THE OLYMPIAD CORNER
No. 267

R.E. Woodrow

How the time seems to fly by! Here it is another year past and the start of a new volume of *CRUX with MAYHEM*, with a new in-coming Editor-in-Chief—Václav (Vazz) Linek—coming on board to take over leadership. I would like to take this opportunity to express my particular thanks to Jim Totten, the out-going Editor-in-Chief and to his Associate Editor, Bruce Crofoot, for the careful proof-reading and correction of the typos and errors I let slip through. It is also appropriate, I think, to thank those who have contributed problem sets, comments, solutions and generalizations for our use in the *Corner*.

Houda Anoun
Miguel Amengual Covas
Mohammed Aassila
Michel Bataille
Robert Bilinski
David Bradley
Pierre Bornsztein
Ricardo Barroso Campos
Bruce Crofoot
José Luis Díaz-Barrero
J. Chris Fisher
Ovidiu Furdui
Joan P. Hutchinson
Geoffrey A. Kandall
Ioannis Katsikis
Matti Lehtinen
Andy Liu
Pavlos Maragoudakis
Vedula N. Murty
Henry Ricard
D.J. Smeeenk
Christopher Small
Jim Totten
Edward T.H. Wang
Li Zhou

And a special thanks to Joanne Canape, whose skill at turning my scribbles, notes, requests, and pleas into a clean well-presented *LaTeX* file, usually on short notice, continues to amaze me.

As a first problem set this number, we give the problems of the Italian Team Selection Test given at Pisa in May 2005. My thanks go to Felix Recio, Canadian Team Leader to the IMO in Mexico, for collecting them for our use.

ITALIAN TEAM SELECTION TEST
20–21 May 2005, Pisa

1. A stage course is attended by \(n\) students \((n \geq 4)\). The day before the final test, each group of three students conspires against another student, to throw him or her out of the competition. Prove that there is a student against whom there are at least \(\sqrt{(n-1)(n-2)}\) conspirators. [From Slovenia 2004.]
2. (a) Prove that in a triangle the sum of the distances from the centroid to the three sides is greater than or equal to three times the radius of the incircle. Determine when exact equality holds.

(b) Determine the points in a triangle such that the sum of the distances from the sides is minimal.

3. For a positive integer $n$, let $\psi(n) = \sum_{k=1}^{n} \gcd(k, n)$.

(a) Prove that $\psi(mn) = \psi(m) \psi(n)$ for $m$ and $n$ relatively prime positive integers.

(b) Prove that, for every positive integer $a$, the equation $\psi(x) = ax$ has at least one solution.

[From IMO Short List 2004.]

4. Let $S_n = \{1, 2, \ldots, n\}$. Let $f : S_{1600} \to S_{1600}$ be such that

$$f^{(2005)}(x) = x, \quad \text{for } x = 1, 2, \ldots, 1600 \text{ and } f(1) = 1. \quad (1)$$

(a) Prove that $f$ has at least another fixed point.

(b) Determine those $n > 1600$ such that any $f : S_n \to S_n$ satisfying $(1)$ has at least two fixed points.

5. A circle $\gamma$ and a line $\ell$ have no points in common. Let $AB$ be the diameter of $\gamma$ perpendicular to $\ell$, with $B$ closer to $\ell$ than $A$. Let $C$ be a point on $\gamma$ different from $A$ and $B$. The line $AC$ intersects $\ell$ at $D$. The line $DE$ is tangent to $\gamma$ at $E$, with $B$ and $E$ on the same side of $AC$. The line $BE$ intersects $\ell$ at $F$, and $G$ is the other intersection of the line $AF$ with $\gamma$. Let $H$ be symmetric to $G$ with respect to $AB$. Prove that $F, C$, and $H$ are on the same line. [From IMO Short List 2004.]

6. Let $N$ be a positive integer. Alberto and Barbara write a number on the blackboard taking turns, according to the following rules: Alberto starts by writing 1 on the blackboard; subsequently, if a player wrote a number $n$ on the blackboard, then, on the next move, the other player chooses to write either $n + 1$ or $2n$ as long as the number is not greater than $N$. The player who writes $N$ on the blackboard wins.

(a) Determine which player has a winning strategy for $N = 2005$.

(b) Determine which player has a winning strategy for $N = 2004$.

(c) Determine for how many integers $N$, $1 \leq N \leq 2005$, Barbara has a winning strategy. [From IMO Short List 2004.]
Next, we give the 11th Form of the Final Round of the XXXI Russian Mathematical Olympiad for 2004–2005. My thanks again go to Felix Rejto, Canadian Team Leader to the IMO in Mexico, for collecting them for our use.

**XXXI RUSSIAN MATHEMATICAL OLYMPIAD**

**Final Round - 11th Form**

1. (I. Rubanov) Let \( \{a_1, a_2, \ldots, a_{50}, b_1, b_2, \ldots, b_{50} \} \) be a set of 100 real numbers. Suppose that the equation
\[
|x - a_1| + \cdots + |x - a_{50}| = |x - b_1| + \cdots + |x - b_{50}|
\]
has \( N \) solutions (\( N \) is finite). Find the maximal value of \( N \).

2. (I. Bogdanov) Different numbers are written on the reverse sides of 2005 cards (one number on each card). In one step, one may select three cards and be told the set of three numbers written on them. Find the least number of steps necessary to determine with certainty which number is written on each card.

