SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

3189. [2006 : 514, 517] *Proposed by K.R.S. Sastry, Bangalore, India.*

In $\triangle ABC$, let A be the largest of the three angles. Let α denote the measure of angle A, and let h, w, and m denote the lengths of the altitude, the internal angle bisector, and the median, all measured from A to the side BC.

(a) Determine the area of $\triangle ABC$ in terms of α, h, and w.

(b) Determine the area of $\triangle ABC$ in terms of α, m, and w.

[*Ed:* The reader may wish to look at Mayhem problem M63 and the accompanying solution [2003 : 427–428].]

Solution by Geoffrey A. Kandall, Hamden, CT, USA.

Since $[ABC] = \frac{1}{2}bc\sin\alpha$, in both parts of the problem we want to express bc in terms of the given data.

(a) Let T be the foot of the bisector of angle A. We use the standard formula $bc = w^2 + BT \cdot TC$. By the Law of Sines,

$$\frac{BT}{\sin\left(\frac{1}{2}\alpha\right)} = \frac{c}{\sin\angle ATB} = \frac{cw}{h}\,.$$

Similarly,

$$\frac{TC}{\sin\left(\frac{1}{2}\alpha\right)} = \frac{b}{\sin\angle CTA} = \frac{bw}{h}\,.$$

Thus, $bc = w^2 + w^2\sin^2\left(\frac{1}{2}\alpha\right) \cdot bc/h^2$; whence,

$$bc = \frac{h^2w^2}{h^2 - w^2\sin^2\left(\frac{1}{2}\alpha\right)}\,.$$

Consequently,

$$[ABC] = \frac{h^2w^2\sin\alpha}{2\left(h^2 - w^2\sin^2\left(\frac{1}{2}\alpha\right)\right)}\,.$$

(b) Here we use the standard formulas $4m^2 = b^2 + c^2 + 2bc\cos\alpha$ and $b + c = 2bc\cos\left(\frac{1}{2}\alpha\right)/w$. We modify the former to

$$4m^2 = (b+c)^2 - 2bc(1-\cos\alpha) = (b+c)^2 - 4bc\sin^2\left(\tfrac{1}{2}\alpha\right)$$

and substitute the value of $b+c$ from the latter to obtain

$$4m^2 \;=\; \frac{4b^2c^2}{w^2}\cos^2\!\left(\tfrac{1}{2}\alpha\right) - 4bc\sin^2\!\left(\tfrac{1}{2}\alpha\right);$$

that is,

$$(bc)^2 - \left[w^2\tan^2\!\left(\tfrac{1}{2}\alpha\right)\right](bc) - m^2w^2\sec^2\!\left(\tfrac{1}{2}\alpha\right) \;=\; 0\,.$$

Consequently,

$$bc \;=\; \frac{w}{2}\left(w\tan^2\!\left(\tfrac{1}{2}\alpha\right) + \sqrt{w^2\tan^4\!\left(\tfrac{1}{2}\alpha\right) + 4m^2\sec^2\!\left(\tfrac{1}{2}\alpha\right)}\right)\;;$$

hence,

$$[ABC] \;=\; \frac{w\sin\alpha}{4}\left(w\tan^2\!\left(\tfrac{1}{2}\alpha\right) + \sqrt{w^2\tan^4\!\left(\tfrac{1}{2}\alpha\right) + 4m^2\sec^2\!\left(\tfrac{1}{2}\alpha\right)}\right)\,.$$

Also solved by ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam (part (a) only); CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA (part (a) only); WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; JOEL SCHLOSBERG, Bayside, NY, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

The idea for the problem was suggested by M63 [2003 : 427–428], which dealt with a particular right triangle. The proposer believes that there might even be an interesting formula for the area of $\triangle ABC$ given α together with the length of two cevians from A; presumably, the cevians should be described by the ratio into which their feet divide BC.

3190. [2006 : 514, 517] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Let A be a point on the circle Γ, and let P be a point outside Γ. Construct a line ℓ through P which intersects Γ at B and C such that

$$2(BC) \;=\; AB + AC\,.$$

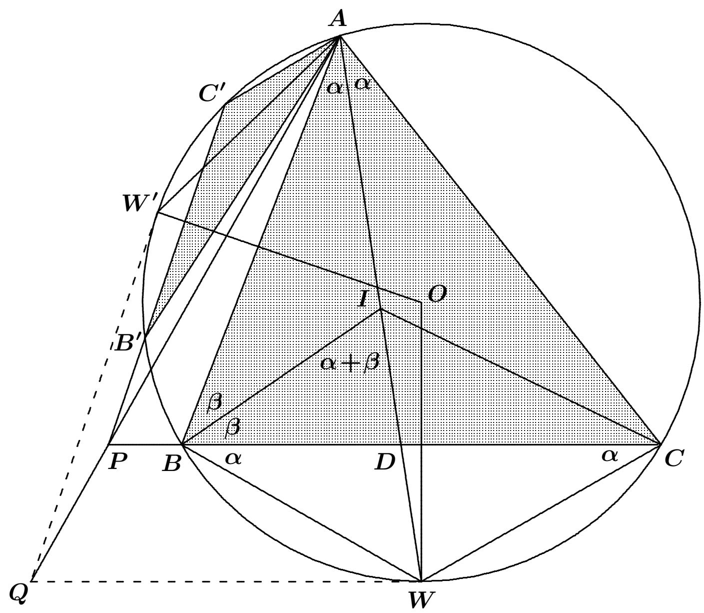
Solution by Claudio Arconcher, Jundiaí, Brazil.

