Mayhem Solutions

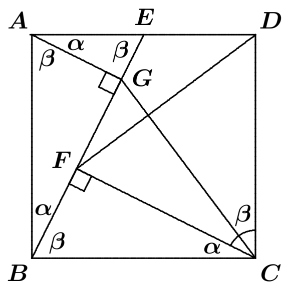
We would like to apologize to RICHARD I. HESS, Rancho Palos Verdes, CA, USA, whose solutions to M265–M268 were misfiled and did not surface until the November issue was being printed.

M269. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Let $ABCD$ be a square. Let E be the mid-point of the side AD, let F be the point on EB such that CF is perpendicular to EB, and let G be the point on EB such that AG is perpendicular to EB. Show that $DF = CG$.

Solution by Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina.

We will prove that the statement is true, more generally, when E is an arbitrary point on AD.



We first have $\angle AEB = \angle FBC = \beta$, since $AD \parallel BC$. Moreover, since $\triangle EGA$, $\triangle AGB$, and $\triangle BFC$ are all right triangles and $ABCD$ is a square, we get $\angle GAB = \angle FBC = \angle FCD = \beta$, and $\angle EAG = \angle ABF = \angle FCB = \alpha = 90^\circ - \beta$.

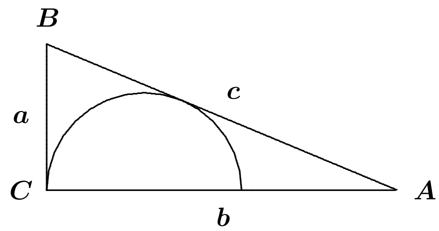
Now, since $\triangle AGB$ and $\triangle BFC$ are right triangles, with $AB = BC$, and $\angle GAB = \angle FBC$, we see that $\triangle AGB$ is congruent to $\triangle BFC$. Hence, $FC = BG$.

Finally, we have $\triangle BGC$ congruent to $\triangle CFD$, because $BG = FC$, $BC = DC$, and $\angle GBC = \angle FCD$. Therefore, $CG = DF$.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; HASAN DENKER, Istanbul, Turkey; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; TAICHI MAEKAWA, Takatsuki City, Osaka, Japan (4 solutions); KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India; DANIEL TSAI, student, Taipei American School, Taipei, Taiwan; and TITU ZVONARU, Comăneşti, Romania.

Amengual Covas, Denker, and Zvonaru also solved this more general case of the problem.

M270. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*



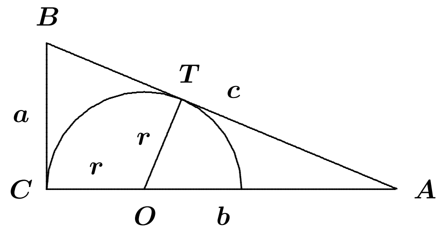
A right triangle has legs of lengths a and b and a hypotenuse of length c. A semicircle has its diameter on the side of length b and is tangent to the other two sides. Determine the radius of the semicircle in terms of a, b, and c.

Solution by Natalia Desy, student, Palembang, Indonesia.

In triangle ABC, let T denote the point of tangency of side AB to the inscribed semicircle with centre O and radius r.

Since $\angle OTA = 90^\circ$, we see that $\sin A = r/(b-r)$; but we also have $\sin A = a/c$. Equating these two expressions for $\sin A$ yields $cr = ab - ar$. Solving for r gives

$$r = \frac{ab}{a+c}.$$



Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; DENISE CORNWELL, student, Angelo State University, San Angelo, TX, USA; HASAN DENKER, Istanbul, Turkey; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; GUSTAVO KRIMKER, Universidad CAECE, Buenos Aires, Argentina; TAICHI MAEKAWA, Takatsuki City, Osaka, Japan (6 solutions); KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India; DANIEL TSAI, student, Taipei American School, Taipei, Taiwan; and TITU ZVONARU, Comănești, Romania.

M271. *Proposé par Yakub N. Aliyev, Université d'Etat de Bakou, Bakou, Azerbaïdjan.*

Sachant que dans un hexagone convexe $ABCDEF$, les côtés BC, DE et FA sont respectivement parallèles aux diagonales AD, CF et EB, on désigne respectivement par K, L et M les intersections des droites AB avec CD, CD avec EF, et EF avec AB ; on désigne enfin par P, Q et R les intersections respectives de CF avec BE, de BE avec AD, et de AD avec CF. Montrer que KP, MR et LQ se coupent en un même point.

Solution par Saturnino Campo Ruiz, "Fray Luis de León" de Salamanca, Salamanca, Espagne, modifié par le rédacteur.

