THE OLYMPIAD CORNER

No. 266

R.E. Woodrow

We begin this number with the six problems of the 2004 Chinese Mathematical Olympiad, Macau. Thanks go to Christopher Small, Canadian team Leader to the IMO in Athens, Greece, for collecting them for our use.

2004 CHINESE MATHEMATICAL OLYMPIAD
January 8-9, 2004, Macau

1. The vertices E, F, G, and H of a convex quadrilateral $EFGH$ are points on the sides AB, BC, CD, and DA, respectively, of a convex quadrilateral $ABCD$ satisfying

$$\frac{AE}{EB}\cdot\frac{BF}{FC}\cdot\frac{CG}{GD}\cdot\frac{DH}{HA} \;=\; 1\,.$$

The points A, B, C, and D are on the sides H_1E_1, E_1F_1, F_1G_1, and G_1H_1, respectively, of a convex quadrilateral $E_1F_1G_1H_1$ satisfying $E_1F_1 \parallel EF$, $F_1G_1 \parallel FG$, $G_1H_1 \parallel GH$, and $H_1E_1 \parallel HE$.

Given that $\frac{E_1A}{AH_1} = \lambda$, find the value of $\frac{F_1C}{CG_1}$.

2. Given a positive integer c, let x_1, x_2, $\dots$ be a sequence satisfying the following conditions: $x_1 = c$, and for $n = 2, 3, \dots,$

$$x_n \;=\; x_{n-1} + \left\lfloor \frac{2x_{n-1} - (n+2)}{n} \right\rfloor + 1\,,$$

where $\lfloor x \rfloor$ denotes the largest integer not exceeding x.

Find a formula for x_n in terms of n and c.

3. Let M be a set of n points in a plane satisfying the following conditions:

(i) There are 7 points in M which are the 7 vertices of a convex 7-gon.

(ii) For any 5 points in M, if they are the 5 vertices of a convex 5-gon, then there is at least 1 point in M in the interior of the 5-gon.

Find the minimum value of n for which such a set M exists.

4. For any real number a and positive integer n, prove the following:

(a) There exists a unique sequence of real numbers x_0, x_1, $\dots$, x_n, x_{n+1} satisfying $x_0 = x_{n+1} = 0$ and, for $i = 1, 2, \dots, n$,

$$\tfrac{1}{2}(x_{i+1} + x_{i-1}) \;=\; x_i + x_i^3 - a^3\,.$$

(b) The sequence in (a) satisfies $|x_i| \le |a|$ for $i = 0, 1, 2, \dots, n+1$.

5. Given any positive integer $n \geq 2$, let a_i ($i = 1, 2, \ldots, n$) be positive integers satisfying $a_1 < a_2 < \cdots < a_n$ and $\sum\limits_{i=1}^{n} \frac{1}{a_i} \leq 1$. Prove that for any real number x,

$$\left(\sum_{i=1}^{n} \frac{1}{a_i^2 + x^2}\right)^2 \leq \frac{1}{2} \cdot \frac{1}{a_1(a_1 - 1) + x^2} .$$

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6. Prove that all but finitely many positive integers n can be represented as a sum of 2004 positive integers, $n = a_1 + a_2 + \cdots + a_{2004}$, such that $1 \leq a_1 < a_2 < \cdots < a_{2004}$ and $a_i \mid a_{i+1}$ for $i = 1, 2, \ldots, 2003$.

As a second Olympiad set, we give the four problems of the Singapore Mathematical Olympiad 2004 (Open Section, Special Round). Thanks again go to Christopher Small for collecting them.

SINGAPORE MATHEMATICAL OLYMPIAD 2004
Open Section, Special Round
June 26, 2004

1. Let m and n be integers such that $m \geq n > 1$. Let $F_1, \ldots, F_k$ be a collection of n-element subsets of $\{1, \ldots, m\}$ such that $F_i \cap F_j$ contains at most 1 element, $1 \leq i < j \leq k$. Show that

$$k \leq \frac{m(m-1)}{n(n-1)} .$$

2. Find the number of ordered pairs (a, b), where a and b are integers and $1 \leq a, b \leq 2004$, such that the equation $x^2 + ax + b = 167y$ has integer solutions in x and y. Justify your answer.

3. Let AD be the common chord of two circles Γ_1 and Γ_2. A line through D intersects Γ_1 at B and Γ_2 at C. Let E be a point on the segment AD different from A and D. The line CE intersects Γ_1 at P and Q. The line BE intersects Γ_2 at M and N.

(i) Prove that P, Q, M, and N lie on the circumference of a circle Γ_3.

(ii) If the centre of Γ_3 is O, prove that OD is perpendicular to BC.

4. If $0 < x_1, x_2, \ldots, x_n \leq 1$, where $n \geq 1$, show that

$$\frac{x_1}{1 + (n-1)x_1} + \frac{x_2}{1 + (n-1)x_2} + \cdots + \frac{x_n}{1 + (n-1)x_n} \leq 1 .$$

Next, we give the four problems of the 18th Nordic Mathematical Contest 2004. Thanks again go to Christopher Small for collecting them.

