SKOLIAD No. 106

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Please send your solutions to the problems in this edition by ***1 May, 2008***. A copy of **MATHEMATICAL MAYHEM Vol. 8** will be presented to one pre-university reader who sends in solutions before the deadline. The decision of the editor is final.

The editor would like to apologize to Jixuan Wang, student, Don Mills Collegiate Institute, Toronto, ON, who sent in a correct solution to problem 4 (Gilbert's beautiful sum) of Skoliad No. 100 [2007 : 67, 68]. (The official solution was given in [2007 : 394].) Wang's solution was misfiled, thus preventing its timely acknowledgement.

Nos questions proviennent ce mois-ci du Concours de Mathématiques des Maritimes 2006. Nous remercions David Horrocks de l'Université de l'Île du Prince Edouard et John Grant McLoughlin de l'Université du Nouveau-Brunswick.

Concours de Mathématiques des Maritimes 2006

1. À la ligne d'arrivée de la course de 100 m, Alice devance Bob de 10 m et Bob devance Charlie de 20 m. En supposant que chaque coureur court à vitesse constante, de combien Alice a-t-elle devancé Charlie ?

2. Trouver des entiers positifs x et y tels que $\sqrt{x}+\sqrt{y}=\sqrt{2007}$.

3. Une expédition à la planète Bizarro découvre l'énoncé suivant inscrit dans le sable.

$$3x^2-25x+66 = 0 \qquad \Longrightarrow \qquad x = 4 \quad \text{ou} \quad x = 9\,.$$

Quelle est la base pour le système de numération de la planète Bizarro ?

4. Trouver la distance des deux points d'intersection de deux cercles de rayon 1 et 2, respectivement, qui se coupent de sorte que le plus grand passe par le centre du plus petit.

5. On écrit au tableau les entiers positifs de 1 à n. Un des nombres est éffacé. La moyenne des $n-1$ qui restent est $46\frac{20}{23}$. Déterminer la valeur de n ainsi que le nombre éffacé.

6. Les points $P_1(0,1)$, $P_2(0,0)$, $P_3(1,0)$, et $P_4(1,1)$ sont les sommets d'un carré. Pour $n \geq 5$, soit P_n le point défini comme suit, où $r(n)$ dénote le reste de la division de n par 8 :

$$P_n = \begin{cases} \text{point milieu du segment } P_{n-3}P_{n-4} & \text{si } r(n) = 1,\, 2, \text{ or } 3, \\ \text{point milieu du segment } P_{n-4}P_{n-7} & \text{si } r(n) = 4, \\ \text{point milieu du segment } P_{n-1}P_{n-4} & \text{si } r(n) = 5, \\ \text{point milieu du segment } P_{n-4}P_{n-5} & \text{si } r(n) = 0,\, 6, \text{ ou } 7. \end{cases}$$

Trouver les coordonnées de P_{2007}.

2006 Maritime Mathematics Competition

1. In a 100-meter race, Alice beat Bob by 10 meters and Bob beat Charlie by 20 meters. Assuming that each runner ran at a constant speed, by how much did Alice beat Charlie?

2. Find positive integers x and y such that $\sqrt{x} + \sqrt{y} = \sqrt{2007}$.

3. An expedition to the planet Bizarro finds the following equation scrawled in the dust.

$$3x^2 - 25x + 66 = 0 \qquad \Longrightarrow \qquad x = 4 \quad \text{or} \quad x = 9\,.$$

What base is used for the number system on Bizarro?

4. Two circles, one of radius 1, the other of radius 2, intersect so that the larger circle passes through the centre of the smaller circle. Find the distance between the two points at which the circles intersect.

5. The positive integers from 1 up to n (inclusive) are written on a blackboard. After one number is erased, the average (arithmetic mean) of the remaining $n-1$ numbers is $46\frac{20}{23}$. Determine n and the number that was erased.

6. Points $P_1(0,1)$, $P_2(0,0)$, $P_3(1,0)$, and $P_4(1,1)$ are the vertices of a square. For $n \geq 5$, let P_n be defined as below, where $r(n)$ is the remainder when n is divided by 8.

$$P_n = \begin{cases} \text{mid-point of } P_{n-3}P_{n-4} & \text{if } r(n) = 1,\, 2, \text{ or } 3, \\ \text{mid-point of } P_{n-4}P_{n-7} & \text{if } r(n) = 4, \\ \text{mid-point of } P_{n-1}P_{n-4} & \text{if } r(n) = 5, \\ \text{mid-point of } P_{n-4}P_{n-5} & \text{if } r(n) = 0,\, 6, \text{ or } 7. \end{cases}$$

Find the coordinates of P_{2007}.