Par la réciproque du Théorème de Pascal, l'hexagone $BCFADE$, dont les côtés opposés sont parallèles, peut être inscrit dans une conique, donc les sommets de l'hexagone $ABEFCD$ sont sur une conique. Par le Théorème de Pascal, les côtés opposés de l'hexagone $ABEFCD$ se coupent en trois points collinéaires. Par le Théorème de Desargues, $\triangle PQR$ et $\triangle KLM$ sont en perspective axiale si et seulement si ils sont en perspective centrale, donc KP, MR, et LQ se coupent en un même point.

Une solution incorrecte a aussi été soumise.

M272. *Proposed by John Grant McLoughlin, University of New Brunswick, Fredericton, NB.*

Let $ABCD$ be a parallelogram, and let P be a point situated on AB. If the ratio of the area of triangle ABC to that of quadrilateral $APCD$ is m/n, determine the ratio of AP to PB.

Composite of solutions submitted by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam; Hasan Denker, Istanbul, Turkey; Richard I. Hess, Rancho Palos Verdes, CA, USA; Kunal Singh, student, Kendriya Vidyalaya School, Shillong, India; and Titu Zvonaru, Comăneşti, Romania.

Let $[\mathcal{P}]$ denote the area of polygon $\mathcal{P}$. The ratio of the area of triangle ABC to that of quadrilateral $APCD$ is $[ABC]/[APCD] = m/n$. Then

$$\frac{n}{m} \;=\; \frac{[APCD]}{[ABC]}\,.$$

Since $[APCD] = [ACD] + [APC]$ and $[ACD] = \frac{1}{2}[ABCD] = [ABC]$, we have

$$\frac{n}{m} \;=\; \frac{[ACD]+[APC]}{[ABC]} \;=\; 1+\frac{[APC]}{[ABC]}\,.$$

Since triangles APC and ABC share a common height with respect to their bases AP and AB, we see that $[APC]/[ABC] = AP/AB$. Therefore,

$$\frac{n}{m} \;=\; 1+\frac{AP}{AB} \;=\; 1+\frac{AP}{AP+PB}\,.$$

Then

$$\frac{AP}{AP+PB} \;=\; \frac{n-m}{m}\,;$$

that is,

$$m\cdot AP \;=\; (n-m)AP+(n-m)PB\,.$$

Thus,

$$\frac{AP}{PB} \;=\; \frac{n-m}{2m-n}\,.$$

There were two incorrect solutions submitted.

M273. *Proposed by John Grant McLoughlin, University of New Brunswick, Fredericton, NB.*

The letters A, B, C, D, E, F, G, and H represent distinct digits. Determine their values given that the two products shown are true. (Note that the first digit of a number must be non-zero.)

$$\begin{array}{r} ABCD \\ \times E \\ \hline DCBA \end{array} \qquad\qquad \begin{array}{r} BFDG \\ \times G \\ \hline GDFB \end{array}$$

Solution by the proposer, modified by the editor.

Consider the second product above. From the thousands digit, we see that $G\cdot B \le G$ and, hence, $B = 1$. Then, since $G\cdot G$ ends in $B = 1$, we have $G = 9$. Now the product is $1FD9\cdot 9 = 9DF1$. We cannot have $F > 1$, because the product $1FD9\cdot 9$ would then have 5 digits instead of 4, and we cannot have $F = 1$, since $B = 1$. Thus $F = 0$. From $10D9\cdot 9 = 9D01$, we find that the only choice for D is $D = 8$.

The first product is now $A1C8 \cdot E = 8C1A$, with A, C, and E having values in $\{2, 3, 4, 5, 6, 7\}$. Since $A1C8 \cdot E$ is even, we see that $8C1A$ is even, which implies that A is even. We then note that the digit 8 in $8C1A$ comes from adding a carried digit to the product $A \cdot E$. The carried digit is at most 1, since it comes from $1C8 \cdot E$; therefore, $A \cdot E$ is either 7 or 8. Since A is even, we must have $A \cdot E = 8$. Now, either $A = 2$ and $E = 4$, or $A = 4$ and $E = 2$. If $A = 4$ and $E = 2$, we have $41C8 \cdot 2 = 8C14$, which is impossible. We conclude that $A = 2$ and $E = 4$, which gives us $21C8 \cdot 4 = 8C12$. Finally, we verify that the only choice for C is $C = 7$.