18th NORDIC MATHEMATICAL CONTEST 2004
April 1, 2004

1. Let 27 balls be numbered from 1 to 27 and distributed into three bowls: one red, one blue, and one yellow. How many balls can there be in the red bowl, given that the means of the numbers on the balls in the red, blue, and yellow bowls are 15, 3, and 18, respectively?

2. Let $f_1 = 0$, $f_2 = 1$, and $f_{n+2} = f_{n+1} + f_n$ for $n = 1, 2, \ldots,$ be the sequence of Fibonacci numbers. Show that there exists a (strictly) increasing infinite arithmetic sequence of integers which has no numbers in common with the Fibonacci sequence.

3. Let $x_{11}, x_{21}, \ldots, x_{n1}$, for $n > 2$, be a sequence of integers which are not all equal. For $k = 1, 2, 3, \ldots,$ let $x_{i,k+1} = \frac{1}{2}(x_{ik} + x_{i+1,k})$ for $i = 1, 2, \ldots, n-1$, and $x_{n,k+1} = \frac{1}{2}(x_{nk} + x_{1k})$. Show that if n is odd, there exist indices j and k such that x_{jk} is not an integer.

4. Let a, b, c, and R be the side lengths and the circumradius of a triangle. Show that

$$\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} \geq \frac{1}{R^2}\,.$$

Finally, to provide extra problems for the (Canadian) winter break, we give the problems proposed to the jury but not used at the 2004 International Mathematical Olympiad (IMO) in Athens. Thanks go to Christopher Small, Canadian Team Leader, for obtaining them for us.

2004 IMO (ATHENS)
Problems Proposed but not Used
Algebra

A1. An infinite sequence $a_0, a_1, a_2, \ldots$ of real numbers satisfies the condition $a_n = |a_{n+1} - a_{n+2}|$ for every $n \geq 0$, with a_0 and a_1 positive and distinct. Can this sequence be bounded?

A2. Does there exist a function $s : \mathbb{Q} \to \{-1, 1\}$ such that, if x and y are distinct rational numbers satisfying $xy = 1$ or $x + y \in \{0, 1\}$, then $s(x)s(y) = -1$? Justify your answer.

A3. Let $a,\,b,\,c > 0$ and $ab + bc + ca = 1$. Prove the inequality

$$\sqrt[3]{\frac{1}{a} + 6b} + \sqrt[3]{\frac{1}{b} + 6c} + \sqrt[3]{\frac{1}{c} + 6a} \;\le\; \frac{1}{abc}\,.$$

A4. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the equation

$$f\left(x^2 + y^2 + 2f(xy)\right) \;=\; \left(f(x+y)\right)^2$$

for all $x,\,y \in \mathbb{R}$.

A5. Let $n > 1$, and let $a_1,\,a_2,\,\ldots,\,a_n$ be positive real numbers. Denote by g_n their geometric mean, and by $A_1,\,A_2,\,\ldots,\,A_n$ the sequence of arithmetic means defined by

$$A_k \;=\; \frac{a_1 + a_2 + \cdots + a_k}{k}\,,$$

for $k = 1,\,2,\,\ldots,\,n$. Let G_n be the geometric mean of $A_1,\,A_2,\,\ldots,\,A_n$. Prove the inequality

$$n\sqrt[n]{\frac{G_n}{A_n}} + \frac{g_n}{G_n} \;\le\; n + 1\,,$$

and establish the cases of equality.

Combinatorics

C1. There are 10001 students at a university. Some students join together to form several clubs (a student may belong to different clubs). Some clubs join together to form several societies (a club may belong to different societies). There are a total of k societies. Suppose that the following conditions hold:

(i) Each pair of students are in exactly one club.

(ii) For each student and each society, the student is in exactly one club of the society.

(iii) Each club has an odd number of students. In addition, a club with $2m + 1$ students (m is a positive integer) is in exactly m societies.

Find all possible values of k.

C2. Let n and k be positive integers. There are given n circles in the plane. Every two of them intersect at two distinct points, and all points of intersection they determine are pairwise distinct. Each intersection point must be coloured with one of n distinct colours so that each colour is used at least once and exactly k distinct colours occur on each circle. Find all values of $n \ge 2$ and k for which such a colouring is possible.

C3. The following operation is allowed on a finite graph: Choose an arbitrary cycle of length 4 (if there is any), choose an arbitrary edge in that cycle, and delete it from the graph. For a fixed integer $n \geq 4$, find the least number of edges of a graph that can be obtained by repeated applications of this operation from the complete graph on n vertices (where each pair of vertices are joined by an edge).

C4. Consider a matrix of size $n \times n$ whose entries are real numbers of absolute value not exceeding 1, such that the sum of all entries is 0. Let n be an even positive integer. Determine the least number C such that every such matrix necessarily has a row or a column with the sum of its entries not exceeding C in absolute value.

C5. Let N be a positive integer. Two players A and B, taking turns, write numbers from the set $\{1, \ldots, N\}$ on a blackboard. Player A begins the game by writing 1 on his first move. Then, if a player has written n on a certain move, his adversary is allowed to write $n+1$ or $2n$ (provided the number he writes does not exceed N). The player who writes N wins. We say that N is of type A or of type B according as A or B has a winning strategy.

