}{p(p+1)^{n+1}}\,. \qquad (2)$$

From (1) and (2), we have $\ln(x_n) = n^\alpha \ln\left(1 + \frac{1}{n} - \frac{1}{n}\sum_{p=1}^{\infty} \frac{1}{p(p+1)^{n+1}}\right)$.

Since $f(x) = 1/x^{n+1}$ is concave up on $[1, \infty)$, we have

$$0 \le \sum_{p=1}^{\infty} \frac{1}{p(p+1)^{n+1}} \le \sum_{p=1}^{\infty} \frac{1}{(p+1)^{n+1}} \le \int_1^{\infty} \frac{dx}{x^{n+1}} = \frac{1}{n}.$$

Hence, $\ln(x_n) = n^\alpha \ln\left(1 + \frac{1}{n} + o\left(\frac{1}{n}\right)\right) \sim n^{\alpha-1}$ as $n \to \infty$.

[*Ed.:* Following the usual convention, $f = o(g)$ means $f/g \to 0$ as $n \to \infty$ and $f \sim g$ means $f/g \to 1$ as $n \to \infty$. See, for example, page 89 of the book: *A Course of Pure Mathematics* by G.H. Hardy.]

This yields

$$\lim_{n\to\infty} \ln(x_n) = \begin{cases} 1, & \text{if } \alpha = 1, \\ 0, & \text{if } \alpha \in [0, 1), \end{cases}$$

and the result follows immediately.

Also solved by ARKADY ALT, San Jose, CA, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; XAVIER ROS, student, Universitat Politècnica de Catalunya, Barcelona, Spain; JOEL SCHLOSBERG, Bayside, NY, USA; and the proposer.