Thus, $A = 2$, $B = 1$, $C = 7$, $D = 8$, $E = 4$, $F = 0$, and $G = 9$, and the products are

$$\begin{array}{r} 2178 \\ \times 4 \\ \hline 8712 \end{array} \qquad \begin{array}{r} 1089 \\ \times 9 \\ \hline 9801 \end{array}$$

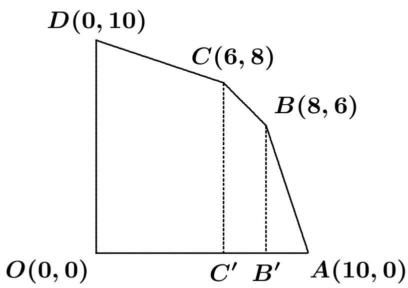
Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; DANIEL TSAI, student, Taipei American School, Taipei, Taiwan; and TITU ZVONARU, Comănești, Romania. One incorrect solution was also submitted.

M274. *Proposed by Neven Jurič, Zagreb, Croatia.*

Determine the area of the polygon whose vertices are all the points on the circle $x^2 + y^2 = 100$ where both coordinates are integers.

Solution by Titu Zvonaru, Comănești, Romania.

Let O be the centre of the polygon. Considering the vertices of the polygon in the first quadrant only, we obtain only the vertices $A(10, 0)$, $B(8, 6)$, $C(6, 8)$, and $D(10, 0)$. Since the polygon is symmetrical about both the x-axis and y-axis, its area is four times the area of pentagon $OABCD$. To find the area of $OABCD$, we sum the areas of the trapezoids $OC'CD$ and $C'B'BC$ and the triangle $B'AB$, where B' and C' are the orthogonal projections of B and C, respectively, onto the x-axis.



If we denote the area of polygon $\mathcal{P}$ by $[\mathcal{P}]$, then

$$\begin{aligned} [OABCD] &= [OC'CD] + [C'B'BC] + [B'AB]\,, \\ &= \tfrac{1}{2} \cdot 6 \cdot (10 + 8) + \tfrac{1}{2} \cdot 2 \cdot (8 + 6) + \tfrac{1}{2} \cdot 2 \cdot 6 \;=\; 74\,. \end{aligned}$$

Hence, the area of the polygon is $4(74) = 296$.

Also solved by DENISE CORNWELL, student, Angelo State University, San Angelo, TX, USA; HASAN DENKER, Istanbul, Turkey; NATALIA DESY, student, Palembang, Indonesia; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and DANIEL TSAI, student, Taipei American School, Taipei, Taiwan. There was one incorrect solution submitted.

M275. *Proposed by K.R.S. Sastry, Bangalore, India.*

A primitive Pythagorean triangle (PPT) is a right triangle whose sides have lengths which are integers with a greatest common divisor of 1. Among all pairs of non-congruent PPTs which have congruent incircles with an integer radius, find a pair for which this radius is minimized.

Solution by Titu Zvonaru, Comănești, Romania.

It is known that a PPT has hypotenuse of length $m^2 + n^2$ and legs of length $2mn$ and $m^2 - n^2$, where m and n are integers of different parity with $\gcd(m, n) = 1$.

Let ABC be a PPT, and let r be its inradius. Let $s = \frac{1}{2}(a + b + c)$ (the semiperimeter of $\triangle ABC$). On the one hand, the area of $\triangle ABC$ is $\frac{1}{2}(2mn)(m^2 - n^2) = mn(m^2 - n^2)$; on the other hand, the area is rs. Thus,

$$\begin{aligned} r = \frac{mn(m^2 - n^2)}{s} &= \frac{mn(m^2 - n^2)}{\frac{1}{2}(2mn + m^2 - n^2 + m^2 + n^2)} \\ &= \frac{mn(m^2 - n^2)}{mn + m^2} = n(m - n)\,. \end{aligned}$$

Searching for a pair of PPTs with minimal r, we find that:

(i) If $r = 1$, then $n(m - n) = 1$, which implies that $n = 1$ and $m = 2$. Therefore, we only have one PPT when $r = 1$.

(ii) If $r = 2$, then $n(m - n) = 2$, which implies that $(m, n) = (3, 1)$ or $(m, n) = (3, 2)$. Since 1 and 3 have same parity, we again have only one PPT when $r = 2$.

(iii) If $r = 3$, then $n(m - n) = 3$, which implies that $(m, n) = (4, 1)$ or $(m, n) = (4, 3)$. These lead to the required pair of PPTs.

Therefore, the pair of PPTs having congruent incircles with minimal integer radius are the triangle with side lengths 8, 15, 17 and the triangle with side lengths 7, 24, 25.

Also solved by HASAN DENKER, Istanbul, Turkey; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; DANIEL TSAI, student, Taipei American School, Taipei, Taiwan; and the proposer.