# MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.



## Mayhem Problems

*Veuillez nous transmettre vos solutions aux problèmes du présent numéro avant le* **premier avril 2008**. *Les solutions reçues après cette date ne seront prises en compte que s'il nous reste du temps avant la publication des solutions.*

*Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais.*

*La rédaction souhaite remercier Jean-Marc Terrier, de l'Université de Montréal, d'avoir traduit les problèmes.*

**M319**. *Proposé par Dragoljub Milošević, Pranjani, Serbie.*

Si, dans un triangle rectangle, on désigne par $h$ l'hypoténuse et par $a$ la hauteur, montrer que

$$\frac{a}{h}+\frac{h}{a} \geq \frac{5}{2}.$$

Quand y a-t-il égalité ?

**M320**. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Si $p$ et $q$ forment une paire de nombres premiers jumeaux, montrer que les nombres $p^4+4$ et $q^4+4$ ne sont jamais relativement premiers.

**M321**. *Proposé par José Luis Díaz-Barrero, Université Polytechnique de Catalogne, Barcelone, Espagne.*

Déterminer tous les entiers positifs $n$ et $k$ pour lesquels on a

$$\frac{\binom{n}{n-1}^6+\binom{n-2}{k}^6+\binom{n+3}{n+1}^3}{3\binom{n-2}{k}^2\binom{n+3}{2}} = n^2.$$

**M322**. *Proposé par Panos E. Tsaoussoglou, Athènes, Grèce.*

Soit $a$, $b$ et $c$ trois nombres réels positifs. Montrer que

$$\frac{a^3+b^3+c^3}{3abc}+\frac{8abc}{(a+b)(b+c)(c+a)} \;\geq\; 2\,.$$

**M323**. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Trouver toutes les solutions réelles $(x,y)$ de l'équation

$$20\sin x - 21\cos x \;=\; 81y^2 - 18y + 30\,.$$

**M324**. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Deux fonctions $f$, $g:\mathbb{R}\longrightarrow\mathbb{R}$ sont définies par

$$f(x) \;=\; 3x-1+|2x+1| \qquad \text{et} \qquad g(x) \;=\; \tfrac{1}{5}(3x+5-|2x+5|)\,.$$

Montrer que $g\circ f = f\circ g$ et $(f\circ f)^{-1} = g\circ g$.

**M325**. *Proposé par Bruce Shawyer, Université Memorial de Terre-Neuve, St. John's, NL.*

Soit $a$, $b$ et $c$ trois chiffres différents de zéro. Pour calculer la fraction $\frac{ab}{ca}$, où $ab$ et $ca$ représentent les entiers à deux chiffres $10a+b$ et $10c+a$, un étudiant applique faussement la loi de simplification, simplifiant le $a$ du numérateur avec le $a$ du dénominateur. Par exemple, si $a = 6$, $b = 5$ et $c = 2$, l'étudiant obtiendrait $65/26 = 5/2$ (en "simplifiant" les 6 !).

Déterminer tous les triplets $(a,b,c)$ pour lesquels cet étudiant obtiendrait un résultat juste.

**M319**. *Proposed by Dragoljub Milošević, Pranjani, Serbie.*

If $h$ and $a$ are the hypotenuse and altitude, respectively, of a right-angled triangle, prove that

$$\frac{a}{h}+\frac{h}{a} \;\geq\; \frac{5}{2}\,.$$

When does equality hold?

**M320**. *Proposed by Mihály Bencze, Brasov, Romania.*

If $p$ and $q$ are any pair of twin primes, show that the numbers $p^4+4$ and $q^4+4$ are never relatively prime.