Next, we give the solutions to the 6th Annual CNU Regional High School Mathematics Contest (2005) [2007 : 129–131].

1. There are 8 girls and 6 boys at the Math Club at Central High School. The Club needs to send a delegation to a conference, and the delegation must contain exactly two girls and two boys. The number of possible delegations that can be formed from the membership of the club is

(A) 480 (B) 420 (C) 576 (D) 1680

Identical solutions by Natalia Desy, student, Palembang, Indonesia; Jaclyn Chang, student, John Ware Junior High School, Calgary, AB; and Jixuan Wang, student, Don Mills Collegiate Institute, Toronto, ON.

The number of delegations is $C_2^8 \times C_2^6 = \frac{8!}{2!6!} \times \frac{6!}{2!4!} = 420$.

4. The remainder of 7^{100} divided by 9 is

(A) 3 (B) 4 (C) 7 (D) 5

Solution by the editor.

Let us examine $7^k/9$ for the first few positive integer values of k:

k	1	2	3	4
$7^k/9$	$\frac{7}{9}$	$\frac{49}{9}$	$\frac{343}{9}$	$\frac{2401}{9}$
remainder	7	4	1	7

One can check the next few values of k to see that the remainders cycle through the values 7, 4, and 1. In particular, every third remainder is 1. Thus, $7^{99}/9$ has a remainder of 1, and $7^{100}/9$ has a remainder of 7.

The above argument is NOT a proof that 7 is the correct answer, but it does produce the right answer. For the sake of completeness, we provide a proof below.

We shall prove by induction that all numbers of the form 7^{3n} divided by 9 give a remainder of 1, or $7^{3n} \equiv 1 \pmod 9$ for all positive integers n.

First of all, we have $7^3/9 = 343/9 = 38 + \frac{1}{9}$; hence, the statement is true for $n = 1$.

Let us suppose the statement is true for some positive integer n. That is, there exists a number t such that $7^{3n}/9 = t + \frac{1}{9}$. We then have

$$\frac{7^{3(n+1)}}{9} = \frac{7^{3n}}{9} \cdot 7^3 = \left(t + \frac{1}{9}\right) \cdot 7^3 = 7^3 t + \frac{7^3}{9} = 7^3 t + 38 + \frac{1}{9}.$$

Hence, the statement is true for $n + 1$ if it is true for n. Since it is true for $n = 1$, it is true for all positive integers n.

There were three incomplete solutions submitted.

7. When $(x^{\frac{1}{2}} - x^{\frac{2}{3}})^7$ is multiplied out and simplified, one of the terms has the form Kx^4 where K is a constant. Find K.

(A) 7 (B) -7 (C) 35 (D) -35

Solution by Natalia Desy, student, Palembang, Indonesia.

We use the Binomial Theorem. The expansion of $(x^{\frac{1}{2}} - x^{\frac{2}{3}})^7$ has terms of the form $C_i^7(x^{\frac{1}{2}})^{7-i}(-x^{\frac{2}{3}})^i$, or $(-1)^iC_i^7x^{\frac{7}{2}+\frac{i}{6}}$. Thus, the term equal to x^4 is the one with i such that $\frac{7}{2} + \frac{i}{6} = 4$, or $i = 3$. Its coefficient is $(-1)^3C_3^7 = -35$.

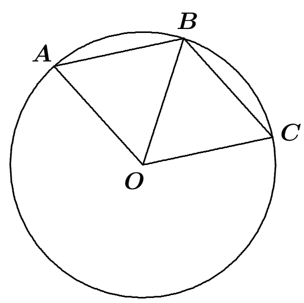
Also solved by JACLYN CHANG, student, John Ware Junior High School, Calgary, AB; and JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON.

8. Two points are picked at random on the unit circle $x^2 + y^2 = 1$. What is the probability the the chord joining the two points has length at least 1?

(A) $\frac{1}{4}$ (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) $\frac{2}{3}$

Solution by Jixuan Wang, student, Don Mills Collegiate Institute, Toronto, ON.

