BOOK REVIEWS

John Grant McLoughlin

How Euler Did It
By C. Edward Sandifer, Mathematical Association of America, 2007
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*Reviewed by **J. Chris Fisher**, University of Regina, Regina, SK*

How Euler Did It brings together the 40 essays that appeared as monthly columns under this title from November 2003 to February 2007 on *MAA Online* (`http://www.maa.org/news/howeulerdidit.html`). The essays average about 5 pages in length. Most begin with a short introduction that provides the historical and mathematical setting and explains why Sandifer finds the month's topic interesting. Then comes an account of some aspect of Euler's work, usually using Euler's notation and sometimes even his own words in translation. The author gently guides the reader through arguments that might involve little more than high-school tools, but display a brilliance that demands some effort from the reader. The exposition maintains a light-hearted tone and would appeal to anybody with a mathematical bent, while the details are kept to a level that is suitable for any reader of ***CRUX with MAYHEM***. Each essay concludes with a list of references that always includes the original source, which can be found on-line in the *Euler Archive* (`http://www.eulerarchive.org`).

A brief biographical note informs us that Sandifer can read the works of Euler in their original Latin, French, and German. Evidently he is familiar with many of those works, having also written *The Early Mathematics of Leonhard Euler* (also published by the MAA) and helped edit *Leonhard Euler: Life, Work and Legacy*. The author does an admirable job of providing background information about 18^{th} century mathematics and mathematicians; he reminds us more than once that Euler used 18^{th} century techniques and wrote for an 18^{th} century audience. This gives him an opportunity to contrast Euler's methods with today's and to describe the Euler impact.

The first essay presents an overview of Euler's work, while the last is a report on a meeting of the Euler Society. The other 38 have been grouped into four parts: 6 on geometry, another 6 on number theory, 5 on combinatorics, and 21 on analysis. Three of the essays that were placed under the analysis heading—probably by a publisher's error—deal with scientific applications which, we learn, occupied much of Euler's time and effort.

Many of the essays were inspired by questions raised by members of the Euler Society. For example, why do some sources credit Lambert with the first proof that e is irrational while others credit Euler? The answer was not easily found: there are 866 items in the authoritative Eneström list of Euler's publications, many of which are rather large books, and there are also 1000 of his letters extant. It turned out that Euler really did prove that e is irrational, although the details are somewhat buried in a long paper on continued fractions from 1737.

Another pair of essays inspired by the Euler Society involve the "Euler formula" $V - E + F = 2$. Some half-truths have arisen concerning what Euler did and what he proved. To uncover the whole truth, Sandifer had to study two papers of Euler together with a paper of Descartes from 1649, a century before Euler's work, which established a formula that is equivalent to Euler's.

Some of Sandifer's stories are based on items he just stumbled across while looking for something else. One example of this is a 1770 paper in which Euler anticipated orthogonal matrices some 80 years before the concept came into mathematics. Euler failed to say what he was thinking when he formulated the problem that led him to these results.

I learned something from every essay, even those on topics about which I already knew something. I learned about the importance of letters for the exchange of ideas in the 18^{th} century. I learned that number theory attracted little interest at the time, so that the impact of Euler's substantial work in the area was delayed for a generation or two. ***CRUX with MAYHEM*** readers might recall G.C. Shephard's article on Euler's triangle theorem [1999 : 148–153]: *If O is any point in the plane of triangle ABC that does not lie on a side, and D, E, and F are the feet of the cevians AO, BO, and CO, respectively, then*

$$\frac{AO}{OD} \cdot \frac{BO}{OE} \cdot \frac{CO}{OF} = \frac{AO}{OD} + \frac{BO}{OE} + \frac{CO}{OF} + 2\,,$$

where the quotients are positive when O is between the vertex and foot, and negative otherwise. Euler provided his first proof of the theorem in 1780, just three years before he died. (In his version of the story, Sandifer raises the question of how a person who has been blind for some 15 years could possibly discover such a theorem! He can explain only that Euler had assistants who wrote down his proofs.) Both Shephard and Sandifer complain of the awkwardness of that first proof. Sandifer, however, provides Euler's beautiful alternative proof that Euler seems to have appended to his paper at a later date. He speculates that Euler lacked sufficient time before his death to revise and polish this paper; thus, it gives us a glimpse of how Euler discovered things as he wrote a paper and how he came back later to improve his solutions.