**M321**. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Determine all positive integers $n$ and $k$ for which we have

$$\frac{\binom{n}{n-1}^6 + \binom{n-2}{k}^6 + \binom{n+3}{n+1}^3}{3\binom{n-2}{k}^2\binom{n+3}{2}} \;=\; n^2\,.$$

**M322**. *Proposed by Panos E. Tsaoussoglou, Athens, Greece.*

Let $a$, $b$, and $c$ be positive real numbers. Prove that

$$\frac{a^3+b^3+c^3}{3abc} + \frac{8abc}{(a+b)(b+c)(c+a)} \;\geq\; 2\,.$$

**M323**. *Proposed by Mihály Bencze, Brasov, Romania.*

Find all real solutions $(x,y)$ to the equation

$$20\sin x - 21\cos x \;=\; 81y^2 - 18y + 30\,.$$

**M324**. *Proposed by Mihály Bencze, Brasov, Romania.*

Let functions $f$, $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) \;=\; 3x - 1 + |2x+1| \qquad \text{and} \qquad g(x) \;=\; \tfrac{1}{5}(3x + 5 - |2x+5|)\,.$$

Prove that $g \circ f = f \circ g$ and $(f \circ f)^{-1} = g \circ g$.

**M325**. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Let $a$, $b$, and $c$ be non-zero digits. A student takes the fraction $\frac{ab}{ca}$, where $ab$ and $ca$ represent the two-digit integers $10a+b$ and $10c+a$, and applies a (false) cancellation law, cancelling the $a$ from the numerator with the $a$ from the denominator. For example, if $a=6$, $b=5$, and $c=2$, the student would obtain $65/26 = 5/2$ (by 'cancelling' the 6s!).

Determine all triples $(a,b,c)$ for which this student actually obtains the correct answer.

# Mayhem Solutions

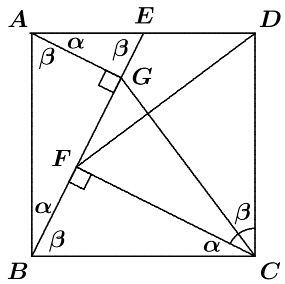
We would like to apologize to RICHARD I. HESS, Rancho Palos Verdes, CA, USA, whose solutions to M265–M268 were misfiled and did not surface until the November issue was being printed.

**M269**. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Let $ABCD$ be a square. Let $E$ be the mid-point of the side $AD$, let $F$ be the point on $EB$ such that $CF$ is perpendicular to $EB$, and let $G$ be the point on $EB$ such that $AG$ is perpendicular to $EB$. Show that $DF = CG$.

*Solution by Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina.*

We will prove that the statement is true, more generally, when $E$ is an arbitrary point on $AD$.




We first have $\angle AEB = \angle FBC = \beta$, since $AD \parallel BC$. Moreover, since $\triangle EGA$, $\triangle AGB$, and $\triangle BFC$ are all right triangles and $ABCD$ is a square, we get $\angle GAB = \angle FBC = \angle FCD = \beta$, and $\angle EAG = \angle ABF = \angle FCB = \alpha = 90^\circ - \beta$.

Now, since $\triangle AGB$ and $\triangle BFC$ are right triangles, with $AB = BC$, and $\angle GAB = \angle FBC$, we see that $\triangle AGB$ is congruent to $\triangle BFC$. Hence, $FC = BG$.

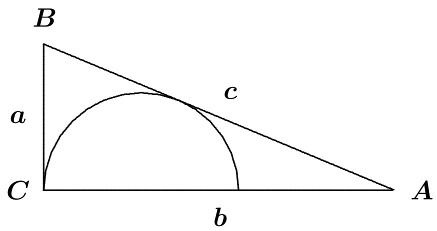
Finally, we have $\triangle BGC$ congruent to $\triangle CFD$, because $BG = FC$, $BC = DC$, and $\angle GBC = \angle FCD$. Therefore, $CG = DF$.

*Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; HASAN DENKER, Istanbul, Turkey; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; TAICHI MAEKAWA, Takatsuki City, Osaka, Japan (4 solutions); KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India; DANIEL TSAI, student, Taipei American School, Taipei, Taiwan; and TITU ZVONARU, Comănești, Romania.*

*Amengual Covas, Denker, and Zvonaru also solved this more general case of the problem.*

**M270**. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

A right triangle has legs of lengths $a$ and $b$ and a hypotenuse of length $c$. A semicircle has its diameter on the side of length $b$ and is tangent to the other two sides. Determine the radius of the semicircle in terms of $a$, $b$, and $c$.