Since we do not need coordinates, we consider only a circle with radius 1. Let the first point be B. Let A and C be the points on the circle which are exactly 1 unit away from B. Now, $AB = BC = OC = OB = OA = 1$. Thus, the triangles AOB and BOC are equilateral, and the arc AC has angle $120°$. This is exactly one-third of the circle. Thus, two-thirds of the points on the circle are at least a distance of 1 from B. This result is independent of the position of B. Therefore, the probability we seek is $2/3$.



Also solved by JACLYN CHANG, student, John Ware Junior High School, Calgary, AB.

11. Let m be a constant. The graphs of the lines $y = x - 2$ and $y = mx + 3$ intersect at a point whose x–coordinate and y–coordinate are both positive if and only if

(A) $m = 1$ (B) $m < 1$ (C) $m > -\frac{3}{2}$ (D) $-\frac{3}{2} < m < 1$

Identical solutions by Natalia Desy, student, Palembang, Indonesia; and Jixuan Wang, student, Don Mills Collegiate Institute, Toronto, ON.

Setting the two equations equal, we get $x - 2 = mx + 3$, which yields $(m - 1)x = -5$; that is, $x = \dfrac{5}{1-m}$. For x to be positive, $1 - m$ must be positive; that is, $m < 1$. Replacing x in the first equation by $\dfrac{5}{1-m}$, we see that $y = \dfrac{5}{1-m} - 2$. In order for y to be positive, we need $m > -3/2$. Hence, $-3/2 < m < 1$.

Also solved by JACLYN CHANG, student, John Ware Junior High School, Calgary, AB.

13. Let $f(x)$ be a function such that $f(x) + 2f(-x) = \sin x$ for every real number x. What is the value of $f\left(\frac{\pi}{2}\right)$?

(A) -1 (B) $-\frac{1}{2}$ (C) $\frac{1}{2}$ (D) 1

Solution by Jixuan Wang, student, Don Mills Collegiate Institute, Toronto, ON.

First, notice that we know nothing of the relationship between $f(x)$ and $f(-x)$. Thus, we will need two equations to solve for $f\left(\frac{\pi}{2}\right)$.

To create two equations, we set x to be $\frac{\pi}{2}$ and $-\frac{\pi}{2}$ in the given equation:

$$\begin{aligned} f\left(\tfrac{\pi}{2}\right) + 2f\left(-\tfrac{\pi}{2}\right) &= \sin\left(\tfrac{\pi}{2}\right) = 1\,, \\ f\left(-\tfrac{\pi}{2}\right) + 2f\left(\tfrac{\pi}{2}\right) &= \sin\left(-\tfrac{\pi}{2}\right) = -1\,. \end{aligned}$$

Combining these two equations gives us $3f\left(\frac{\pi}{2}\right) = -3$ or $f\left(\frac{\pi}{2}\right) = -1$.

Two incorrect solutions were also submitted.

15. $\sqrt{7+4\sqrt{3}} - \sqrt{7-4\sqrt{3}} =$

(A) 4 (B) $2\sqrt{3}$ (C) $\sqrt{6}$ (D) 2

Solution by Jaclyn Chang, student, John Ware Junior High School, Calgary, AB, expanded by the editor.

We have

$$\begin{aligned} &\left(\sqrt{7+4\sqrt{3}} - \sqrt{7-4\sqrt{3}}\right)^2 \\ &\qquad = (7+4\sqrt{3}) - 2\sqrt{(7+4\sqrt{3})(7-4\sqrt{3})} + (7-4\sqrt{3}) \\ &\qquad = 14 - 2\sqrt{49 - 16\cdot 3} \\ &\qquad = 14 - 2\sqrt{1} = 14 - 2 = 12\,. \end{aligned}$$

Hence,

$$\sqrt{7+4\sqrt{3}} - \sqrt{7-4\sqrt{3}} = \sqrt{12} = 2\sqrt{3}\,.$$

Also solved by NATALIA DESY, student, Palembang, Indonesia; and JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON.

29. One root of $mx^2 - 10x + 3 = 0$ is two thirds of the other root. What is the sum of the roots?

(A) $\frac{3}{2}$ (B) $\frac{5}{2}$ (C) $\frac{7}{2}$ (D) $\frac{5}{4}$

Solution by Jaclyn Chang, student, John Ware Junior High School, Calgary, AB, modified by the editor.