3. (I. Emelyanov) The excircles of \( \triangle ABC \) touch the corresponding sides at \( A', B', \) and \( C' \). The circumcircles of triangles \( A'B'C, AB'C', \) and \( A'BC' \) meet the circumcircle of \( \triangle ABC \) for the second time at \( C_1, A_1, \) and \( B_1 \), respectively. Prove that \( \triangle A_1B_1C_1 \) is similar to the triangle whose vertices are the tangent points of the incircle of \( \triangle ABC \) with its sides.

4. (V. Senderov) Positive integers \( x, y, \) and \( z \) (where \( x > 2, y > 1 \)) satisfy \( x^y + 1 = z^2 \). Let \( p \) denote the number of different prime divisors of \( x \), and let \( q \) denote the number of different prime divisors of \( y \). Prove that \( p \geq q + 2 \).

5. (N. Agakhanov) Does there exist a bounded function \( f: \mathbb{R} \to \mathbb{R} \) with \( f(1) > 0 \) such that
\[
f^2(x + y) \geq f^2(x) + 2f(xy) + f^2(y)
\]
for all \( x, y \in \mathbb{R} \)?

6. (A. Akopyan) The edges of 12 rectangular parallelepipeds \( P_1, P_2, \ldots, P_{12} \) are parallel to the \( x-, y-, \) and \( z- \) axes. Is it possible that \( P_2 \) intersects (that is, has a point in common with) each of the other 11 parallelepipeds except for \( P_1 \) and \( P_5 \); \( P_3 \) intersects each of the others except for \( P_2 \) and \( P_4 \); \ldots; \( P_{12} \) intersects each of the others except for \( P_{11} \) and \( P_1 \); and \( P_1 \) intersects each of the others except for \( P_{12} \) and \( P_2 \). (The surface of a parallelepiped belongs to the parallelepiped.)

7. (A. Zaslavsky, M. Isaev, and D. Tsvetov) Let \( ABCD \) be a quadrilateral having an incircle and having no two sides parallel. Let \( O \) be its incentre. Prove that \( O \) lies on two lines joining the mid-points of the opposite sides of \( ABCD \) if and only if \( OA \cdot OC = OB \cdot OD \).
8. (S. Berlov) Seated at a round table are 100 representatives of 25 countries (four persons from each country). Prove that they can be partitioned into four groups satisfying the following conditions: (i) each group contains a representative of each country, (ii) each person has no neighbours among the members of his group.

Our last set of problems comes from the Selected Problems from the Taiwan Mathematical Olympiad of July 2005. Again thanks go to Felix Recio for collecting them for us.

**TAIWAN MATHEMATICAL OLYMPIAD**

**Selected Problems**

**July 6, 2005**

1. A \( \triangle ABC \) is given with side lengths \( a, b, \) and \( c \). A point \( P \) lies inside \( \triangle ABC \), and the distances from \( P \) to three sides are \( p, q, \) and \( r \), respectively. Prove that

\[
R \leq \frac{a^2 + b^2 + c^2}{18 \sqrt{pqr}},
\]

where \( R \) is the circumradius of \( \triangle ABC \). When does equality hold?

2. Suppose that \( G \) is a graph of order \( n \) which does not contain a complete graph of order \( k \) as a subgraph. Prove that \( G \) contains at most

\[
\frac{k - 2}{k - 1} \cdot \frac{n^2 - r^2}{2} + \binom{r}{2}
\]

edges, where \( n \equiv r \pmod{(k - 1)} \) and \( 0 \leq r \leq k - 2 \).

3. The IMO will take place on the 13\textsuperscript{th} and 14\textsuperscript{th} of July 2005. Prove that the sum

\[
\sum_{\substack{1 \leq i,j,k \leq 3 \atop i \neq j \neq k}} \csc^{13} \left( \frac{2^i \pi}{7} \right) \csc^{14} \left( \frac{2^j \pi}{7} \right) \csc^{2005} \left( \frac{2^k \pi}{7} \right)
\]

is a rational number.

4. Let \( a \pmod{b} \) denote \( a = b \left\lfloor \frac{a}{b} \right\rfloor \); that is, the remainder when \( a \) is divided by \( b \). Find all solutions in positive integers \( (x, y, z) \) to the equations

\[
xy \pmod{z} = yz \pmod{x} = zx \pmod{y} = 2.
\]

5. Given \( \triangle ABC \), a circle \( \Gamma \) with centre \( O \) passes through \( B \) and \( C \) and intersects sides \( AC \) and \( AB \) at points \( D \) and \( E \), respectively. Let \( F \) be the intersection of \( BD \) and \( CE \). The line \( OF \) intersects the circumcircle of \( \triangle ABC \) at \( P \). Prove that the incentre of \( \triangle PBD \) coincides with the incentre of \( \triangle PCE \).
6. Find all positive integers \( n > 3 \) with the following property: there exists a positive integer \( M_n \) such that, for any given \( n \) positive real numbers \( a_1, a_2, \ldots, a_n \), the following equality holds:

\[
\frac{a_1 + a_2 + \cdots + a_n}{\sqrt[n]{a_1 a_2 \cdots a_n}} \leq M_n \left( \frac{a_2}{a_1} + \frac{a_3}{a_2} + \cdots + \frac{a_n}{a_{n-1}} + \frac{a_1}{a_n} \right).
\]

We have had some readers challenge the value 47 as the least possible answer to problem 4 of the Category B Belarus Mathematical Olympiad [2006: 438; 2007: 421].