Analysis. Suppose there is a point D on side BC of $\triangle ABC$ such that $BD = \frac{1}{2}AB$ and $DC = \frac{1}{2}AC$. We then would have $2(BC) = AB + AC$, which is the desired condition. With this choice of D, we have

$$\frac{AB}{BD} \;=\; \frac{AC}{DC} \;=\; \frac{2}{1}\,;$$

whence, AD is the bisector of $\angle BAC$. The incentre I of $\triangle ABC$ therefore lies on AD; whence (since BI bisects $\angle CBA$), $AB : BD = AI : ID = 2 : 1$, or

$$AI \;=\; 2ID\,.$$



Let W be the point of intersection of the circumcircle Γ of $\triangle ABC$ with the interior angle bisector of $\angle BAC$. Because $\triangle ABW \sim \triangle BDW$ [they share the angle at W and $\angle WBD = \angle WBC = \angle WAC = \angle WAB$, marked α in the figure], we have $AB : BD = BW : DW = 2 : 1$, so that $BW = 2DW$. Furthermore, since $BW = IW$ [$\triangle WBI$ is isosceles because the angles at B and at I both equal $\alpha + \beta$, as shown in the figure], we have

$$IW = 2DW = 2ID.$$

It follows that

$$AW = \tfrac{4}{3}AD.$$

Finally, note that the radius OW of Γ is perpendicular to side BC (because W is the mid-point of arc BC), which implies that the tangent to Γ at W is parallel to the line BC. That tangent, therefore, meets the line AP in a point, say Q, for which $AQ = \frac{4}{3}AP$.

Construction. Construct the point Q on the extension of AP beyond P so that $PQ = \frac{1}{3}PA$. Draw either tangent from Q to the circle Γ, calling W the point of tangency. The line through P parallel to QW is then a line that meets Γ in points B and C for which $2(BC) = AB + AC$, as was to be constructed. Since there are two lines tangent to the circumcircle from Q, there are two solution lines—the second is denoted by $B'C'$ in the figure.

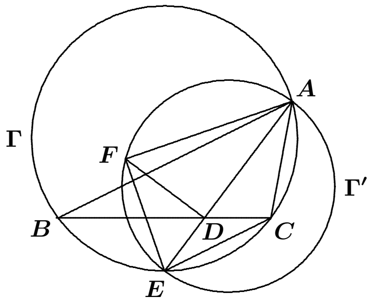
Also solved by MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; SOTIRIS LOURIDAS, Aegaleo, Greece; and the proposer.

Triangles for which $2(BC) = AB + AC$ have many interesting properties. Note that problem 3197 (later in this issue) deals with such a triangle. Some references are given there.

3191. [2006 : 515, 517] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Let Γ be the circumcircle of $\triangle ABC$, let AD be the internal angle bisector of $\angle BAC$ with D on BC, and let E be the point where AD meets Γ for the second time. Let Γ' be the circle with AE as diameter, and let F be a point of Γ' such that $DF \perp AE$. Prove that $EF = EC$.

Solution by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam.



Since F is on the circle Γ', we have $\angle AFE = 90^\circ$. Since $DF \perp AE$, we obtain $EF^2 = ED \cdot EA$. Hence, $\triangle ACE \sim \triangle CDE$. Thus, $\dfrac{AE}{CE} = \dfrac{CE}{DE}$; that is, $CE^2 = AE \cdot DE$. Therefore,

$$EF^2 = ED \cdot EA = CE^2,$$

which gives $EF = EC$.

Also solved by MOHAMMED AASSILA, Strasbourg, France; MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; CLAUDIO ARCONCHER, Jundiaí, Brazil; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GEOFFREY A. KANDALL, Hamden, CT, USA; TAICHI MAEKAWA, Takatsuki City, Osaka, Japan; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; JOEL SCHLOSBERG, Bayside, NY, USA; BOB SERKEY, Leonia, NJ, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

Many of the submitted solutions were equally concise!

3192. [2006 : 515, 517] *Proposed by Mihály Bencze, Brasov, Romania.*

Let $k \in (0,1)$, and let the sequence $\{B_n\}_{n=0}^{\infty}$ be defined by $B_0 = k$, $B_1 = k^2$, and $B_{n+2} = kB_{n+1} + k^2 B_n$ for integers $n \geq 0$. Find $\sum\limits_{n=0}^{\infty} \dfrac{B_n}{n+1}$.

Solution by Michel Bataille, Rouen, France, modified by the editor.

We denote by $\{f_n\}$ the Fibonacci sequence, defined by $f_0 = 0$, $f_1 = 1$, and $f_{n+1} = f_n + f_{n-1}$ for $n \geq 1$. One can show by induction that $B_n = k^{n+1} f_{n+1}$ for all non-negative integers n. Thus, we need to find $\sum\limits_{n=0}^{\infty} f_{n+1} \dfrac{k^{n+1}}{n+1} = \sum\limits_{n=1}^{\infty} f_n \dfrac{k^n}{n}$.