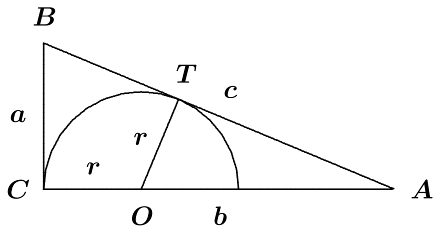


*Solution by Natalia Desy, student, Palembang, Indonesia.*

In triangle $ABC$, let $T$ denote the point of tangency of side $AB$ to the inscribed semicircle with centre $O$ and radius $r$.

Since $\angle OTA = 90°$, we see that $\sin A = r/(b-r)$; but we also have $\sin A = a/c$. Equating these two expressions for $\sin A$ yields $cr = ab - ar$. Solving for $r$ gives

$$r = \frac{ab}{a+c}.$$




*Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; DENISE CORNWELL, student, Angelo State University, San Angelo, TX, USA; HASAN DENKER, Istanbul, Turkey; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; GUSTAVO KRIMKER, Universidad CAECE, Buenos Aires, Argentina; TAICHI MAEKAWA, Takatsuki City, Osaka, Japan (6 solutions); KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India; DANIEL TSAI, student, Taipei American School, Taipei, Taiwan; and TITU ZVONARU, Comăneşti, Romania.*

**M271**. *Proposé par Yakub N. Aliyev, Université d'Etat de Bakou, Bakou, Azerbaïdjan.*

Sachant que dans un hexagone convexe $ABCDEF$, les côtés $BC$, $DE$ et $FA$ sont respectivement parallèles aux diagonales $AD$, $CF$ et $EB$, on désigne respectivement par $K$, $L$ et $M$ les intersections des droites $AB$ avec $CD$, $CD$ avec $EF$, et $EF$ avec $AB$ ; on désigne enfin par $P$, $Q$ et $R$ les intersections respectives de $CF$ avec $BE$, de $BE$ avec $AD$, et de $AD$ avec $CF$. Montrer que $KP$, $MR$ et $LQ$ se coupent en un même point.

*Solution par Saturnino Campo Ruiz, "Fray Luis de León" de Salamanca, Salamanca, Espagne, modifié par le rédacteur.*

Par la réciproque du Théorème de Pascal, l'hexagone $BCFADE$, dont les côtés opposés sont parallèles, peut être inscrit dans une conique, donc les sommets de l'hexagone $ABEFCD$ sont sur une conique. Par le Théorème de Pascal, les côtés opposés de l'hexagone $ABEFCD$ se coupent en trois points collinéaires. Par le Théorème de Desargues, $\triangle PQR$ et $\triangle KLM$ sont en perspective axiale si et seulement si ils sont en perspective centrale, donc $KP$, $MR$, et $LQ$ se coupent en un même point.

*Une solution incorrecte a aussi été soumise.*

**M272**. *Proposed by John Grant McLoughlin, University of New Brunswick, Fredericton, NB.*

Let $ABCD$ be a parallelogram, and let $P$ be a point situated on $AB$. If the ratio of the area of triangle $ABC$ to that of quadrilateral $APCD$ is $m/n$, determine the ratio of $AP$ to $PB$.