4. (I. Voronovich) Pairwise distinct positive integers \( a, b, c, d, e, f, g, h \), and \( n \) satisfy the equalities \( n = ab + cd = ef + gh \).

Find the smallest possible value of \( n \).

**Comments and solutions by Stan Wagon, Macalester College, St. Paul, MN, USA; and John P. Robertson, National Council on Compensation Insurance, Inc., Boca Raton, FL, USA.**

Using a computer search for other solutions to \( n = ab + cd = ef + gh \), Wagon found that \( 18 = 2 \cdot 9 = 3 \cdot 6 \) and \( 20 = 1 \cdot 20 = 4 \cdot 5 \), which implies that \( 38 = 2 \cdot 9 + 1 \cdot 20 = 3 \cdot 6 + 4 \cdot 5 \), improving considerably on 47. Robertson did a more general search finding \( 31 = 1 \cdot 7 + 4 \cdot 6 = 2 \cdot 8 + 3 \cdot 5 \). Robertson notes he uses the smallest possible set of \( a, b, \ldots, h \) (smallest in the sense of lexicographic ordering of possible sorted \( a, b, \ldots, h \)). He generated all \( n = ab + cd \) for \( a = 4 \) to 50, with \( b, c, d \) smaller than \( a \) and distinct, and looked for an \( n \) with two representations using 8 distinct elements. Since \( 50 > 31 \) and \( n > a \), there cannot be any solution smaller than 31. The checking involved about 700,000 cases, taking a minute or two on a personal computer.

**Next we give the solution by Przemyslaw Mazur, Jan Sobieski High School, Krakow, Poland.**

The answer is \( 31 = 8 \cdot 2 + 3 \cdot 5 = 6 \cdot 4 + 1 \cdot 7 \).

To prove minimality we will show that:

(i) The minimum possible value of \( S = ab + cd + ef + gh \) is 60.

(ii) In the case where \( S = 60 \), the terms in the sum are 8, 14, 18, and 20. Therefore, there is no way to rearrange them into pairs giving the same sums.

To prove both claims, let us assume that \( a, b, c, d, e, f, g, \) and \( h \) minimize the sum \( S \). If any of the eight numbers—let us call it \( x \)—is greater than 8, then one of the numbers from \( \{1, 2, 3, 4, 5, 6, 7, 8\} \) must remain unused—let us call it \( y \). Replacing \( x \) with \( y \) decreases \( S \), a contradiction.
This means that \( \{a, b, c, d, e, f, g, h\} = \{1, 2, 3, 4, 5, 6, 7, 8\} \) which are grouped into four pairs that give the terms. Assume that 8 is paired with something other than 1, say 8 is paired with \( w \). Naturally, 1 is paired with yet another number, say \( z \). Changing pairs \((8, w) \) and \((1, z) \) into \((8, 1) \) and \((w, z) \) decreases \( S \) (because \( w > 1 \) and \( z < 8 \)), a contradiction.

Repeating the above argument, we prove that the pairs are \((8, 1)\), \((7, 2)\), \((6, 3)\), and \((5, 4)\). This completes the proof.


1. Let \( n \) be an integer, \( n > 1 \). Define

\[
A = \frac{\sqrt{n+1}}{n} + \frac{\sqrt{n+4}}{n+3} + \frac{\sqrt{n+7}}{n+6} + \frac{\sqrt{n+10}}{n+9} + \frac{\sqrt{n+13}}{n+12},
\]

and

\[
B = \frac{1}{\sqrt{n-1}} + \frac{1}{\sqrt{n+2}} + \frac{1}{\sqrt{n+5}} + \frac{1}{\sqrt{n+8}} + \frac{1}{\sqrt{n+11}}.
\]

Determine which of the following relations holds (depending on \( n \)): \( A > B \), \( A = B \), or \( A < B \).

Solved by Houida Anoun, Bordeaux, France; Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; Ioannis Katsikis, Athens, Greece; Andrea Munaro, student, University of Trento, Trento, Italy; Vedula N. Murty, Dover, PA, USA; and Titu Zvonaru, Comănești, Romania. We give Anoun's write-up.

Let \( p \) be a positive integer such that \( p > 1 \). We can easily prove that

\[\frac{\sqrt{p+1}}{p} < \frac{1}{\sqrt{p-1}} \quad \text{(1)}\]

In fact, this is equivalent to \((p-1)(p+1) = p^2 - 1 < p^2\), which is true.

By applying (1), we have for each \( i \in \{0, \ldots, 4\} \),

\[\frac{\sqrt{n+3i+1}}{n+3i} < \frac{1}{\sqrt{n+3i-1}};\]

hence, we can straightforwardly deduce that \( A < B \). This result can be generalized as follows:

\[\sum_{i=0}^{k} \frac{\sqrt{n+3i+1}}{n+3i} < \sum_{i=1}^{k} \frac{1}{\sqrt{n+3i-1}}.\]

2. Let \( a, b, \) and \( c \) denote the sides of a triangle opposite the angles \( A, B, \) and \( C, \) respectively. Let \( r \) be the inradius and \( R \) the circumradius of the triangle. If \( \angle A \geq 90^\circ \), prove that

\[\frac{r}{R} \geq \frac{a \sin A}{a + b + c}.\]
Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-­-Laffitte, France; Ioannis Katsikis, Athens, Greece; Andrea Munaro, student, University of Trento, Trento, Italy; Vedula N. Murty, Dover, PA, USA; Daniel Tsai, student, Taipei American School, Taipei, Taiwan; and Titu Zvonaru, Comănești, Romania. We first give Bataille’s solution.