Let $\alpha = \frac{1}{2}\left(1+\sqrt{5}\right)$ and $\beta = \frac{1}{2}\left(1-\sqrt{5}\right)$. Dropping the condition $k \in (0,1)$, we will prove that the series above converges if and only if $\beta \leq k < -\beta$, and that, in this case, it converges to $\dfrac{1}{\sqrt{5}} \ln\left(\dfrac{1-\beta k}{1-\alpha k}\right)$.

It is well known that the power series $\sum\limits_{n=0}^{\infty} f_n x^x$ converges if and only if $|x| < |\beta|$, in which case $\sum\limits_{n=0}^{\infty} f_n x^n = \frac{x}{1-x-x^2}$, or equivalently,

$$\sum_{n=1}^{\infty} f_n x^{n-1} \;=\; \frac{1}{1-x-x^2}\,.$$

By integration, it follows that the radius of convergence of the power series $\sum\limits_{n=1}^{\infty} f_n \frac{x^n}{n}$ is $|\beta|$, and that, for $|x| < |\beta|$,

$$\begin{aligned}\sum_{n=1}^{\infty} f_n \frac{x^n}{n} &= \int_0^x \frac{1}{1-t-t^2}\,dt \;=\; \frac{1}{\sqrt{5}} \int_0^x \left(\frac{1}{t+\alpha} - \frac{1}{t+\beta}\right) dt \\ &= \frac{1}{\sqrt{5}} \ln\left(\frac{1-\beta x}{1-\alpha x}\right)\,.\end{aligned}$$

As a result, $\sum\limits_{n=0}^{\infty} \frac{B_n}{n+1}$ diverges for $|k| > |\beta|$ and converges for $|k| < |\beta|$, and in the case of convergence, we have

$$\sum_{n=0}^{\infty} \frac{B_n}{n+1} \;=\; \sum_{n=1}^{\infty} f_n \frac{k^n}{n} \;=\; \frac{1}{\sqrt{5}} \ln\left(\frac{1-\beta k}{1-\alpha k}\right)\,.$$

It remains to examine the cases $k = \beta$ and $k = -\beta$. From $k = -\beta$, since

$$\lim_{k\to-\beta^-} \ln\left(\frac{1-\beta k}{1-\alpha k}\right) \;=\; \infty\,,$$

it follows from Abel's Limit Theorem that $\sum\limits_{n=1}^{\infty} f_n \frac{(-\beta)^n}{n}$ diverges. Now consider $k = \beta$. Using $f_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n)$, we obtain

$$f_n \frac{\beta^n}{n} \;=\; \frac{1}{\sqrt{5}} \left(\frac{(-1)^n}{n} - \frac{(\beta^2)^n}{n}\right)\,.$$

Since $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n}$ and $\sum\limits_{n=1}^{\infty} \frac{(\beta^2)^n}{n}$ both converge, the same is true for $\sum\limits_{n=1}^{\infty} f_n \frac{\beta^n}{n}$, and (using Abel's Limit Theorem again)

$$\sum_{n=1}^{\infty} f_n \frac{\beta^n}{n} \;=\; \lim_{k\to\beta^+} \ln\left(\frac{1-\beta k}{1-\alpha k}\right) \;=\; \ln\left(\frac{1-\beta^2}{2}\right)\,.$$

The proof is complete.

Also solved by BRIAN D. BEASLEY, Presbyterian College, Clinton, SC, USA; JOHN HAWKINS and DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and JOE HOWARD, Portales, NM, USA.

There were also two partially incorrect answers and three incomplete solutions which did not consider the values of k for which the given series is convergent.

3193. [2006 : 517] *Proposed by Mihály Bencze, Brasov, Romania.*

Let ABC be a triangle, and let A_1, B_1, C_1 be on sides BC, CA, AB, respectively, such that

$$\frac{AC_1}{C_1B} = \frac{BA_1}{A_1C} = \frac{CB_1}{B_1A} = k\,,$$

where k is a positive constant. Let H and H_1 be the orthocentres of $\triangle ABC$ and $\triangle A_1B_1C_1$, respectively, and let O and O_1 be their respective circumcentres. Prove that $OO_1 \parallel HH_1$.

I. *Solution by Michel Bataille, Rouen, France.*

From the definition of A_1, we have $\overrightarrow{A_1B} + k\overrightarrow{A_1C} = \overrightarrow{0}$; whence,

$$(1+k)\overrightarrow{O_1A_1} = \overrightarrow{O_1B} + k\overrightarrow{O_1C}\,.$$

Similar relations hold for $\overrightarrow{O_1B_1}$ and $\overrightarrow{O_1C_1}$.

Now, from the well-known relations $\overrightarrow{O_1H_1} = \overrightarrow{O_1A_1} + \overrightarrow{O_1B_1} + \overrightarrow{O_1C_1}$ and $\overrightarrow{OH} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$, we deduce that

$$(1+k)\overrightarrow{O_1H_1} = \overrightarrow{O_1B} + k\overrightarrow{O_1C} + \overrightarrow{O_1C} + k\overrightarrow{O_1A} + \overrightarrow{O_1A} + k\overrightarrow{O_1B}\,;$$

that is,

$$\overrightarrow{O_1H_1} = \overrightarrow{O_1A} + \overrightarrow{O_1B} + \overrightarrow{O_1C} = 3\overrightarrow{O_1O} + \overrightarrow{OH}\,.$$

Thus, $\overrightarrow{HH_1} = 2\overrightarrow{O_1O}$, which implies the desired result.