*Composite of solutions submitted by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam; Hasan Denker, Istanbul, Turkey; Richard I. Hess, Rancho Palos Verdes, CA, USA; Kunal Singh, student, Kendriya Vidyalaya School, Shillong, India; and Titu Zvonaru, Comăneşti, Romania.*

Let $[\mathcal{P}]$ denote the area of polygon $\mathcal{P}$. The ratio of the area of triangle $ABC$ to that of quadrilateral $APCD$ is $[ABC]/[APCD] = m/n$. Then

$$\frac{n}{m} \;=\; \frac{[APCD]}{[ABC]}\,.$$

Since $[APCD] = [ACD] + [APC]$ and $[ACD] = \frac{1}{2}[ABCD] = [ABC]$, we have

$$\frac{n}{m} \;=\; \frac{[ACD]+[APC]}{[ABC]} \;=\; 1+\frac{[APC]}{[ABC]}\,.$$

Since triangles $APC$ and $ABC$ share a common height with respect to their bases $AP$ and $AB$, we see that $[APC]/[ABC] = AP/AB$. Therefore,

$$\frac{n}{m} \;=\; 1+\frac{AP}{AB} \;=\; 1+\frac{AP}{AP+PB}\,.$$

Then

$$\frac{AP}{AP+PB} \;=\; \frac{n-m}{m}\,;$$

that is,

$$m\cdot AP \;=\; (n-m)AP+(n-m)PB\,.$$

Thus,

$$\frac{AP}{PB} \;=\; \frac{n-m}{2m-n}\,.$$

*There were two incorrect solutions submitted.*

**M273**. *Proposed by John Grant McLoughlin, University of New Brunswick, Fredericton, NB.*

The letters $A$, $B$, $C$, $D$, $E$, $F$, $G$, and $H$ represent distinct digits. Determine their values given that the two products shown are true. (Note that the first digit of a number must be non-zero.)

$$\begin{array}{r} ABCD \\ \times E \\ \hline DCBA \end{array} \qquad\qquad \begin{array}{r} BFDG \\ \times G \\ \hline GDFB \end{array}$$

*Solution by the proposer, modified by the editor.*

Consider the second product above. From the thousands digit, we see that $G\cdot B \le G$ and, hence, $B = 1$. Then, since $G\cdot G$ ends in $B = 1$, we have $G = 9$. Now the product is $1FD9\cdot 9 = 9DF1$. We cannot have $F > 1$, because the product $1FD9\cdot 9$ would then have 5 digits instead of 4, and we cannot have $F = 1$, since $B = 1$. Thus $F = 0$. From $10D9\cdot 9 = 9D01$, we find that the only choice for $D$ is $D = 8$.

The first product is now $A1C8 \cdot E = 8C1A$, with $A$, $C$, and $E$ having values in $\{2, 3, 4, 5, 6, 7\}$. Since $A1C8 \cdot E$ is even, we see that $8C1A$ is even, which implies that $A$ is even. We then note that the digit 8 in $8C1A$ comes from adding a carried digit to the product $A \cdot E$. The carried digit is at most 1, since it comes from $1C8 \cdot E$; therefore, $A \cdot E$ is either 7 or 8. Since $A$ is even, we must have $A \cdot E = 8$. Now, either $A = 2$ and $E = 4$, or $A = 4$ and $E = 2$. If $A = 4$ and $E = 2$, we have $41C8 \cdot 2 = 8C14$, which is impossible. We conclude that $A = 2$ and $E = 4$, which gives us $21C8 \cdot 4 = 8C12$. Finally, we verify that the only choice for $C$ is $C = 7$.

Thus, $A = 2$, $B = 1$, $C = 7$, $D = 8$, $E = 4$, $F = 0$, and $G = 9$, and the products are

$$\begin{array}{r} 2178 \\ \times 4 \\ \hline 8712 \end{array} \qquad \begin{array}{r} 1089 \\ \times 9 \\ \hline 9801 \end{array}$$

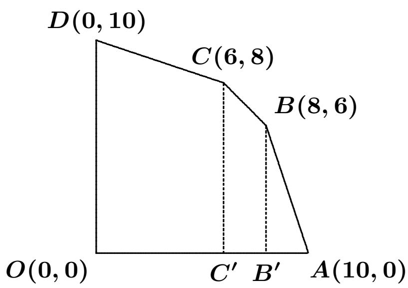
*Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; DANIEL TSAI, student, Taipei American School, Taipei, Taiwan; and TITU ZVONARU, Comăneşti, Romania. One incorrect solution was also submitted.*

**M274**. *Proposed by Neven Jurič, Zagreb, Croatia.*

Determine the area of the polygon whose vertices are all the points on the circle $x^2 + y^2 = 100$ where both coordinates are integers.