(a) Determine whether $N = 2004$ is of type A or of type B.

(b) Find the least $N > 2004$ whose type is different from the one of 2004.

C6. For an $n \times n$ matrix A, let X_i be the set of entries in row i, and Y_j the set of entries in column j, for $1 \leq i, j \leq n$. We say that A is *golden* if $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ are distinct sets. Find the least integer n such that there exists a 2004×2004 golden matrix with entries in the set $\{1, 2, \ldots, n\}$.

C7. For a finite graph G, let $f(G)$ be the number of triangles and $g(G)$ the number of tetrahedra formed by edges of G. Find the least constant c such that $g(G)^3 \leq c \cdot f(G)^4$ for every graph G.

Geometry

G1. The circle Γ and the line ℓ do not intersect. Let AB be the diameter of Γ perpendicular to ℓ, with B closer to ℓ than A. An arbitrary point C different from both A and B is chosen on Γ. The line AC intersects ℓ at D. The line DE is tangent to Γ at E, with B and E on the same side of AC. Let BE intersect ℓ at F, and let AF intersect Γ at $G \neq A$. Prove that the reflection of G in AB lies on the line CF.

G2. Let O be the circumcentre of an acute-angled triangle ABC with $\angle B < \angle C$. The line AO meets the side BC at D. The circumcentres of the triangles ABD and ACD are E and F, respectively. Extend the sides BA and CA beyond A, and choose on the respective extensions points G and H such that $AG = AC$ and $AH = AB$. Prove that the quadrilateral $EFGH$ is a rectangle if and only if $\angle ACB - \angle ABC = 60°$.

G3. Let $A_1A_2\dots A_n$ be a regular n-gon. The points $B_1, \dots, B_{n-1}$ are defined as follows:

(i) If $i = 1$ or $i = n - 1$, then B_i is the mid-point of the side A_iA_{i+1};

(ii) If $i \neq 1$, $i \neq n-1$, and S is the intersection point of A_1A_{i+1} and A_nA_i, then B_i is the intersection point of the bisector of the angle A_iSA_{i+1} with A_iA_{i+1}.

Prove that $\angle A_1B_1A_n + \angle A_1B_2A_n + \cdots + \angle A_1B_{n-1}A_n = 180°$.

G4. Let $\mathcal{P}$ be a convex polygon. Prove that there is a convex hexagon which is contained in $\mathcal{P}$ and which occupies at least 75 percent of the area of $\mathcal{P}$.

G5. For a given triangle ABC, let X be a variable point on the line BC such that C lies between B and X and the incircles of the triangles ABX and ACX intersect at two distinct points P and Q. Prove that the line PQ passes through a point independent of X.

G6. A cyclic quadrilateral $ABCD$ is given. The lines AD and BC intersect at E, with C between B and E; the diagonals AC and BD intersect at F. Let M be the mid-point of the side CD, and let $N \neq M$ be a point on the circumcircle of the triangle ABM such that $AN/BN = AM/BM$. Prove that the points E, F, and N are collinear.

Number Theory

N1. Let $\tau(n)$ denote the number of positive divisors of the positive integer n. Prove that there exist infinitely many positive integers a such that the equation $\tau(an) = n$ does not have a positive integer solution n.

N2. The function ψ from the set $\mathbb{N}$ of positive integers into itself is defined by the equality

$$\psi(n) = \sum_{k=1}^{n} (k, n), \qquad \text{for } n \in \mathbb{N},$$

where (k, n) denotes the greatest common divisor of k and n.

(a) Prove that $\psi(mn) = \psi(m)\psi(n)$ for every two relatively prime m, $n \in \mathbb{N}$.

(b) Prove that, for each $a \in \mathbb{N}$, the equation $\psi(x) = ax$ has a solution.

(c) Find all $a \in \mathbb{N}$ such that the equation $\psi(x) = ax$ has a unique solution.

N3. A function f from the set of positive integers $\mathbb{N}$ into itself is such that, for all m, $n \in \mathbb{N}$, the number $(m^2 + n)^2$ is divisible by $\big(f(m)\big)^2 + f(n)$. Prove that $f(n) = n$ for each $n \in \mathbb{N}$.

N4. Let k be a fixed integer greater than 1, and let $m = 4k^2 - 5$. Show that there exist positive integers a and b such that the sequence $\{x_n\}$ defined by $x_0 = 1$, $x_1 = b$, and $x_{n+2} = x_{n+1} + x_n$ for $n = 0, 1, 2, \ldots$ has all of its terms relatively prime to m.

N5. Given an integer $n > 1$, denote by P_n the product of all positive integers x less than n and such that n divides $x^2 - 1$. For each $n > 1$, find the remainder of P_n on division by n.

N6. Let p be an odd prime and n a positive integer. In the coordinate plane, eight distinct points with integer coordinates lie on a circle with diameter of length p^n. Prove that there exists a triangle with vertices at three of the given points such that the squares of its side lengths are integers divisible by p^{n+1}.