II. *Solution by Chip Curtis, Missouri Southern State University, Joplin, MO, USA.*

By Theorem 276 [1, p.175], the proportionality condition implies that triangles $A_1B_1C_1$ and ABC have the same centroid.

On the other hand, by Theorem 257 [1, p.165], in any triangle, the circumcentre O, the centroid G, and the orthocentre H are collinear, with $2OG = GH$. Hence, triangle OGO_1 is similar to triangle HGH_1, so that OO_1 is parallel to HH_1.

References

[1] R. Johnson, *Modern Geometry*, Houghton-Mifflin, 1929.

Also solved by APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; TAICHI MAEKAWA, Takatsuki City, Osaka, Japan; JOEL SCHLOSBERG, Bayside, NY, USA; SKIDMORE COLLEGE PROBLEM SOLVING GROUP, Skidmore College Saratoga Springs, NY, USA; D.J. SMEENK, Zaltbommel, the Netherlands; TITU ZVONARU, Comănești, Romania; and the proposer.

3194. [2005 : 515, 517] *Proposed by Mihály Bencze, Brasov, Romania.*

Let n be any positive integer, and let $x_k, y_k \in \mathbb{R}$ for $k = 1, 2, \ldots, n$. Prove that

$$\min\left\{\sum_{k=1}^{n} x_k^2, \sum_{k=1}^{n} y_k^2\right\} \cdot \sum_{k=1}^{n}(x_k - y_k)^2 \ \geq \sum_{1\leq i<j\leq n} (x_iy_j - x_jy_i)^2 .$$

Solution by Michel Bataille, Rouen, France, modified by the editor.

Without loss of generality, we suppose that $\sum\limits_{k=1}^{n} x_k^2 \leq \sum\limits_{k=1}^{n} y_k^2$. The proposed inequality then reduces to

$$\left(\sum_{k=1}^{n} x_k^2\right)\left(\sum_{k=1}^{n}(x_k - y_k)^2\right) \ \geq \sum_{1\leq i<j\leq n} (x_iy_j - x_jy_i)^2 .$$

From the well-known identity

$$\left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) = \sum_{1\leq i<j\leq n} (a_ib_j - a_jb_i)^2 + \left(\sum_{k=1}^{n} a_kb_k\right)^2 ,$$

we obtain the inequality

$$\left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) \ \geq \sum_{1\leq i<j\leq n} (a_ib_j - a_jb_i)^2$$

(for all real numbers $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$). Setting $a_k = x_k$ and $b_k = x_k - y_k$ for $k = 1, 2, \ldots, n$ yields

$$\begin{aligned}\left(\sum_{k=1}^{n} x_k^2\right)\left(\sum_{k=1}^{n}(x_k - y_k)^2\right) &\geq \sum_{1\leq i<j\leq n} \big(x_i(x_j - y_j) - x_j(x_i - y_i)\big)^2 \\ &= \sum_{1\leq i<j\leq n} (x_iy_j - x_jy_i)^2 .\end{aligned}$$

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposer.

3195. [2006 : 515, 518] *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

(a) Let n be a natural number, $n \geq 3$. Prove that there is a real number $q_n > 1$ such that for any real numbers $a_1, a_2, \ldots, a_n \in [1/q_n, q_n]$,

$$\frac{a_1}{a_2 + a_3} + \frac{a_2}{a_3 + a_4} + \cdots + \frac{a_n}{a_n + a_1} \ \geq \ \frac{n}{2} .$$

(b)★ Does there exist a real number $q > 1$ such that the inequality in (a) holds for any natural number $n \geq 3$ and for any real numbers $a_1, a_2, \ldots, a_n \in [1/q, q]$?

Solution to part (a) *by the proposer, modified by the editor.*

It is easy to see that the inequality in (a) is equivalent to

$$\sum_{\text{cyclic}} \frac{2q_n^2 a_1 - a_2 - a_3}{a_2 + a_3} \;\geq\; n(q_n^2 - 1)\,. \tag{1}$$

Since $2q_n^2 a_1 - a_2 - a_3 = (q_n^2 a_1 - a_2) + (q_n^2 a_1 - a_3) \geq 0$, the Cauchy-Schwarz Inequality may be applied to get

$$\begin{aligned}&\Bigl(\sum_{\text{cyclic}} (2q_n^2 a_1 - a_2 - a_3)\Bigr)^2 \\ &\qquad \leq\; \Bigl(\sum_{\text{cyclic}} (a_2 + a_3)(2q_n^2 a_1 - a_2 - a_3)\Bigr)\Bigl(\sum_{\text{cyclic}} \frac{2q_n^2 a_1 - a_2 - a_3}{a_2 + a_3}\Bigr)\,.\end{aligned}$$