*Solution by Titu Zvonaru, Comăneşti, Romania.*

Let $O$ be the centre of the polygon. Considering the vertices of the polygon in the first quadrant only, we obtain only the vertices $A(10, 0)$, $B(8, 6)$, $C(6, 8)$, and $D(10, 0)$. Since the polygon is symmetrical about both the $x$-axis and $y$-axis, its area is four times the area of pentagon $OABCD$. To find the area of $OABCD$, we sum the areas of the trapezoids $OC'CD$ and $C'B'BC$ and the triangle $B'AB$, where $B'$ and $C'$ are the orthogonal projections of $B$ and $C$, respectively, onto the $x$-axis.




If we denote the area of polygon $\mathcal{P}$ by $[\mathcal{P}]$, then

$$\begin{aligned} [OABCD] &= [OC'CD] + [C'B'BC] + [B'AB]\,, \\ &= \tfrac{1}{2} \cdot 6 \cdot (10 + 8) + \tfrac{1}{2} \cdot 2 \cdot (8 + 6) + \tfrac{1}{2} \cdot 2 \cdot 6 \;=\; 74\,. \end{aligned}$$

Hence, the area of the polygon is $4(74) = 296$.

*Also solved by DENISE CORNWELL, student, Angelo State University, San Angelo, TX, USA; HASAN DENKER, Istanbul, Turkey; NATALIA DESY, student, Palembang, Indonesia; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and DANIEL TSAI, student, Taipei American School, Taipei, Taiwan. There was one incorrect solution submitted.*

**M275**. *Proposed by K.R.S. Sastry, Bangalore, India.*

A primitive Pythagorean triangle (PPT) is a right triangle whose sides have lengths which are integers with a greatest common divisor of 1. Among all pairs of non-congruent PPTs which have congruent incircles with an integer radius, find a pair for which this radius is minimized.

*Solution by Titu Zvonaru, Comăneşti, Romania.*

It is known that a PPT has hypotenuse of length $m^2 + n^2$ and legs of length $2mn$ and $m^2 - n^2$, where $m$ and $n$ are integers of different parity with $\gcd(m, n) = 1$.

Let $ABC$ be a PPT, and let $r$ be its inradius. Let $s = \frac{1}{2}(a + b + c)$ (the semiperimeter of $\triangle ABC$). On the one hand, the area of $\triangle ABC$ is $\frac{1}{2}(2mn)(m^2 - n^2) = mn(m^2 - n^2)$; on the other hand, the area is $rs$. Thus,

$$\begin{aligned} r = \frac{mn(m^2 - n^2)}{s} &= \frac{mn(m^2 - n^2)}{\frac{1}{2}(2mn + m^2 - n^2 + m^2 + n^2)} \\ &= \frac{mn(m^2 - n^2)}{mn + m^2} = n(m - n)\,. \end{aligned}$$

Searching for a pair of PPTs with minimal $r$, we find that:

(i) If $r = 1$, then $n(m - n) = 1$, which implies that $n = 1$ and $m = 2$. Therefore, we only have one PPT when $r = 1$.

(ii) If $r = 2$, then $n(m - n) = 2$, which implies that $(m, n) = (3, 1)$ or $(m, n) = (3, 2)$. Since 1 and 3 have same parity, we again have only one PPT when $r = 2$.

(iii) If $r = 3$, then $n(m - n) = 3$, which implies that $(m, n) = (4, 1)$ or $(m, n) = (4, 3)$. These lead to the required pair of PPTs.

Therefore, the pair of PPTs having congruent incircles with minimal integer radius are the triangle with side lengths 8, 15, 17 and the triangle with side lengths 7, 24, 25.

*Also solved by HASAN DENKER, Istanbul, Turkey; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; DANIEL TSAI, student, Taipei American School, Taipei, Taiwan; and the proposer.*

# Problem of the Month

Ian VanderBurgh

As the final Problem of the Month for 2007, we have a mean sort of problem . . .