If we let \( F \) be the area of \( \triangle ABC \) and let \( h_a \) be the altitude from \( A \), the proposed inequality is successively equivalent to

\[
\begin{align*}
    r(a + b + c) &\leq aR \sin A, \\
    2F &\leq \frac{1}{2}a \cdot 2R \sin A, \\
    ah_a &\leq \frac{1}{2}a^2, \\
    h_a &\leq \frac{1}{2}a.
\end{align*}
\]

Now, the median \( m_a \) from \( A \) satisfies

\[
m_a^2 = \frac{1}{2}(b^2 + c^2) - \frac{1}{4}a^2 = \frac{1}{2}(a^2 + 2bc \cos A) - \frac{1}{4}a^2 = \frac{1}{4}a^2 + bc \cos A \leq \frac{1}{4}a^2,
\]

where the final inequality follows from \( \cos A \leq 0 \) (since \( \angle A \geq 90^\circ \)).

Thus, \( m_a \leq \frac{1}{2}a \), and (2) follows from the obvious inequality \( h_a \leq m_a \).

Next we give the version of Bornsztein.

Soit \( S \) l’aire du triangle. Il est bien connu que

\[
S = \frac{1}{2}(a+b+c)r = \frac{1}{2}\sqrt{(a+b+c)(a+b-c)(a-b+c)(-a+b+c)}.
\]

D’autre part, d’après la loi des sinus, on a \( \frac{\sin(A)}{a} = \frac{1}{2R} \). L’inégalité désirée est donc équivalente à

\[
a^4 \geq (a + b + c)(a + b - c)(a - b + c)(-a + b + c). \tag{1}
\]

Or, comme \( \angle A \geq 90^\circ \), on a \( a^2 = b^2 + c^2 - 2bc \cos(A) \geq b^2 + c^2 \). Par la suite, d’après l’inégalité arithmético-­-géométrique,

\[
(b + c)^2 = b^2 + c^2 + 2bc \leq 2(b^2 + c^2) \leq 2a^2.
\]

Et ainsi

\[
(a + b + c)(a + b - c)(a - b + c)(-a + b + c) = (a^2 - (b - c)^2)((b + c)^2 - a^2) \leq a^2 \cdot a^2
\]

ce qui prouve que (1) est vraie. On peut noter qu’il y a égalité si et seulement si \( b = c \) et \( a^2 = b^2 + c^2 \); c’est à dire, le triangle est rectangle et isocèle en \( A \).

3. Prove that the equation \( x^3 + 2px^2 + 2p^2x + p = 0 \) cannot have three distinct real roots, for any real number \( p \).
Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain; Geoffrey A. Kandall, Hamden, CT, USA; Ioannis Katsikis, Athens, Greece; Vedula N. Murty, Dover, PA, USA; Daniel Tsai, student, Taipei American School, Taipei, Taiwan; and Titu Zvonaru, Comănești, Romania. We first give the solution by Tsai.

We prove the slightly stronger result that $x^3 + 2px^2 + 2p^2x + p = 0$ cannot have at least two distinct real roots, for any real number $p$. Assume to the contrary that the solutions (counting multiplicity) $\alpha$, $\beta$, and $\gamma$ to $x^3 + 2px^2 + 2p^2x + p = 0$ are all real such that there are at least two distinct ones. Then

$$x^3 + 2px^2 + 2p^2x + p = x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)x - \alpha\beta\gamma$$

for any real number $x$, and equating coefficients gives

$$2p = - (\alpha + \beta + \gamma)$$
$$2p^2 = \alpha\beta + \beta\gamma + \gamma\alpha$$
and $$p = -\alpha\beta\gamma$$

Squaring (1) yields $4p^2 = \alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \beta\gamma + \gamma\alpha)$ and by (2), we get $\alpha^2 + \beta^2 + \gamma^2 = 0$, which cannot be.

Next we give the approach of Kandall.

Suppose the polynomial $x^3 + 2px^2 + 2p^2x + p$ has three distinct real roots $x_1 < x_2 < x_3$. Then, by Rolle’s Theorem, its derivative $3x^2 + 4px + 2p^2$ has two distinct real roots: one between $x_1$ and $x_2$, the other between $x_2$ and $x_3$. Therefore,

$$(4p)^2 - 4(3)(2p^2) = -8p^2 > 0,$$

which is impossible.

Note that, if $p$ and $q$ are any real numbers, then the same argument applies to a polynomial $x^3 + 2px^2 + 2p^2x + q$.

4. Let $ABCD$ be a cyclic quadrilateral with $AB = 2AD$ and $BC = 2CD$. Let $d = AC$ and $\alpha = \angle BAD$ be given. Express the area of $ABCD$ in terms of $d$ and $\alpha$.

Comments and solutions by Michel Bataille, Rouen, France; Geoffrey A. Kandall, Hamden, CT, USA; Ioannis Katsikis, Athens, Greece; Andrea Munaro, student, University of Trento, Trento, Italy; Vedula N. Murty, Dover, PA, USA; and Titu Zvonaru, Comănești, Romania.