Thus, to obtain (1), it is sufficient to prove the inequality

$$\begin{aligned}&\Bigl(\sum_{\text{cyclic}} (2q_n^2 a_1 - a_2 - a_3)\Bigr)^2 \\ &\qquad \geq\; n(q_n^2 - 1)\sum_{\text{cyclic}} (a_2 + a_3)(2q_n^2 a_1 - a_2 - a_3)\,.\end{aligned} \tag{2}$$

Since

$$\sum_{\text{cyclic}} (2q_n^2 a_1 - a_2 - a_3) \;=\; \sum_{\text{cyclic}} 2(q_n^2 - 1)a_1 \;=\; 2(q_n^2 - 1)\sum_{\text{cyclic}} a_1$$

and

$$\sum_{\text{cyclic}} (a_2 + a_3)(2q_n^2 a_1 - a_2 - a_3) \;=\; 2q_n^2 \sum_{\text{cyclic}} a_1(a_2 + a_3) - \sum_{\text{cyclic}} (a_1 + a_2)^2\,,$$

the following inequality is equivalent to (2):

$$\frac{4}{n}(q_n^2 - 1)\Bigl(\sum_{\text{cyclic}} a_1\Bigr)^2 \;\geq\; 2q_n^2 \sum_{\text{cyclic}} a_1(a_2 + a_3) - \sum_{\text{cyclic}} (a_1 + a_2)^2\,. \tag{3}$$

We have

$$2\sum_{\text{cyclic}} a_1(a_2 + a_3) \;=\; 2\sum_{\text{cyclic}} (a_1 + a_2)(a_2 + a_3) - \sum_{\text{cyclic}} (a_1 + a_2)^2\,;$$

hence, inequality (3) can be written in the form

$$\frac{4}{n}(q_n^2-1)\Big(\sum_{\text{cyclic}} a_1\Big)^2 \geq 2q_n^2 \sum_{\text{cyclic}}(a_1+a_2)(a_2+a_3) - (q_n^2+1)\sum_{\text{cyclic}}(a_1+a_2)^2 . \quad (4)$$

Using the substitution $b_i = a_i + a_{i+1}$ for $i = 1, 2, \ldots, n$, inequality (4) reduces to

$$\frac{1}{n}(q_n^2-1)\Big(\sum_{\text{cyclic}} b_1\Big)^2 \geq 2q_n^2\sum_{\text{cyclic}} b_1b_2 - (q_n^2+1)\sum_{\text{cyclic}} b_1^2 . \quad (5)$$

Since

$$\Big(\sum_{\text{cyclic}} b_1\Big)^2 = n\sum_{\text{cyclic}} b_1^2 - \sum_{j<k}(b_j-b_k)^2 ,$$

inequality (5) is equivalent to

$$n\sum_{\text{cyclic}}(b_1-b_2)^2 \geq \left(1-\frac{1}{q_n^2}\right)\sum_{j<k}(b_j-b_k)^2 . \quad (6)$$

But, for $j<k$, we have

$$\sum_{\text{cyclic}}(b_1-b_2)^2 \geq \sum_{i=j}^{k-1}(b_i-b_{i+1})^2 \geq \frac{1}{k-j}\Big(\sum_{i=j}^{k-1}(b_i-b_{i+1})\Big)^2 \geq \frac{1}{n-1}(b_j-b_k)^2$$

(where we have used the Cauchy-Schwarz Inequality). Summing over j and k with $j<k$ yields

$$\frac{n(n-1)}{2}\sum_{\text{cyclic}}(b_1-b_2)^2 \geq \frac{1}{n-1}\sum_{j<k}(b_j-b_k)^2 .$$

Comparing this inequality with (6), we see that we may obtain (6) by choosing $1-\frac{1}{q_n^2} = \frac{2}{(n-1)^2}$; that is,

$$q_n = \frac{1}{\sqrt{1-2/(n-1)^2}} = \frac{n-1}{\sqrt{n^2-2n-1}} .$$

Since $q_n > \frac{1}{\sqrt{1-2/n^2}} = \frac{n}{\sqrt{n^2-2}} > 1$, we can also choose $q_n = \frac{n}{\sqrt{n^2-2}}$.

The proposer also offered the following remarks.

1. Using the substitution $b_i = x_i - b$ for all $i = 1, 2, \ldots, n$, where $b = \dfrac{b_1 + b_2 + \cdots + b_n}{n}$ and $x_1 + x_2 + \cdots + x_n = 0$, inequality (5) becomes

$$\frac{\sum\limits_{\text{cyclic}} x_1 x_2}{\sum\limits_{\text{cyclic}} x_1^2} \leq \frac{q_n^2 + 1}{2q_n^2}.$$

According to Fan's Inequality [1]

$$\frac{\sum\limits_{\text{cyclic}} x_1 x_2}{\sum\limits_{\text{cyclic}} x_1^2} \leq \cos\frac{2\pi}{n};$$

thus, we may choose $q_n = \dfrac{1}{\sqrt{2\cos\frac{2\pi}{n} - 1}}$. This is the largest value of q_n such that (5) holds for all positive b_i.

2. The inequality

$$\frac{a_1}{a_2 + a_3} + \frac{a_2}{a_3 + a_4} + \cdots + \frac{a_n}{a_1 + a_n} \geq \frac{n}{2}$$

is the well-known Shapiro Inequality. It is valid for any positive real numbers $a_1, a_2, \ldots, a_n$, for even $n \leq 12$ and odd $n \leq 23$.