Next we given an alternate solution for problem 3 of the Italy Team Selection Contest 1999 [2002 : 356–357], for which a solution was given earlier [2004 : 492–494].

3. (a) Determine all the strictly monotone functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$f\big(x + f(y)\big) \;=\; f(x) + y\,, \quad \forall\, x,\, y \in \mathbb{R}\,. \tag{1}$$

(b) Prove that for every integer $n > 1$, there do not exist strictly monotone functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$f\big(x + f(y)\big) \;=\; f(x) + y^n\,, \quad \forall\, x,\, y \in \mathbb{R}\,. \tag{2}$$

Alternate solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

(a) Adding z to both sides of (1) and taking f, we get

$$f\Big(z + f\big(x + f(y)\big)\Big) \;=\; f\big(y + z + f(x)\big)\,. \tag{3}$$

Applying (1) to each side of (3), we have $f(z) + x + f(y) = f(y+z) + x$, or $f(y) + f(z) = f(y+z)$, for all y, $z \in \mathbb{R}$. This is the well-known Cauchy Equation, for which the monotone solutions are $f(x) = cx$, where $c = f(1)$. Substituting $f(x) = cx$ into (1), we find that $c(x + cy) = cx + y$. Hence, $c = \pm 1$ and $f(x) = \pm x$.

(b) Taking f of both sides of (2), we get

$$f\Big(0 + f\big(x + f(y)\big)\Big) \;=\; f\big(y^n + f(x)\big)\,. \tag{4}$$

Applying (2) to each side of (4), we have $f(0) + \big(x + f(y)\big)^n = f(y^n) + x^n$, or

$$\sum_{k=1}^{n} \binom{n}{k} x^{n-k} f(y)^k \;=\; f(y^n) - f(0)\,, \tag{5}$$

for all x, $y \in \mathbb{R}$. Note that the right side of (5) does not depend on x. Therefore, as the coefficient of nx^{n-1} on the left side of (5), $f(y) = 0$ for all $y \in \mathbb{R}$. But $f \equiv 0$ is clearly not a solution of (2).

Note that the monotonicity assumption was not needed here (part (b)).

We turn to readers' solutions to problems from the Thai Mathematical Olympiad 2002 given in the November 2006 *Corner* [2006 : 440–442].

3. Find the maximum real number K such that

$$\frac{1}{ka+b} + \frac{1}{kb+a} \;\ge\; \frac{K}{a+b}$$

for all a, $b > 0$ and all $k \in [0,\pi]$.

Solved by Michel Bataille, Rouen, France; and Pierre Bornsztein, Maisons-Laffitte, France. We give Bataille's write-up.

The maximal value of K is $\frac{4}{\pi+1}$.

First, if $a = b > 0$ and $k = \pi$, then

$$(a+b)\cdot\left(\frac{1}{ka+b} + \frac{1}{kb+a}\right) \;=\; \frac{4}{\pi+1}\,.$$

Hence, we have $K \le 4/(\pi+1)$ for any K satisfying the conditions.

To complete the proof, we show that

$$\frac{1}{ka+b} + \frac{1}{kb+a} \;\ge\; \frac{4}{\pi+1}\cdot\frac{1}{a+b}$$

for all a, $b > 0$ and $k \in [0,\pi]$. Using the HM–GM Inequality, we get

$$\begin{aligned}\frac{1}{2}\left(\frac{1}{ka+b} + \frac{1}{kb+a}\right) \;&\ge\; \frac{2}{ka+b+kb+a}\\ &=\; \frac{2}{k+1}\cdot\frac{1}{a+b} \;\ge\; \frac{2}{\pi+1}\cdot\frac{1}{a+b}\,,\end{aligned}$$

for all a, $b > 0$ and $k \in [0,\pi]$. The result follows.

4. Let x_1 and x_2 be consecutive integers (that is, $x_2 = x_1 + 1$). For each integer $n \ge 3$, let x_n be the remainder when $x_{n-1}^2 + x_{n-2}^2$ is divided by 7. If $x_{2545} = 1$, determine the value of x_4.

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; Geoffrey A. Kandall, Hamden, CT, USA; and Ioannis Katsikis, Athens, Greece. We give Kandall's write-up.

For each positive integer n, let $\overline{x}_n$ be the residue of x_n modulo 7 (so that $0 \le \overline{x}_n \le 6$). It is evident from the table below that $x_n = x_{n+3}$ for all $n \ge 6$. Since $2545 \equiv 7 \pmod 3$, we have $x_{2545} = x_7$. There are only two sequences with $x_7 = 1$, and in each case $x_4 = 1$.