1. Let $ABC$ be an acute triangle, and let $P$ be a point on side $AB$. Draw lines through $P$ parallel to $AC$ and $BC$, and let them cut $BC$ and $AC$ at $X$ and $Y$, respectively. Construct (with straightedge and compass) the point $P$ which gives the shortest length $XY$. Prove that the shortest $XY$ is perpendicular to the median of $ABC$ through $C$.

\begin{center}
\textbf{Solved by Andrea Munaro, student, University of Trento, Trento, Italy; and D.J. Smeenk, Zaltbommel, the Netherlands. We present Smeenk’s solution.}
\end{center}

Let $\lambda$ and $\mu$ be the positive real numbers such that $AP = \lambda AB = \lambda c$ and $BP = \mu AB = \mu c$. Then $\lambda + \mu = 1$, and we also have

\[ PY = \lambda a, \quad AY = \lambda b, \quad BX = \mu a, \quad XC = \lambda a, \quad YC = \mu b. \]

Let $f(\lambda, \mu)$ be the square of the distance from $X$ to $Y$. If $\gamma = \angle C = \angle XPY$, then, by the Law of Cosines in $\triangle PXY$ and $\triangle ABC$, we have

\[
 f(\lambda, \mu) = XY^2 = \lambda^2 a^2 + \mu^2 b^2 - 2\lambda\mu ab \cos \gamma \\
 = \lambda^2 a^2 + \mu^2 b^2 - \lambda \mu (a^2 + b^2 - c^2) \\
= \lambda^2 a^2 + (\lambda^2 - 2\lambda + 1)b^2 + \lambda(\lambda - 1)(a^2 + b^2 - c^2) \\
= \lambda^2(2a^2 + 2b^2 - c^2) - \lambda(a^2 + 3b^2 - c^2) + b^2 \\
= \lambda^2 \cdot 4m_c^2 - \lambda(a^2 + 3b^2 - c^2) + b^2.
\]

Setting $\frac{df}{d\lambda} = 0$, we get

\[ \lambda = \frac{a^2 + 3b^2 - c^2}{8m_c^2} \quad \text{(and} \quad \mu = 1 - \lambda = \frac{3a^2 + b^2 - c^2}{8m_c^2}). \]

Therefore,

\[ CX = \lambda a = \frac{a^2 + 3b^2 - c^2}{8m_c^2} \quad \text{and} \quad CY = \mu b = \frac{3a^2 + b^2 - c^2}{8m_c^2}. \quad (1) \]

Let $M$ be the mid-point of $AB$ and set $\gamma_1 = \angle ACM$ and $\gamma_2 = \angle BCM$.

By the Law of Cosines in $\triangle ACM$, 

\[
 \cos \gamma_1 = \frac{\lambda a^2 + \mu c^2 - \lambda^2 b^2}{2\lambda a \mu c} = \frac{a^2 + 3b^2 - c^2}{2m_c},
\]

\[
 \cos \gamma_2 = \frac{\lambda b^2 + \mu c^2 - \mu^2 a^2}{2\mu b \lambda c} = \frac{3a^2 + b^2 - c^2}{2m_c},
\]

\[
 \cos \gamma = \frac{\lambda a^2 + \mu c^2 - \lambda^2 b^2}{2\lambda a \mu c} = \frac{a^2 + 3b^2 - c^2}{2m_c},
\]

\[
 \cos \gamma = \frac{\lambda a^2 + \mu c^2 - \mu^2 a^2}{2\mu b \lambda c} = \frac{3a^2 + b^2 - c^2}{2m_c},
\]
\[ AM^2 = CA^2 + CM^2 - 2CA \cdot CM \cdot \cos \gamma_1, \]

or \[ c^2 = 4b^2 + 4m_e^2 - 8am_e \cos \gamma_1. \]

Similarly, we get \( c^2 = 4a^2 + 4m_e^2 - 8am_e \cos \gamma_2. \) Since \( 4m_e^2 = 2a^2 + 2b^2 - c^2, \) we have

\[ \cos \gamma_1 = \frac{a^2 + 3b^2 - c^2}{4bm_e} \quad \text{and} \quad \cos \gamma_2 = \frac{3a^2 + b^2 - c^2}{4am_e}, \]

from which we have

\[ \frac{\cos \gamma_1}{\cos \gamma_2} = \frac{a(a^2 + 3b^2 - c^2)}{b(3a^2 + b^2 - c^2)}. \]

Let the line through \( M \) perpendicular to \( CM \) intersect \( BC \) at \( D \) and \( AC \) at \( E. \) Applying the Law of Sines to \( \triangle CDE, \) we obtain

\[ \frac{CD}{CE} = \frac{\sin \angle CED}{\sin \angle CDE} = \frac{\cos \gamma_1}{\cos \gamma_2} = \frac{a(a^2 + 3b^2 - c^2)}{b(3a^2 + b^2 - c^2)}. \quad (2) \]

By (1) and (2), we have

\[ \frac{CX}{CY} = \frac{a(a^2 + 3b^2 - c^2)}{b(3a^2 + b^2 - c^2)} = \frac{CD}{CE}. \]

Since \( \triangle CXY \) and \( \triangle CDE \) also have the angle at \( C \) in common, we see that they must be similar. Since \( CM \perp ED, \) it follows that \( CM \perp XY. \)

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Now we turn to readers' solutions to problems of the Hungarian National Olympiad 2003–2004 (Specialized Mathematics Classes), First Round, Grades 11–12 which appeared in \([2007: 84]\).