**Problem** (2001 American Invitational Mathematics Examination)

A finite set $S$ of distinct real numbers has the following properties: the mean of $S \cup \{1\}$ is $13$ less than the mean of $S$ and the mean of $S \cup \{2001\}$ is $27$ more than the mean of $S$. Find the mean of $S$.

Have you guessed that this was a problem from a 2001 contest? One notational reminder: $S \cup \{1\}$ means the *union* of $S$ and $\{1\}$. In other words, it is the set that we get by adding $1$ to the list of numbers already in $S$. (Actually, there is an additional technicality here that we'll look at quickly after the solution of the problem.)

Problems involving means usually require us to remember the fact that the mean of a list of numbers equals the sum of the numbers divided by the number of numbers in the list. In fact, that's all we really need to know here.

*Solution:* Let $n$ be the number of numbers in the set $S$ (in mathematical language, $n$ is the *cardinality* of $S$). Let $u$ be the sum of the numbers in the set $S$. Then the mean of the numbers in $S$ equals $\frac{u}{n}$. When the number 1 is added to the set $S$, the new mean is $\frac{u+1}{n+1}$ since the sum of the number in $S$ increases by 1 and the number of numbers in $S$ also increases by $1$. Similarly, when the number $2001$ is added to the set $S$, the new mean is $\frac{u+2001}{n+1}$.

Therefore, the given information tells us

$$\begin{aligned} \frac{u+1}{n+1} - \frac{u}{n} &= -13\,, \\ \frac{u+2001}{n+1} - \frac{u}{n} &= 27\,. \end{aligned}$$

We need to find the mean of $S$, in other words $\frac{u}{n}$. In order to find $\frac{u}{n}$, it looks like we pretty much have to solve this system of two equations in two unknowns for $u$ and $n$. However, there is a slight wrinkle: the AIME exam from which this problem is taken did not permit the use of a calculator! (For some of us, this may be more of a concern than for others.) So let's try to solve this system of equations in a clever way.

Can you see a manipulation that we can perform that will allow us to solve for $n$ almost immediately? Try fiddling around for a couple of minutes before reading on.

Did you get it? What happens when we subtract the first equation from the second? When we do this, we get

$$\frac{u+2001}{n+1} - \frac{u+1}{n+1} - \frac{u}{n} + \frac{u}{n} \;=\; 27 - (-13)\,,$$

which simplifies to $\dfrac{2000}{n+1} = 40$; that is, $n+1 = 50$, or $n = 49$.

Now we need to find $u$. Substituting $n = 49$ into the first equation, we obtain

$$\frac{u+1}{50} - \frac{u}{49} \;=\; -13\,.$$

Being without a calculator, the idea of trying to get a common denominator on the left side seems a bit scary. Remember, though, that we really need $\dfrac{u}{49}$ (not $u$):

$$\begin{aligned}
\frac{u}{50} + \frac{1}{50} - \frac{u}{49} &= -13\,,\\
\frac{49}{50}\left(\frac{u}{49}\right) - \frac{50}{50}\left(\frac{u}{49}\right) &= -13 - \frac{1}{50}\,,\\
-\frac{1}{50}\left(\frac{u}{49}\right) &= -13 - \frac{1}{50}\,,\\
\frac{u}{49} &= 50(13) + 1 \;=\; 651\,.
\end{aligned}$$

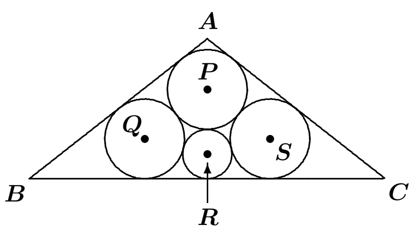
(That worked out pretty well—no ugly calculations!) Therefore, the mean of $S$ is 651.

Those of you who have written the AIME before will be relieved by this answer, as it is an integer between 000 and 999, as per AIME prescription.