$\overline{x}_1$	$\overline{x}_2$	$\overline{x}_3$	$\overline{x}_4$	$\overline{x}_5$	$\overline{x}_6$	$\overline{x}_7$	$\overline{x}_8$	$\overline{x}_9$	$\overline{x}_{10}$	$\overline{x}_{11}$
0	1	1	2	5	1	5	5	1	5	5
1	2	5	1	5	5	1	5	5	1	5
2	3	6	3	3	4	4	4	4	4	4
3	4	4	4	4	4	4	4	4	4	4
4	5	6	5	5	1	5	5	1	5	5
5	6	5	5	1	5	5	1	5	5	1
6	0	1	1	2	5	1	5	5	1	5

6. Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a non-constant polynomial with integer coefficients. Assume that $p(-1) = 0$ and $p(\sqrt{2})$ is an integer. Show that there is an integer k such that $p(k) + a_k$ is even.

Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Ioannis Katsikis, Athens, Greece. We present Bornsztein's approach.

We prove that $k = 1$ works.

We have $p(1) = \sum a_{2i} + \sum a_{2i+1}$. But $\sum a_{2i} = \sum a_{2i+1}$, because $p(-1) = 0$. Thus, $p(1)$ is even. Since $\sqrt{2}$ is irrational and $p(\sqrt{2})$ is an integer, we have $\sum a_{2i+1}2^i = 0$. It follows that a_1 is even. Then $p(1) + a_1$ is even, as claimed.

7. Find all integers n with the property that both $n + 2002$ and $n - 2002$ are perfect squares.

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; Geoffrey A. Kandall, Hamden, CT, USA; and Ioannis Katsikis, Athens, Greece. We give the solution of Bornsztein.

Assume that $n + 2002 = a^2$ and $n - 2002 = b^2$ for some non-negative integers a and b. Then $n = \frac{1}{2}(a^2 + b^2)$. Also, $a^2 - b^2 = 4004$, which gives $(a-b)(a+b) = 2^2 \times 7 \times 11 \times 13$.

Note that $0 < a - b \le a + b$ and that $a - b$ and $a + b$ have the same parity. Then, since their product is even, they are even. This means that $(a-b, a+b)$ is one of the pairs $(2, 2002)$, $(14, 286)$, $(22, 182)$, and $(26, 154)$. The corresponding pairs (a, b) are $(1002, 1000)$, $(150, 136)$, $(102, 80)$, and $(90, 64)$. The desired values of n are then 1002002, 20498, 8402, and 6098.

9. Find the greatest integer which divides

$$(a-b)(b-c)(c-d)(d-a)(a-c)(b-d)$$

for any integers a, b, c, d.

Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Ioannis Katsikis, Athens, Greece. We give Bornsztein's approach.

We prove that 12 is the desired maximum.

Let $f(a,b,c,d) = (a-b)(b-c)(c-d)(d-a)(a-c)(b-d)$. The greatest common divisor of the products $f(a,b,c,d)$ is a divisor of 12, because $f(0,1,2,3) = -12$.

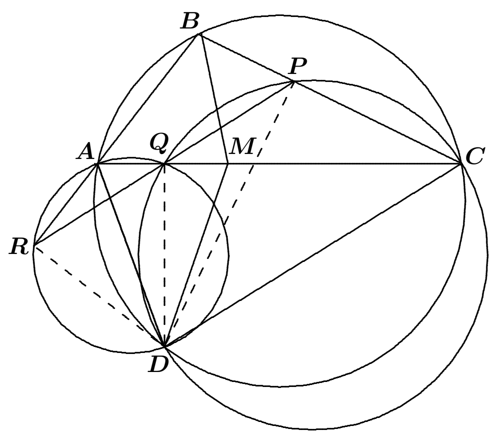
On the other hand, among any four integers at least two have the same residue modulo 3, which ensures that $f(a,b,c,d) \equiv 0 \pmod{3}$. Moreover, among four integers, either two are even and two are odd, or at least three have the same parity. In both cases, $f(a,b,c,d)$ is a multiple of 4. Therefore, the product $f(a,b,c,d)$ is always a multiple of 12, and we are done.

Next we turn to the solutions sent in by readers to the short-listed problems of the 44^{th} International Mathematical Olympiad, which appear in [2006 : 501–503].

G1. Let $ABCD$ be a cyclic quadrilateral. Let P, Q, R be the feet of the perpendiculars from D to the lines BC, CA, AB, respectively. Show that $PQ = QR$ if and only if the bisectors of $\angle ABC$ and $\angle ADC$ are concurrent with AC.

Solution by Michel Bataille, Rouen, France.

Let A, B, C denote the angles of $\triangle ABC$. Let M be the point of intersection of the internal bisector of $\angle ABC$ with AC. Recall that M is the point of the line segment AC characterized by $\dfrac{MA}{MC} = \dfrac{BA}{BC}$.



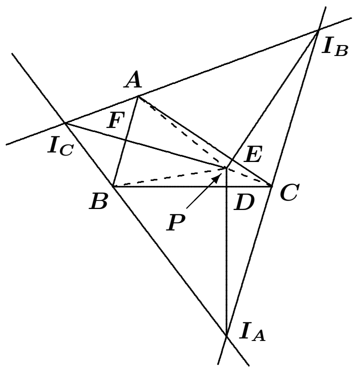
Points Q and R lie on the circle with diameter AD; hence, by the Law of Sines, $QR = DA\sin\angle RAQ = DA\sin A$. Similarly, $QP = DC\sin C$. Then

$$\frac{QR}{QP} = \frac{DA\sin A}{DC\sin C} = \frac{DA}{DC}\cdot\frac{BC}{BA} = \frac{DA}{DC}\cdot\frac{MC}{MA}.$$

Now $PQ = QR$ if and only if $\frac{MA}{MC} = \frac{DA}{DC}$, which is equivalent to DM being the internal bisector of $\angle ADC$.

