Problem of the Month

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As the final Problem of the Month for 2007, we have a mean sort of problem ...

Problem (2001 American Invitational Mathematics Examination)

A finite set S of distinct real numbers has the following properties: the mean of $S \cup \{1\}$ is 13 less than the mean of S and the mean of $S \cup \{2001\}$ is 27 more than the mean of S. Find the mean of S.

Have you guessed that this was a problem from a 2001 contest? One notational reminder: $S \cup \{1\}$ means the *union* of S and $\{1\}$. In other words, it is the set that we get by adding 1 to the list of numbers already in S. (Actually, there is an additional technicality here that we'll look at quickly after the solution of the problem.)

Problems involving means usually require us to remember the fact that the mean of a list of numbers equals the sum of the numbers divided by the number of numbers in the list. In fact, that's all we really need to know here.

Solution: Let n be the number of numbers in the set S (in mathematical language, n is the *cardinality* of S). Let u be the sum of the numbers in the set S. Then the mean of the numbers in S equals $\frac{u}{n}$. When the number 1 is added to the set S, the new mean is $\frac{u+1}{n+1}$ since the sum of the number in S increases by 1 and the number of numbers in S also increases by 1. Similarly, when the number 2001 is added to the set S, the new mean is $\frac{u+2001}{n+1}$.

Therefore, the given information tells us

$$\begin{aligned} \frac{u+1}{n+1} - \frac{u}{n} &= -13\,, \\ \frac{u+2001}{n+1} - \frac{u}{n} &= 27\,. \end{aligned}$$

We need to find the mean of S, in other words $\frac{u}{n}$. In order to find $\frac{u}{n}$, it looks like we pretty much have to solve this system of two equations in two unknowns for u and n. However, there is a slight wrinkle: the AIME exam from which this problem is taken did not permit the use of a calculator! (For some of us, this may be more of a concern than for others.) So let's try to solve this system of equations in a clever way.

Can you see a manipulation that we can perform that will allow us to solve for n almost immediately? Try fiddling around for a couple of minutes before reading on.

Did you get it? What happens when we subtract the first equation from the second? When we do this, we get

$$\frac{u+2001}{n+1} - \frac{u+1}{n+1} - \frac{u}{n} + \frac{u}{n} \;=\; 27 - (-13)\,,$$

which simplifies to $\frac{2000}{n+1} = 40$; that is, $n+1 = 50$, or $n = 49$.

Now we need to find u. Substituting $n = 49$ into the first equation, we obtain

$$\frac{u+1}{50} - \frac{u}{49} \;=\; -13\,.$$

Being without a calculator, the idea of trying to get a common denominator on the left side seems a bit scary. Remember, though, that we really need $\frac{u}{49}$ (not u):

$$\begin{aligned} \frac{u}{50} + \frac{1}{50} - \frac{u}{49} &= -13\,, \\ \frac{49}{50}\left(\frac{u}{49}\right) - \frac{50}{50}\left(\frac{u}{49}\right) &= -13 - \frac{1}{50}\,, \\ -\frac{1}{50}\left(\frac{u}{49}\right) &= -13 - \frac{1}{50}\,, \\ \frac{u}{49} &= 50(13) + 1 \;=\; 651\,. \end{aligned}$$

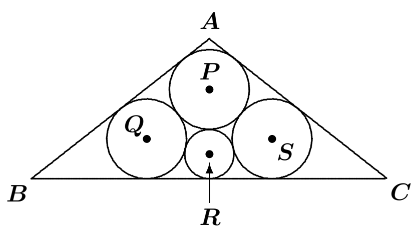
(That worked out pretty well—no ugly calculations!) Therefore, the mean of S is 651.

Those of you who have written the AIME before will be relieved by this answer, as it is an integer between 000 and 999, as per AIME prescription.

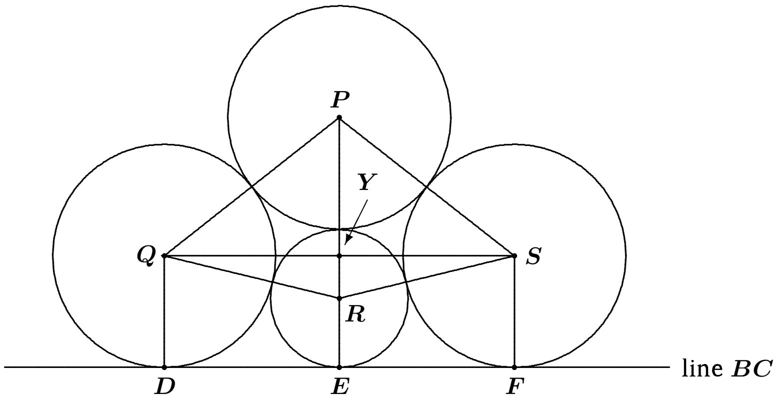
I mentioned a technicality before we launched into the solution. When we look at the union $S \cup \{1\}$, technically we add 1 to the set S only if 1 does not already appear in S. (For example, $\{1,\,2,\,3\} \cup \{1\} = \{1,\,2,\,3\}$.) Do we need to worry about this here? In fact, we don't: since the mean actually changes when we perform each of the two unions, the numbers 1 and 2001 could not have been in S to begin with.

Last month, I left you with a challenge problem, adapted from this year's Hypatia Contest. We repeat the problem, followed by its solution:

In the diagram, the circles with centres P, Q and S all have radius 1. Each is tangent to two sides of the isosceles $\triangle ABC$ and to the circle with centre R; the circle with centre P is tangent to both of the other circles of radius 1. What is the radius of the circle with centre R?



Solution to November's Challenge Problem: Drop perpendiculars to BC from Q, R, and S at D, E, and F, respectively. Since the circles with centres Q, R, and S are tangent to BC, we see that D, E, and F are the points of tangency of these circles to BC. Thus, $QD = SF = 1$. Let $RE = r$.



Join QR, RS, SP, PQ, and PR. Since we are connecting centres of tangent circles, then $PQ = PS = 2$ and $QR = RS = PR = 1 + r$. By symmetry, PRE is a straight line (that is, PE passes through R). Join QS. Since QD and SF are perpendicular to BC, then QS is parallel to BC. Thus, QS is perpendicular to PR, meeting at Y. Since $QD = 1$, then $YE = 1$. Since $RE = r$, then $YR = 1 - r$. Since $QR = 1 + r$, $YR = 1 - r$, and $\triangle QYR$ is right-angled at Y, then, by the Pythagorean Theorem,

$$\begin{aligned} QY^2 = QR^2 - YR^2 &= (1+r)^2 - (1-r)^2 \\ &= (1 + 2r + r^2) - (1 - 2r + r^2) = 4r \,. \end{aligned}$$

Since $PR = 1 + r$ and $YR = 1 - r$, then $PY = PR - YR = 2r$. Since $\triangle PYQ$ is right-angled at Y, then

$$\begin{aligned} PY^2 + YQ^2 &= PQ^2 \,, \\ (2r)^2 + 4r &= 2^2 \,, \\ 4r^2 + 4r &= 4 \,, \\ r^2 + r - 1 &= 0 \,. \end{aligned}$$

By the quadratic formula, $r = \dfrac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2} = \dfrac{-1 \pm \sqrt{5}}{2}$. Since $r > 0$, then $r = \dfrac{-1 + \sqrt{5}}{2}$ (which is the reciprocal of the famous "golden ratio").