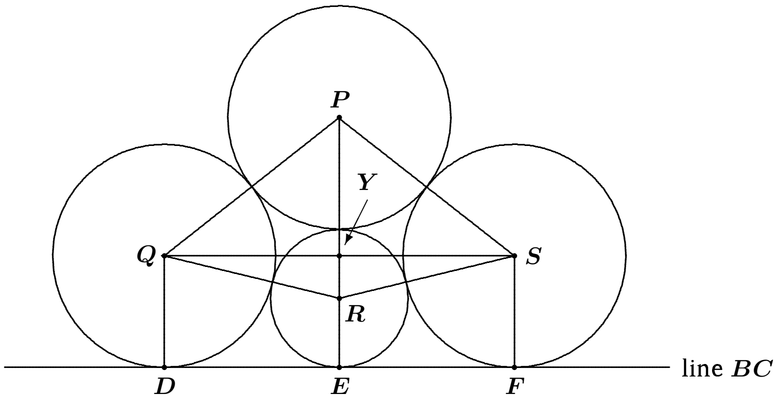
I mentioned a technicality before we launched into the solution. When we look at the union $S \cup \{1\}$, technically we add 1 to the set $S$ only if 1 does not already appear in $S$. (For example, $\{1,\ 2,\ 3\} \cup \{1\} = \{1,\ 2,\ 3\}$.) Do we need to worry about this here? In fact, we don't: since the mean actually changes when we perform each of the two unions, the numbers 1 and 2001 could not have been in $S$ to begin with.

Last month, I left you with a challenge problem, adapted from this year's Hypatia Contest. We repeat the problem, followed by its solution:

In the diagram, the circles with centres $P$, $Q$ and $S$ all have radius 1. Each is tangent to two sides of the isosceles $\triangle ABC$ and to the circle with centre $R$; the circle with centre $P$ is tangent to both of the other circles of radius 1. What is the radius of the circle with centre $R$?



*Solution to November's Challenge Problem:* Drop perpendiculars to $BC$ from $Q$, $R$, and $S$ at $D$, $E$, and $F$, respectively. Since the circles with centres $Q$, $R$, and $S$ are tangent to $BC$, we see that $D$, $E$, and $F$ are the points of tangency of these circles to $BC$. Thus, $QD = SF = 1$. Let $RE = r$.




Join $QR$, $RS$, $SP$, $PQ$, and $PR$. Since we are connecting centres of tangent circles, then $PQ = PS = 2$ and $QR = RS = PR = 1 + r$. By symmetry, $PRE$ is a straight line (that is, $PE$ passes through $R$). Join $QS$. Since $QD$ and $SF$ are perpendicular to $BC$, then $QS$ is parallel to $BC$. Thus, $QS$ is perpendicular to $PR$, meeting at $Y$. Since $QD = 1$, then $YE = 1$. Since $RE = r$, then $YR = 1 - r$. Since $QR = 1 + r$, $YR = 1 - r$, and $\triangle QYR$ is right-angled at $Y$, then, by the Pythagorean Theorem,

$$\begin{aligned} QY^2 = QR^2 - YR^2 &= (1+r)^2 - (1-r)^2 \\ &= (1 + 2r + r^2) - (1 - 2r + r^2) = 4r\,. \end{aligned}$$

Since $PR = 1 + r$ and $YR = 1 - r$, then $PY = PR - YR = 2r$. Since $\triangle PYQ$ is right-angled at $Y$, then

$$\begin{aligned} PY^2 + YQ^2 &= PQ^2\,, \\ (2r)^2 + 4r &= 2^2\,, \\ 4r^2 + 4r &= 4\,, \\ r^2 + r - 1 &= 0\,. \end{aligned}$$

By the quadratic formula, $r = \dfrac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2} = \dfrac{-1 \pm \sqrt{5}}{2}$. Since $r > 0$, then $r = \dfrac{-1 + \sqrt{5}}{2}$ (which is the reciprocal of the famous "golden ratio").