G3. Let ABC be a triangle, and let P be a point in its interior. Denote by D, E, F the feet of the perpendiculars from P to the lines BC, CA, AB, respectively. Suppose that $AP^2 + PD^2 = BP^2 + PE^2 = CP^2 + PF^2$. Denote by I_A, I_B, I_C the excentres of the triangle ABC. Prove that P is the circumcentre of the triangle $I_AI_BI_C$.

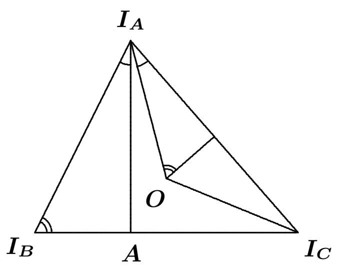
Solution by Michel Bataille, Rouen, France.



Let $a = BC$, $b = CA$, $c = AB$, and $s = \frac{1}{2}(a+b+c)$. From the hypothesis, $BD^2 = BP^2 - PD^2 = AP^2 - PE^2 = AE^2$; hence, $BD = AE$. Similarly, $CE = BF$ and $AF = CD$. Since $BD + DC = a$, $CE + EA = b$, and $AF + FB = c$, we find that $BD = AE = s-c$, $CE = BF = s-a$, and $AF = CD = s-b$, which implies that D, E, and F are the points of tangency of the excircles with the sides BC, CA, and AB, respectively. In particular, D is on the line PI_A, and similar results hold for E and F.

Now, $\angle BCI_A = \frac{1}{2}(\pi - C)$, since I_AI_B is the external bisector of $\angle ACB$. Using this and similar equalities, we obtain the angles of $\triangle I_AI_BI_C$:

$$\angle I_A = \tfrac{1}{2}(\pi - A), \quad \angle I_B = \tfrac{1}{2}(\pi - B), \quad \angle I_C = \tfrac{1}{2}(\pi - C).$$



The internal bisector I_AA is perpendicular to the external bisector I_BI_C; whence, I_AA is the altitude from I_A in $\triangle I_AI_BI_C$. Moreover,

$$I_CI_AA = \tfrac{\pi}{2} - \angle AI_CI_A = \tfrac{1}{2}C$$

and

$$\angle DI_AC = \tfrac{\pi}{2} - \angle BCI_A = \tfrac{1}{2}C.$$

Therefore, I_AD is the image of the altitude I_AA in the internal bisector of $\angle I_BI_AI_C$. It follows that I_AD passes through the circumcentre of $\triangle I_AI_BI_C$ (see the figure for a proof without words). Similarly, I_BE and I_CF pass through this circumcentre. The required result follows.

To round out this number of the *Corner*, we finish our file of solutions from our readers to the remaining short-listed problems of the 44^{th} International Mathematical Olympiad given in [2007 : 19–21].

A4. Let n be a positive integer, and let $x_1 \leq x_2 \leq \cdots \leq x_n$ be real numbers.

(a) Prove that $\left(\sum\limits_{i,j=1}^{n} |x_i - x_j|\right)^2 \leq \dfrac{2(n^2-1)}{3} \sum\limits_{i,j=1}^{n} (x_i - x_j)^2$.

(b) Show that the equality holds if and only if $x_1, \ldots, x_n$ is an arithmetic sequence.

Solution by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain, modified by the editor.

(a) The Cauchy-Schwarz Inequality yields

$$\left(\sum_{i=1}^{n}\sum_{j=1}^{n} |i-j||x_i - x_j|\right)^2 \leq \left(\sum_{i=1}^{n}\sum_{j=1}^{n}(i-j)^2\right)\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(x_i-x_j)^2\right). \quad (1)$$

Evaluating the first term on the right side of (1), we get

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(i-j)^2 = 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(i-j)^2 = 2\sum_{k=1}^{n-1}(n-k)k^2 = \frac{n^2(n^2-1)}{6}.$$

We also get

$$\sum_{i=1}^{n}\sum_{j=1}^{n} |i-j||x_i - x_j| = \frac{n}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} |x_i - x_j|. \quad (2)$$

[Ed. To obtain (2), we proceed as follows:

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{n} |i-j||x_i - x_j| &= 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(j-i)(x_j - x_i) \\
&= 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(j-i)x_j - 2\sum_{i=1}^{n-1} x_i \sum_{j=i+1}^{n}(j-i) \\
&= 2\sum_{j=2}^{n} x_j \sum_{i=1}^{j-1}(j-i) - 2\sum_{i=1}^{n-1} x_i \sum_{j=i+1}^{n}(j-i).
\end{aligned}$$