1. Let \( n \) be a positive integer, and let \( a \) and \( b \) be positive real numbers. Prove that

\[ \log(a^n) + \binom{n}{1} \log(a^{n-1}b) + \binom{n}{2} \log(a^{n-2}b^2) + \cdots + \log(b^n) = \log((ab)^{n-1}). \]

\textit{Solved by Houda Anoun, Bordeaux, France; Michel Bataille, Rouen, France; Pierre Bornztein, Maisons-Laffitte, France; Ioannis Katsikis, Athens, Greece; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Wang's write-up.}

Let \( S \) denote the left side of the identity to be proved. Then

\[ S = \sum_{k=0}^{n} \binom{n}{k} \log(a^{n-k}b^k) = \log \left( \prod_{k=0}^{n} (a^{n-k}b^k) \binom{n}{k} \right) \]

\[ = \log \left( a^{\sum (n-k) \binom{n}{k}} b^{\sum k \binom{n}{k}} \right), \quad (1) \]
where both summations in the last line are from $k = 0$ to $k = n$.

Using the standard well-known method, we differentiate the binomial expansion

$$\left(1 + x\right)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

to obtain

$$n\left(1 + x\right)^{n-1} = \sum_{k=1}^{n} k \binom{n}{k} x^{k-1}.$$  

Multiplying by $x$, we have $nx\left(1 + x\right)^{n-1} = \sum_{k=1}^{n} k \binom{n}{k} x^k$.

Setting $x = 1$, we then obtain

$$\sum_{k=0}^{n} k \binom{n}{k} = \sum_{k=1}^{n} k \binom{n}{k} = n2^{n-1}.$$  

Hence,

$$\sum_{k=0}^{n} (n-k) \binom{n}{k} = n \sum_{k=0}^{n} \binom{n}{k} - \sum_{k=0}^{n} k \binom{n}{k} = n2^n - n2^{n-1} = n2^{n-1}.$$  

Substituting (2) and (3) into (1) yields the result immediately.

2. Let $H$ be a finite set of positive integers none of which has a prime factor greater than 3. Show that the sum of the reciprocals of the elements of $H$ is smaller than 3.

Solved by Houda Aoun, Bordeaux, France; and Pierre Bornsztein, Maisons-Laffitte, France. We give Aoun’s write-up.

Let $H = \{n_1, \ldots, n_k\}$, where each $n_i$ is a positive integer such that $n_i = 2^{a_i} 3^{b_i}$ ($a_i$ and $b_i$ are positive integers).

Let $S(k) = \sum_{i=1}^{k} \frac{1}{n_i}$. We have

$$S(k) = \sum_{i=1}^{k} \frac{1}{2^{a_i} 3^{b_i}}.$$  

Hence,

$$S(k) < \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{1}{2^i 3^j} = \left(\sum_{i=0}^{\infty} \frac{1}{2^i}\right) \cdot \left(\sum_{j=0}^{\infty} \frac{1}{3^j}\right).$$  

Since

$$\sum_{j=0}^{\infty} \frac{1}{3^j} = \frac{3}{2} \quad \text{and} \quad \sum_{i=0}^{\infty} \frac{1}{2^i} = 2,$$

we deduce that $S(k) < 3$. 

3. Consider the three disjoint arcs of a circle determined by three points on the circle. For each of these arcs, draw a circle centred at the mid-point of the arc and passing through the end-points of the arc. Prove that the three circles have a common point.

Solution by D.J. Smeenk, Zaltbommel, the Netherlands.

The following theorem is well-known:
Let \( I \) be the incentre of \( \triangle ABC \). Then the lines \( AI, BI, \) and \( CI \) intersect the circum-circle for the second time in the mid-points \( A_1, B_1, \) and \( C_1 \) of the arcs \( BC, CA, \) and \( AB, \) respectively. Furthermore,

\[
\begin{align*}
A_1B &= A_1I = A_1C, \\
B_1C &= B_1I = B_1A, \\
\text{and} \quad C_1A &= C_1I = C_1B.
\end{align*}
\]

The common point of the three circles is thus the incentre of \( \triangle ABC \).

4. A palace which has a square shape is divided into 2003 \( \times \) 2003 square rooms of the same size which form a square grid. There might be a door between two rooms if they have a common side. The main gate leads to the room at the northwest corner. Someone has entered the palace, walked around for a while and upon returning to the room at the northwest corner for the first time, immediately left the palace. It turned out that this person visited each of the other rooms 100 times, except the room at the southeast corner. How many times did this person visit the room at the southeast corner?

Solution by Pierre Bornstein, Maisons-Laffitte, France.

La pièce a été visitée 99 fois.

Plus généralement, on considère un palace de \( n \times n \) pièces, chacune d'entre elles ayant été visitée exactement \( p \geq 1 \) fois, sauf celle située au coin nord-ouest qui a été visitée exactement deux fois, et celle située au coin sud-est, qui a été visitée exactement \( x \) fois. On va prouver qu'alors \( x = p - 1 \) si \( n \) est impair, et \( x = 2p - 1 \) si \( n \) est pair.