3. The previous results are not enough to solve part (b) of the problem because $\lim\limits_{n\to\infty} q_n = 1$.

References

[1] K. Fan, O. Taussky, and J. Todd, *Discrete Analogs of Inequalities of Wirtinger*, Monatsh. Math. 59 (1955), 73–79.

Part (a) *also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria. Part* (b) *remains open.*

3196. [2006 : 515, 518] *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

Let $x_1, x_2, \ldots, x_n$ be positive real numbers. Prove that

$$x_1^n + x_2^n + \cdots + x_n^n + n(n-1)x_1x_2\cdots x_n$$
$$\geq x_1x_2\cdots x_n(x_1 + x_2 + \cdots + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right).$$

Solution by Gabriel Dospinescu, University of Bucharest, Romania.

We will use induction. For $n = 2$, equality holds. Assuming that the inequality is true for $n-1$ (where $n \geq 3$), we will show that it is true for n.

Using the induction hypothesis, we obtain, for each $k \in \{1, 2, \ldots, n\}$,

$$x_k \sum_{i \neq k} x_i^{n-1} + (n-1)(n-2)x_1 x_2 \cdots x_n \ \geq \ x_1 x_2 \cdots x_n \left(\sum_{i \neq k} x_i \right) \left(\sum_{i \neq k} \frac{1}{x_i} \right) .$$

We write this in a more useful form:

$$x_k \sum_{i=1}^{n} x_i^{n-1} \ - \ x_k^n \ + \ (n-1)(n-2)x_1 x_2 \cdots x_n$$
$$\geq \ x_1 x_2 \cdots x_n \left[\left(\sum_{i=1}^{n} x_i \right) \left(\sum_{i=1}^{n} \frac{1}{x_i} \right) - x_k \sum_{i=1}^{n} \frac{1}{x_i} - \frac{1}{x_k} \sum_{i=1}^{n} x_i + 1 \right] .$$

Summing over k, we find that

$$\left(\sum_{i=1}^{n} x_i \right) \cdot \left(\sum_{i=1}^{n} x_i^{n-1} \right) \ - \ \sum_{i=1}^{n} x_i^n \ + \ n(n-1)(n-2)x_1 x_2 \cdots x_n$$
$$\geq \ x_1 x_2 \cdots x_n \left[(n-2) \left(\sum_{i=1}^{n} x_i \right) \left(\sum_{i=1}^{n} \frac{1}{x_i} \right) + n \right] .$$

Using Suranyi's Inequality (see [1]),

$$\left(\sum_{i=1}^{n} x_i \right) \left(\sum_{i=1}^{n} x_i^{n-1} \right) \ \leq \ (n-1) \sum_{i=1}^{n} x_i^n \ + \ n x_1 x_2 \cdots x_n \, ,$$

we get

$$(n-2) \sum_{i=1}^{n} x_i^n \ + \ n x_1 x_2 \cdots x_n \ + \ n(n-1)(n-2)x_1 x_2 \cdots x_n$$
$$\geq \ x_1 x_2 \cdots x_n \left[(n-2) \left(\sum_{i=1}^{n} x_i \right) \left(\sum_{i=1}^{n} \frac{1}{x_i} \right) + n \right] ,$$

which simplifies to give the desired result.

For $n \geq 3$, equality holds if and only if $x_1 = x_2 = \cdots = x_n$.

References

[1] T. Andreescu, V. Cîrtoaje, G. Dospinescu, M. Lascu, *Old and New Inequalities*, GIL Publishing House, Zalau, Romania, 2004, pp. 110–111.

No other solutions were submitted. (The above solution was the one that accompanied the problem proposal.)

Walther Janous, Ursulinengymnasium, Innsbruck, Austria noted that this problem is Proposition 3.7 in the proposer's paper entitled The Equal Variable Method, which was published in the electonically distributed journal JIPAM (Journal of Inequalities in Pure and Applied

Mathematics, http://jipam.vu.edu.au), *Volume 8 (2007), Issue 1, Article 15, p. 28. That paper gives a reference to* http://www.mathlinks.ro/Forum/viewtopic.php?t=14906, *where the proposer had previously posted the inequality. Solutions to the problem are given at both of these locations.*

3197. [2006 : 516, 518] *Proposed by Paul Deiermann, Southeast Missouri State University, Cape Girardeau, MO, USA.*

If AB is a fixed line segment, find the triangle ABC which has maximum area among those which satisfy $\angle AIO = \pi/2$, where I is the incentre of $\triangle ABC$ and O is its circumcentre. What is this maximum area?

Solution by Michel Bataille, Rouen, France.