All forty columns are pleasant and informative. The book can be enjoyed whether you read it from cover to cover or browse through random essays. But should you buy the book? After all, the whole thing (and more recent material) is readily available on-line. On the one hand, it is surely a good idea to support the MAA (Mathematical Association of America), which published the book. Besides, the book would be an excellent resource for teachers, either for directed reading or for term projects in any number of courses. It would make a great gift for anybody who loves math. On the other hand, the MAA could have and should have done a much better job of editing the manuscript. Although Sandifer thanks the members of the editorial board for their "conscientious and rigorous editing," it looked to me as if they did little more than download the computer files and add a table of contents and index. There are just too many typos—nothing particularly serious, but just what one would expect from an on-line document, not a published text. More serious was the clumsy cross-referencing. Sandifer often refers to other columns by date and title, which is great when one has handy computer links, but inconvenient in a 200-page book. Page numbers, or even chapter numbers, should have been provided for each cross-reference. Moreover, the introductory remarks from many of the columns need revising. In essay 3 he repeats what he said in the previous month's column; in the book that means that consecutive pages contain identical paragraphs. It would have been easy for an editor to deal with these minor problems. We should expect more for our money.

Nonplussed! Mathematical Proof of Implausible Ideas
By Julian Havil, published by Princeton University Press, 2007
ISBN 0-691-12056-0, hardcover, 196+xiii pages, US$24.95
Reviewed by ***Robert D. Poodiack****, Norwich University, Northfield, VT, USA*

Among my extracurricular activities, I travel to Vermont high schools to give talks about paradoxes in probability. Thanks to Julian Havil's wonderful book, I've just found several new ways to expand my lectures.

The 14 chapters of *Nonplussed!* deal primarily with paradoxes in probability and combinatorics, but other examples are pulled in from calculus, mechanics, number theory, and game theory. Many will be familiar to experienced mathematics readers. The Birthday Problem, derangements (permutations in which there is no fixed point), Buffon's needle, and Torricelli's Trumpet (also known as Gabriel's horn—the surface of revolution for $y = 1/x$ about the x-axis for $x \geq 1$) all make appearances.

The chapters on the slightly less-known topics were thoroughly enjoyable. The mathematical work is pitched at the level of an undergraduate mathematics journal, a decision that shows why Havil is a master teacher. Havil's conversational tone entices the reader into areas which may be entertaining but, a few chapters in, quite confounding. The first chapter on "Three Tennis Paradoxes" is thick with algebra and the level of mathematics takes off from there. Havil notes in his introduction that the difficulty level generally rises from chapter to chapter and the reader moves from algebra and basic probability through trigonometry, differential and integral calculus, sequences and series, combinatorics, generating functions, differential equations, and modular arithmetic. Havil optimistically writes that "none of it is beyond a committed senior high school student," but university-level students with some mathematical experience might better appreciate the articles. (I will have to wave my hands over some areas for my high school audiences.) However, Havil never forgets that each problem stems from a great story and even novices will be attracted by the histories of the 14 problems, one associated with each chapter.

In the best chapters, Havil takes a topic with which many readers might be familiar and summarizes or describes it from a new angle. The most accessible may be Chapter 3 on the Birthday Problem.

Many of us learned about this problem as either an amazing fact (it takes a gathering of only 23 people to have a better than 50% chance of having two people with the same birthday) or a lesson in computing probabilities via complements. Havil expands the problem in two interesting directions. First, he dispenses with a misstatement of the problem—how many people are needed to have a better than 50% chance of someone having the same birthday as you? (The answer: a lot more than 23.) After proving the usual result, Havil generalizes the Birthday Problem in various directions. How many people are needed in a group to have three, four, or more with the same birthday? How many people are needed to have more than two birthdays separated by a certain number of days? The answer, via Paul Halmos, brings in the Arithmetic Mean–Geometric Mean Inequality.

Havil concludes the chapter with an application of the Birthday Problem to identifying web browsers and computers. This is just one example of how Havil connects historical problems to modern applications. In the chapter on Parrondo Games—winning games that are constructed from two losing games—Havil mentions a demonstration in a New York Times article from 2000 of how two losing stock portfolios can be combined into a winning one.

As with the best classroom teachers, Havil's enthusiasm for the material carries the reader even through the tough spots. The chapter on hyperdimensions is quite stunning in its rigour, but Havil continues to pull ever more amazing facts and identities from his bag of tricks. I questioned some of the techniques Havil introduced here for their lack of physical meaning, but I became a believer when I saw Gelfond's constant (e^{π}), the Gamma function, and the double factorial being introduced to a new audience.

I'm still not sure I understand the closing chapter on John Conway's Fractran programming language, after several attempts to do so and even after reading Conway's own exposition in his *Book of Numbers*, written with Richard Guy. Havil so obviously enjoys the material, though, that I want to keep trying! (The chapter on Conway's checkerboard game is much easier to digest.)

Havil's presentation on Fractran follows Guy's 1983 article from *Mathematics Magazine*. Most of the chapters explain, interpret, and synthesize previously published articles from other authors. Havil cites all his sources throughout the text, but my one wish would have been for a united bibliography somewhere in the book.

Nonplussed! is a wonderful collection of astounding paradoxes and stunning turnarounds. Although the mathematics can get fairly dense, the brimming delight Havil has for his material repays the work readers must invest. Fans of puzzle, paradox, or game books should find much to enjoy. I look forward to sharing a lot of this with Vermont high school students in the next few years.