On commence par colorier les pièces alternativement en noir et blanc, comme sur un échiquier, de sorte que les deux cases situées au coins nord-ouest et sud-est soient toutes les deux noires. En particulier, on passe toujours d'une pièce blanche à une pièce noire, et réciproquement.

Cas 1. \( n \) est impair.

En tout, il y a alors \( \frac{1}{2}(n^2 + 1) \) pièces noires et \( \frac{1}{2}(n^2 - 1) \) pièces blanches.

Puisque chaque pièce blanche a été visitée exactement \( p \) fois et que la personne est sortie d'une pièce blanche pour entrer dans une pièce noire,
le nombre total de passages d'une pièce blanche à une pièce noire réalisés pendant la promenade est $\frac{1}{2}(n^2 - 1)p$.

Mais, si l'on élimine la première entrée dans la pièce du coin nord-ouest (puisque l'on vient de l'extérieur), cela correspond au nombre total de visites des pièces noires au cours de la promenade. Or, la pièce du coin nord-ouest a été visitée une seule fois (puisque la promenade s'arrête après le premier retour dans cette pièce) et celle du coin sud-est a été visitée $x$ fois, alors que les $\frac{1}{2}(n^2 + 1) - 2$ autres pièces noires ont été visitées chacune exactement $p$ fois. D'où

$$1 + x + (\frac{1}{2}(n^2 + 1) - 2) p = \frac{1}{2}(n^2 - 1)p$$

On en déduit facilement que $x = p - 1$.

**Cas 2.** $n$ est pair.

En tout, il y a alors $\frac{1}{2}n^2$ pièces noires et $\frac{1}{2}n^2$ pièces blanches. Le reste du raisonnement est le même pour arriver à

$$1 + x + (\frac{1}{2}n^2 - 2) p = \frac{1}{2}n^2 p$$

On en déduit facilement que $x = 2p - 1$.

Now we turn to problems of the Finnish High School Math Contest 2004, Final Round, given at [2007: 85].

1. The equations

$$x^2 + 2ax + b^2 = 0 \quad \text{and} \quad x^2 + 2bx + c^2 = 0$$

both have two different real roots. Determine the number of real roots of the equation

$$x^2 + 2cx + a^2 = 0.$$

Solved by Houda Anoun, Bordeaux, France; Pierre Bornsztein, Maisons-Laffitte, France; José Luis Díaz-Barrero, Universitat Politècnica de Catalunyana, Barcelona, Spain; Geoffrey A. Kandall, Hamden, CT, USA; Ioannis Katsikis, Athens, Greece; Daniel Tsai, student, Taipei American School, Taipei, Taiwan; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Titu Zvonaru, Comănești, Romania. We give the write-up of Zvonaru.

Since $x^2 + 2ax + b^2 = 0$ and $x^2 + 2bx + c^2 = 0$ each have different real roots, we have

$$4(a^2 - b^2) > 0 \quad \text{and} \quad 4(b^2 - c^2) > 0.$$ 

Summing these inequalities, we deduce that $a^2 - c^2 > 0$, or $c^2 - a^2 < 0$, which implies that the equation $x^2 + 2cx + a^2 = 0$ has no real roots.
2. Let $a$, $b$, and $c$ be positive integers such that

$$\frac{a\sqrt{3} + b}{b\sqrt{3} + c}$$

is a rational number. Show that

$$\frac{a^2 + b^2 + c^2}{a + b + c}$$

is an integer.

*Solved by Houda Anoun, Bordeaux, France; Pierre Bornsztein, Maisons-Laffitte, France; Geoffrey A. Kandall, Hamden, CT, USA; Ioannis Katsikis, Athens, Greece; Daniel Tsai, student, Taipei American School, Taipei, Taiwan; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Titu Zvonaru, Comănești, Romania. We give Kandall's write-up.*

We first observe that

$$\frac{a\sqrt{3} + b}{b\sqrt{3} + c} = \frac{a\sqrt{3} + b}{b\sqrt{3} + c} \cdot \frac{b\sqrt{3} - c}{b\sqrt{3} - c} = \frac{(3ab - bc) + (b^2 - ac)}{3b^2 - c^2}.\]$$

Since this number is rational, we must have $b^2 = ac$. Then

$$a^2 + b^2 + c^2 = a^2 + ac + c^2 = (a + c)^2 - ac$$

$$= (a + c)^2 - b^2 = (a + c + b)(a + c - b).$$

Consequently,

$$\frac{a^2 + b^2 + c^2}{a + b + c} = a + c - b,$$

an integer.

3. Two circles with radii $r$ and $R$ are externally tangent at a point $P$. Determine the length of the segment cut from the common tangent through $P$ by the other common tangents.

*Solved by Geoffrey A. Kandall, Hamden, CT, USA; Ioannis Katsikis, Athens, Greece; and Titu Zvonaru, Comănești, Romania. We give the write-up of Katsikis.*

Without loss of generality, we may assume that $r \leq R$. Let the circle with radius $r$ have centre $O_1$ and the circle with radius $R$ have centre $O_2$. Let $P$ be their point of tangency. Let the common external tangents meet the circles at $A$, $B$, $C$, and $D$, as in the diagram. Let the internal common tangent meet the external common tangents at $K$ and $L$. 
Let $S$ be the point on $O_2B$ such that $O_1S \perp O_2B$. Then $O_1S = AB$ and $O_2S = R - r$. Also
\[
O_1S = \sqrt{(O_1O_2)^2 - (O_2S)^2} = \sqrt{(R + r)^2 - (R - r)^2} = 2\sqrt{Rr}.
\]
Thus, $KP = \frac{1}{2}AB = \sqrt{Rr}$. Similarly, since $CD = AB = 2\sqrt{Rr}$, we have $PL = \sqrt{Rr}$, which implies that $KL = 2\sqrt{Rr}$.