In the last step, we changed the order of summation in the first sum on the right side of the equation. Now we change the variable of summation from i to $k = j - i$ in the first sum and from j to $k = j - i$ in the second sum:

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{n}|i-j||x_i-x_j| &= 2\sum_{j=2}^{n}x_j\sum_{k=1}^{j-1}k \;-\; 2\sum_{i=1}^{n-1}x_i\sum_{k=1}^{n-i}k \\ &= 2\sum_{j=2}^{n}x_j\frac{j(j-1)}{2} \;-\; 2\sum_{i=1}^{n-1}x_i\frac{(n-i)(n-i+1)}{2} \\ &= \sum_{i=1}^{n}x_i\Big(i(i-1)-(n-i)(n-i+1)\Big) \\ &= n\sum_{i=1}^{n}x_i(2i-1-n)\,.\end{aligned}$$

Similarly, $\sum_{i=1}^{n}\sum_{j=1}^{n}|x_i-x_j| = 2\sum_{i=1}^{n}x_i(2i-1-n)$. Thus, we obtain (2).]

Therefore, (1) becomes

$$\frac{n^2}{4}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}|x_i-x_j|\right)^2 \;\le\; \frac{n^2(n^2-1)}{6}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(x_i-x_j)^2\right)\,,$$

from which the proposed inequality follows.

(b) Equality holds when it holds in (1). Therefore, equality holds when there is some constant $d\in\mathbb{R}$ such that $d|i-j| = |x_i-x_j|$ for all i and j. But $d|i-j| = |x_i-x_j|$ for all i and j if and only if $x_{i+1}-x_i = d$ for all i. Thus, equality holds if and only if x_1, x_2, ..., x_n is an arithmetic sequence.

C4. Let x_1, ..., x_n and y_1, ..., y_n be real numbers. Let $A = (a_{ij})_{1\le i,j\le n}$ be the matrix with entries

$$a_{ij} \;=\; \begin{cases}1 & \text{if } x_i+y_j\ge 0;\\ 0 & \text{if } x_i+y_j<0.\end{cases}$$

Suppose that B is an $n\times n$ matrix with entries 0, 1 such that the sum of the elements in each row and each column of B is equal to the corresponding sum for the matrix A. Prove that $A = B$.

Solution by Michel Bataille, Rouen, France.

We will say that A is associated with the n-tuples $(x_1,x_2,\ldots,x_n)$ and $(y_1,y_2,\ldots,y_n)$, and that B is sum-related to A. Let σ and τ be permutations of $\{1, 2, \ldots, n\}$ such that $x_{\sigma(1)}\le x_{\sigma(2)}\le\cdots\le x_{\sigma(n)}$ and $y_{\tau(1)}\le y_{\tau(2)}\le\cdots\le y_{\tau(n)}$. Then the matrix A' associated with $(x_{\sigma(1)},x_{\sigma(2)},\ldots,x_{\sigma(n)})$ and $(y_{\tau(1)},y_{\tau(2)},\ldots,y_{\tau(n)})$ is given by $a'_{i,j} = a_{\sigma(i),\tau(j)}$. Moreover, if $B = (b_{k,j})_{1\le i,j\le n}$, the matrix B' given by $b'_{i,j} = b_{\sigma(i),\tau(j)}$ is sum-related to A'. If we can prove that $B' = A'$, it will immediately follow that $B = A$. Thus, without loss of generality, we may as well show the property in the case $x_1\le x_2\le\cdots\le x_n$ and $y_1\le y_2\le\cdots\le y_n$.

The proof is by induction on n. The equality $A = B$ is obvious for $n = 1$. Let $n \geq 2$, and assume that the property is true for $(n-1) \times (n-1)$ matrices. Let A be the $n \times n$ matrix associated with $x_1 \leq x_2 \leq \cdots \leq x_n$ and $y_1 \leq y_2 \leq \cdots \leq y_n$, and let B be a matrix that is sum-related to A. If $A = 0$, then we must have $B = 0$. If $A \neq 0$, then certainly $a_{n,n} = 1$ (since $x_n + y_n = \max\{x_i + y_j\}$). We observe the following facts:

- if $i < n$ and $a_{i,n} = 1$, then $a_{i+1,n} = 1$ (since $x_{i+1}+y_n \geq x_i+y_n \geq 0$); and similarly, if $j < n$ and $a_{n,j} = 1$, then $a_{n,j+1} = 1$.
- if $k < n$ and $a_{k,n} = 0$, then the k^{th} row of A is null (since, for $j \leq n$, we have $x_k + y_j \leq x_k + y_n < 0$). Similarly, the ℓ^{th} column is null if $a_{n,\ell} = 0$ (where $\ell < n$). Because B is sum-related to A, it follows that any 1 situated in the n^{th} column of B cannot be in row k in case $a_{k,n} = 0$. Since the n^{th} columns of A and B have the same number of 1s, and these 1s are all situated below all possible 0s, the n^{th} columns of A and B must be equal. In the same way, we can prove that the n^{th} rows of A and B are equal. Now, delete the n^{th} row and column both in A and B and call the resulting matrices A_1 and B_1. Clearly, A_1 is associated with $x_1 \leq x_2 \leq \cdots \leq x_{n-1}$ and $y_1 \leq y_2 \leq \cdots \leq y_{n-1}$, and B_1 is sum-related to A_1. From the induction hypothesis, $B_1 = A_1$. Then $B = A$ follows.