Since the equation has two roots, $m \neq 0$. Then we can divide by m to get the equivalent equation $x^2 - (10/m)x + 3/m = 0$. Now we see that the sum of the roots is $10/m$ and the product of the roots is $3/m$. If the smaller root is denoted by z and the larger root by y, we have the equations

$$zy = \frac{3}{m}, \qquad z + y = \frac{10}{m}, \qquad \text{and} \qquad z = \frac{2}{3}y. \tag{1}$$

Using the third equation to eliminate z from the second equation, we get $\frac{2}{3}y + y = \frac{10}{m}$; that is, $\frac{5}{3}y = \frac{10}{m}$, or $y = \frac{6}{m}$. Then

$$z = \frac{2}{3}y = \frac{2}{3}\left(\frac{6}{m}\right) = \frac{4}{m}.$$

Now we can eliminate both y and z from the first equation of (1) to get $\left(\frac{4}{m}\right)\left(\frac{6}{m}\right) = \frac{3}{m}$. Solving, we find that $m = 8$. Then $y = \frac{6}{m} = \frac{3}{4}$ and $z = \frac{4}{m} = \frac{1}{2}$. Finally, $z + y = \frac{1}{2} + \frac{3}{4} = \frac{5}{4}$.

Also solved by NATALIA DESY, student, Palembang, Indonesia; and JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON.

33. Calculate the expression $1 \times 1! + 2 \times 2! + 3 \times 3! + \cdots + n \times n!$.

(A) $(n^2 + n + 1)n!$ (B) $(n + 1)! - 1$

(C) $(n + 2)! - n!$ (D) $(n!)^2 - 1$

Solution by Natalia Desy, student, Palembang, Indonesia.

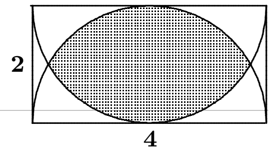
Let p be the given expression. Then

$$\begin{aligned} p + 1 &= 1 + 1 \times 1! + 2 \times 2! + 3 \times 3! + 4 \times 4! + \cdots + n \times n! \\ &= 2! + 2 \times 2! + 3 \times 3! + 4 \times 4! + \cdots + n \times n! \\ &= 3 \times 2! + 3 \times 3! + 4 \times 4! + \cdots + n \times n! \\ &= 3! + 3 \times 3! + 4 \times 4! + \cdots + n \times n! \\ &= 4 \times 3! + 4 \times 4! + \cdots + n \times n! \\ &= 4! + 4 \times 4! + \cdots + n \times n! \end{aligned}$$

Continuing in this way, we get $p + 1 = (n + 1)!$; thus, $p = (n + 1)! - 1$.

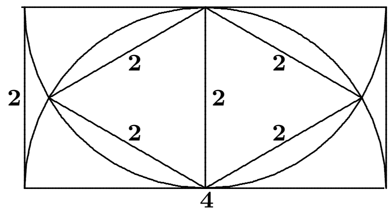
There were two solutions submitted with the correct answer, but without any formal justification.

36. A rectangle has length 4 and height 2. What is the area of the shaded region, which is the intersection of the two semicircles pictured?



(A) $\frac{4\pi}{3} + 2\sqrt{3}$ (B) $\frac{4\pi}{3} - 2\sqrt{3}$ (C) $\frac{8\pi}{3} - 2\sqrt{3}$ (D) $\frac{8\pi}{3} + 2\sqrt{3}$

Solution by Jaclyn Chang, student, John Ware Junior High School, Calgary, AB.



Split the rectangle into two squares, and draw the lines shown in the diagram above. This creates two equilateral triangles within the diagram. Thus, we can determine the shaded area as the sum of the areas of the two equilateral triangles and the four "slices" (each bounded by an arc and a chord which is the side of one of the triangles). The area of each equilateral triangle is $\sqrt{3}$, and the area of each slice is $\frac{2\pi}{3} - \sqrt{3}$. Four slices plus two equilateral triangles add up to $\frac{8\pi}{3} - 2\sqrt{3}$.

Also solved by NATALIA DESY, student, Palembang, Indonesia; and JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON.

That brings us to the end of another issue. This month's winner of a past Volume of Mayhem is Justin Wang. Congratulations, Justin! Continue sending in your contests and solutions.