Let $a = BC$, $b = CA$, and $c = AB$, as usual. It has been shown more than once in this journal that $\angle AIO = \pi/2$ if and only if $2a = b + c$. (See the references below.) It follows that the area F of a triangle satisfying $\angle AIO = \pi/2$ is given by

$$16F^2 = (a+b+c)(a+b-c)(b+c-a)(c+a-b) = 3a^2(3a-2c)(2c-a)\,.$$

Moreover, a must satisfy $a > |b - c|$, which is equivalent to requiring that $2a - 2c < a$ and $2c - 2a < a$. It follows that

$$\tfrac{2}{3}c < a < 2c\,.$$

Without loss of generality, we can set $c = 1$. Let $f(x) = x^2(3x - 2)(2 - x)$. The derivative of f is then

$$f'(x) = -12x\left(x - \tfrac{3+\sqrt{3}}{3}\right)\left(x - \tfrac{3-\sqrt{3}}{3}\right)\,.$$

Since $0 < \frac{3-\sqrt{3}}{3} < \frac{2}{3} < \frac{3+\sqrt{3}}{3} < 2$, we see that f reaches its maximum on $\left(\frac{2}{3}, 2\right)$ when $x = \frac{3+\sqrt{3}}{3}$. As a result, $F \le F_m$ where

$$F_m = \tfrac{\sqrt{3}}{4}\left(f\left(\tfrac{3+\sqrt{3}}{3}\right)\right)^{1/2} = \left(\tfrac{3+\sqrt{3}}{3}\right)\tfrac{\sqrt[4]{3}}{2\sqrt{2}} = \tfrac{\sqrt{9+6\sqrt{3}}}{6}\,.$$

To complete our discussion, we show that this value F_m is the area of an actual triangle ABC constructed on the side AB and satisfying $\angle AIO = \pi/2$. With $AB = 1$, let $a = \frac{3+\sqrt{3}}{3}$ and $b = 2a - 1$. Clearly, $a < b + 1$, but also $a > |b - c|$ (because $a \in \left(\frac{2}{3}, 2\right)$. Thus, we can construct a triangle ABC with sides $AB = 1$, $BC = a$, and $CA = b$. In such a triangle, $\angle AIO = \pi/2$ (because $2a = b + c$) and $F = F_m$ (because $a = \frac{3+\sqrt{3}}{3}$). This triangle therefore achieves the maximal area F_m while satisfying all the required constraints.

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; and the proposer.

A proof that $\angle AIO = \pi/2$ if and only if $2a = b+c$ (for a triangle that is not equilateral, of course) is given in [2005 : 520–521]. *Many other properties of these triangles (that is, triangles with sides in arithmetic progression) are discussed in* [2004 : 382–383] *and in the references given there. They are also the subject of a morsel in Ross Honsberger's* Mathematical Morsels (*Dolciani Mathematical Expositions No. 3, 1978, pages 209–210*), *which is based on problem E411* (Amer. Math. Monthly, *47:10, December, 1940, pages 708–709*), *where yet other properties and references are provided.*

3198. Replacement. [2007 : 41, 44] *Proposed by Michel Bataille, Rouen, France.*

Let $p = 2n+1$ be a prime, and let s be any integer such that $1 \le s \le n$. Prove that:

(a) $4^s \sum\limits_{k=0}^{n-s} \binom{2s+2k-1}{2s-1} \equiv 1 \pmod{p}$,

(b) $4^s \sum\limits_{k=0}^{n-s} \binom{2s+2k}{2s-1} \equiv -1 \pmod{p}$.

A composite of similar solutions by Richard J. McIntosh, University of Regina, Regina, SK; Joel Schlosberg, Bayside, NY, USA; and the proposer.

Motivation. Our argument comes down to proving a simple property of binomial coefficients: for a given integer r and prime p, with $0 \le r < p$,

$$\binom{r+i}{i} \equiv (-1)^i \binom{p-(r+1)}{i} \pmod{p} ,$$

for $i = 0, 1, \ldots, p-(r+1)$; in words, the i^{th} entry in the r^{th} diagonal of the Pascal Triangle is congruent to the i^{th} entry, or its negative, in the $p-(r+1)^{\text{st}}$ row. We shall use the symbol '$\equiv$' for congruences modulo p and shall assume as known the fact that, for $N \ge 1$,

$$\sum_{r \text{ even}} \binom{N}{r} = \sum_{r \text{ odd}} \binom{N}{r} = 2^{N-1} . \tag{1}$$

(a) Let $t = \dfrac{p-(2s-1)}{2}$; that is, $2s-1 = p-2t$. Then the proposed identity becomes

$$2^{p-2t+1} \sum_{k=0}^{t-1} \binom{p-2t+2k}{p-2t} \equiv 1 .$$

Since $2^{p+1} \equiv 2^2$ and $\binom{p-2t+2k}{p-2t} = \binom{p-2t+2k}{2k}$, it suffices to prove that

$$\sum_{k=0}^{t-1} \binom{p-2t+2k}{2k} \equiv 2^{2t-2} .$$

Observe that, because $2k < p$,

$$\begin{aligned}\binom{p-2t+2k}{2k} &= \frac{(p-2t+2k)(p-2t+2k-1)\cdots(p-2t+1)}{(2k)!}\\ &\equiv \frac{(-2t+2k)(-2t+2k-1)\cdots(-2t+1)}{(2k)!}\\ &= \frac{(-1)^{2k}(2t-2k)(2t-2k+1)\cdots(2t-1)}{(2k)!}\\ &= \binom{2t-1}{2k}.\end{aligned}$$

Hence,

$$\sum_{k=0}^{t-1}\binom{p-2t+2k}{2k} \equiv \sum_{k=0}^{t-1}\binom{2t-1}{2k} = 2^{2t-2}$$

by (1), as claimed.