4. The numbers $2005! + 2, 2005! + 3, \ldots, 2005! + 2005$ form a sequence of 2004 consecutive integers, none of which is a prime number. Does there exist a sequence of 2004 consecutive integers containing exactly 12 prime numbers?

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

Si l'on n'impose pas qu'il s'agisse d'entiers positifs, la conclusion est triviale puisqu'il suffit de prendre les entiers positifs jusqu'aux 12 premiers nombres premiers et de compléter la suite par des nombres négatifs.

Soient $n > 0$ un entier et $\pi(n)$ le nombre de nombres premiers inférieurs ou égaux à $n$.

Soit enfin $f(n) = \pi(n + 2004) - \pi(n)$. Alors, $f(n)$ est le nombre de nombres premiers dans $\{n + 1, n + 2, \ldots, n + 2004\}$, ensemble formé de 2004 entiers consécutifs. Il s'agit donc de prouver qu'il existe $n > 0$ tel que $f(n) = 12$.

On commence par noter que $f(1) = \pi(2005) > 12$ et que, d'après le rappel de l'énoncé, $f(2005! + 1) = 0$.

Soit $n > 0$ un entier. On a
\[
f(n + 1) - f(n) = \pi(n + 2005) - \pi(n + 2004) + \pi(n) - \pi(n + 1)
\]
\[
= \begin{cases} 
0 & \text{si } n + 1 \text{ et } n + 2005 \text{ sont tous les deux premiers} \\
1 & \text{si } n + 2005 \text{ est premier et } n + 1 \text{ est composé}, \\
-1 & \text{si } n + 1 \text{ est premier et } n + 2005 \text{ est composé}.
\end{cases}
\]

On en déduit que la fonction $f$ passe de la valeur $f(1)$ à la valeur $f(2005! + 1)$ en ne sautant aucun entier strictement positif. En particulier, elle prend au moins une fois la valeur 12, ce qui permet d'affirmer qu'il existe une suite de 2004 entiers strictement positifs consécutifs qui contienne exactement 12 nombres premiers.

5. Finland is going to change its monetary system again and replace the Euro by the Finnish Mark. The Mark is divided into 100 pennies. There shall be coins of three denominations only, and the number of coins a person has to carry in order to be able to pay for any purchase less than one Mark should be minimal. Determine the coin denominations.
Solution by Pierre Bornsztein, Maisons-Laffitte, France.

Si l'on veut pouvoir payer une somme de 1 penny il faut que l'une des pièces soit de ce montant. On note a et b les autres montants, avec a < b.

Une personne qui possède x pièces de 1 penny, y pièces de a pennies et z pièces de b pennies pourra payer tout prix inférieur ou égal à 99 pennies seulement si avec ces pièces il peut former au moins 100 valeurs différentes (on compte le cas d'un prix nul).

Il faut donc que $(x + 1)(y + 1)(z + 1) \geq 100$.

Or, d'après l'inégalité arithmético-géométrique, cela implique que

$$x + y + z \geq 3 \sqrt[3]{100} - 3 = 10,92\ldots$$

et donc que $x + y + z \geq 11$.

Ainsi, quels que soient a et b, il faudra toujours au moins 11 pièces pour pouvoir payer toute somme inférieure ou égale à 99 pennies.

Réciproquement : Tout entier $p \in \{1, \ldots, 99\}$ se décompose en base 5 sous la forme $p = x + 5y + 25z$, avec $x, y, z \leq 4$. Mais, puisque $p < 100$, on a même $z \leq 3$. Par conséquent, pour $a = 5$ et $b = 25$, on pourra payer tout achat de moins d'un Mark en partant avec 4 pièces de 1 penny, 4 pièces de 5 pennies et 3 pièces de 25 pennies, soit donc un total de 11 pièces et l'on a vu que l'on ne pourra pas faire mieux.

Donc, avec des pièces de 1 penny, 5 pennies et 25 pennies, l'objectif sera atteint.

Cela étant, il n'y a pas unicité car l'objectif sera atteint également avec 4 pièces de 1 penny, 3 pièces de 5 pennies et 4 pièces de 20 pennies (soit donc à nouveau un total de 11 pièces). En effet, si $p \in \{1, \ldots, 99\}$ et $\lfloor \cdot \rfloor$ désigne la partie entière alors, en posant $z = \lfloor \frac{p}{20} \rfloor$ on $a z \leq 4$ et $p = 20z + q$ avec $q \in \{0, \ldots, 19\}$. En particulier, $q = 5y + x$ avec $y \leq 3$ et $x \leq 4$. Ainsi, $p = x + 5y + 20z$ et on pourra donc payer un prix de $p$ pennies avec les pièces que l'on a en poche.

That completes the material for this issue of the Corner. Send me your nice solutions and generalizations.