G5. Let ABC be an isosceles triangle with $AC = BC$, whose incentre is I. Let P be a point on the circumcircle of the triangle AIB lying inside the triangle ABC. The lines through P parallel to CA and CB meet AB at D and E, respectively. The line through P parallel to AB meets CA and CB at F and G, respectively. Prove that the lines DF and EG intersect on the circumcircle of the triangle ABC.

Solved by Ricardo Barroso Campos, University of Seville, Seville, Spain; Michel Bataille, Rouen, France; and Francisco Javier García Capitán, IES Álvarez Cubero, Priego de Córdoba, Spain. We give the solution of Barroso Campos, modified by the editor.

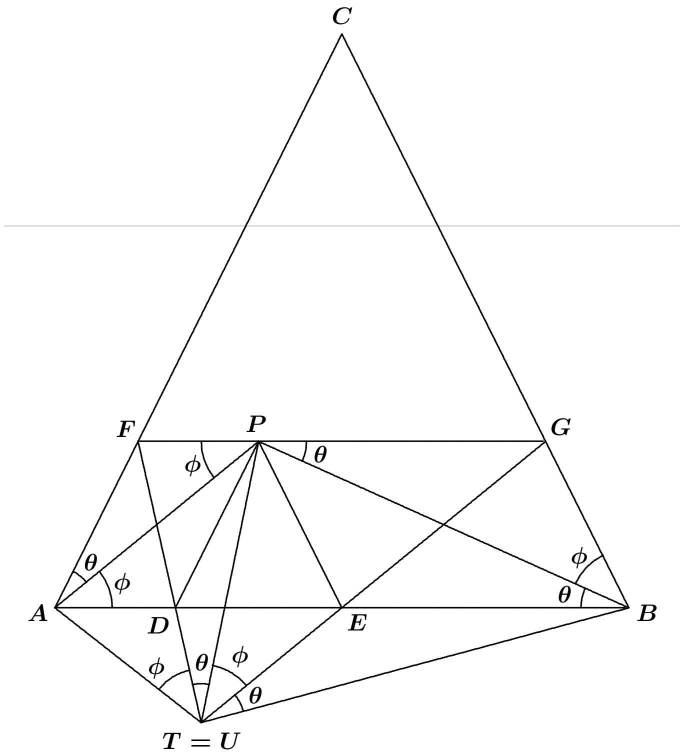
Set $\alpha = \angle CAB = \angle CBA$. Then $\angle ACB = 180^\circ - 2\alpha$. Since P lies on the circumcircle of triangle AIB, we have $\angle APB = \angle AIB = 180^\circ - \alpha$.

Set $\theta = \angle FAP$ and $\phi = \angle PAB$. Then $\phi = \alpha - \theta$,

$$\angle PBA \;=\; 180^\circ - \angle PAB - \angle APB \;=\; 180^\circ - (\alpha - \theta) - (180^\circ - \alpha) \;=\; \theta\,,$$

and $\angle PBG = \angle CBA - \angle PBA = \alpha - \theta = \phi$. Also, since $FG \parallel AB$, we have $\angle APF = \angle PAB = \phi$ and $\angle BPG = \angle PBA = \theta$.

Since $\angle FPE = \angle FGB = 180^\circ - \alpha$ and $\angle FAE = \angle CAB = \alpha$, we see that $\angle FPE + \angle FAE = 180^\circ$; hence, quadrilateral $FPEA$ is cyclic. Let GE produced intersect the circle $FPEA$ at T. Then $\angle FTP = \angle FAP = \theta$, $\angle ATF = \angle APF = \phi$, and $\angle PTG = \angle PTE = \angle PAE = \phi$.



Likewise, quadrilateral $DPGB$ is cyclic. Let FD produced intersect the circle $DPGB$ at U. Then $\angle PUG = \angle PBG = \phi$, $\angle BUG = \angle BPG = \theta$, and $\angle FUP = \angle DUP = \angle DBP = \angle PBA = \theta$.

Now we have $\angle FTP = \angle FUP = \theta$ and $\angle PTG = \angle PUG = \phi$, meaning that segments FP and PG are seen from both U and T with angles θ and ϕ, respectively; hence, $U = T$. Therefore,

$$\begin{aligned} \angle ATB &= \angle ATF + \angle FTP + \angle PTG + \angle GTB \\ &= (\alpha - \theta) + \theta + (\alpha - \theta) + \theta \;=\; 2\alpha\,. \end{aligned}$$

Finally, $\angle ACB + \angle ATB = (180^\circ - 2\alpha) + 2\alpha = 180^\circ$, which implies that T is on the circumcircle of triangle ABC.

That completes the *Corner* for this issue and this volume. As you may have noticed, we are now publishing solutions slightly less than one year after the corresponding problems appeared. We need you to send in your nice solutions and generalizations.