(b) We must prove that

$$\sum_{k=0}^{t-1}\binom{p-2t+2k+1}{2k+1} \equiv -2^{2t-2}.$$

We therefore go through the same steps as in part (a), here using the odd terms. We find that

$$\binom{p-2t+2k+1}{2k+1} \equiv -\binom{2t-1}{2k+1};$$

whence the sum turns out to be negative.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and JOSÉ H. NIETO, Universidad del Zulia, Maracaibo, Venezuela.

3199. [2006 : 516, 518] *Proposed by Michel Bataille, Rouen, France.*

Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(xy) = f\big(f(x)+f(y)\big)$ for all real numbers x and y.

Solution by Joel Schlosberg, Bayside, NY, USA.

Constant functions satisfy the given functional equation trivially. We will show that, conversely, any function which satisfies the equation is a constant.

If $f(z) = f(0)$ for some $z \neq 0$, then for any real number x, we have

$$\begin{aligned}f(x) &= f\left(z\cdot\frac{x}{z}\right) = f\left(f(z)+f\left(\frac{x}{z}\right)\right)\\ &= f\left(f(0)+f\left(\frac{x}{z}\right)\right) = f\left(0\cdot\frac{x}{z}\right) = f(0);\end{aligned}$$

thus, f is constant.

On the other hand, if we assume that $f(z) = f(0)$ only for $z = 0$, then, since $f(0) = f(0 \cdot 0) = f\big(2f(0)\big)$, we have $2f(0) = 0$ and, therefore, $f(0) = 0$. For any $x \neq 0$,

$$f(0) = f(0 \cdot x) = f\big(f(0) + f(x)\big) = f\big(f(x)\big) .$$

Then $f(x) = 0 = f(0)$, a contradiction.

Also solved by ROY BARBARA, Lebanese University, Fanar, Lebanon; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposer. All the solutions submitted were similar to the featured solution.

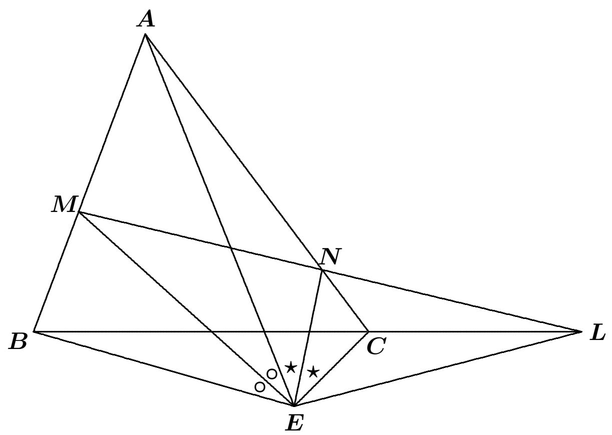
3200. [2006 : 518] *Proposed by Christopher J. Bradley, Bristol, UK.*

Let ABC be a triangle with $\angle B > \angle C$, and let E be the centre of the excircle opposite angle A. Let M and N be points on AB and AC, respectively, such that EM is the internal bisector of $\angle AEB$ and EN is the internal bisector of $\angle AEC$. If MN is extended to meet BC at L, prove that $\angle BEL + \angle CEL = 180^\circ$.

A combined solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Apostolis K. Demis, Varvakeio High School, Athens, Greece; and Joel Schlosberg, Bayside, NY, USA; modified by the editor.

We show that the claim is true under a weaker condition than stated: We only assume that the point E is on none of

1. the perpendicular bisector of the segment BC;
2. the extensions of the segment BC beyond the points B and C;
3. the segments CA and AB and their extensions.



Since EM and EN bisect angles AEB and AEC, respectively, we have

$$\frac{AM}{BM} = \frac{AE}{BE} \quad \text{and} \quad \frac{CN}{AN} = \frac{CE}{AE}.$$

By Menelaus' Theorem,

$$\frac{AM}{BM} \cdot \frac{BL}{CL} \cdot \frac{CN}{AN} = 1,$$

so that

$$\frac{AE}{BE} \cdot \frac{BL}{CL} \cdot \frac{CE}{AE} = 1,$$

or

$$\frac{BL}{CL} = \frac{BE}{CE}.$$

The Law of Sines applied to triangles BEL and CEL yields

$$\frac{BL}{\sin\angle BEL} = \frac{BE}{\sin\angle ELB} \quad \text{and} \quad \frac{CL}{\sin\angle CEL} = \frac{CE}{\sin\angle ELC}.$$

Since $\angle ELB = \angle ELC$, we obtain

$$\sin\angle BEL = \sin\angle CEL,$$

so that

$$\angle BEL = \angle CEL \quad \text{or} \quad \angle BEL + \angle CEL = 180^\circ.$$

Since point L is on the extension of BC, we see that $\angle BEL \neq \angle CEL$ and, therefore,

$$\angle BEL + \angle CEL = 180^\circ,$$

as claimed.

It is easy to show that if point E is on the perpendicular bisector of the segment BC, then the line MN is parallel to the side BC. If point E is on the extensions of the segment BC beyond the points B and C, then point L coincides with the point E.

Also solved by MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer.