New Editor-in-Chief for *CRUX with MAYHEM*

We are pleased to announce that Václav (Vazz) Linek, Department of Mathematics & Statistics, University of Winnipeg, Winnipeg, MB, Canada has accepted the position of Editor-in-Chief of ***Crux Mathematicorum with Mathematical Mayhem***, effective 1 July 2008. From January 1, 2008 through June 30, 2008, Vazz and I will be Co-Editors-in-Chief for the journal.

Let me provide you with some background information about Vazz.

> Vazz was born in the town of Beroun, just outside Prague in the former Czechoslovakia. His parents emigrated to Canada soon after the Prague Spring. He grew up in Calgary, Alberta. He obtained his B.Sc. at the University of Calgary, and received his M.Sc. and Ph.D. degrees at the University of Toronto.
>
> Vazz then went to the University of Waterloo on an NSERC Post-Doctorate Fellowship, and later joined the faculty in the Department of Mathematics and Statistics at the University of Winnipeg in 1995, where he is currently an Associate professor. He also holds an Adjunct Professorship at Carleton University.
>
> He has 20 publications in refereed journals in various areas of combinatorics such as graph theory, combinatorial designs, and combinatorics on words. He has graduated one Masters student at Carleton University. Vazz considers himself mathematically omnivorous, and has an affinity for all branches of mathematics.
>
> He enjoys Nature and being outdoors, especially hiking in the Rocky Mountains, when the opportunity presents itself. His interests include astronomy, history, poetry, politics, cooking, mycology, gardening, recycling, playing Go, and playing the didgeridoo.

Effective immediately, we ask that all problem proposals and solutions be sent to Vazz at the address listed elsewhere. This will assist in a smooth transition. If you submit your proposals and solutions via e-mail to the address in the back inside cover, they will automatically be directed to him.

Jim Totten

Nouveau Rédacteur en Chef pour *CRUX with MAYHEM*

Il nous fait plaisir d'annoncer que Václav (Vazz) Linek, du Département de mathématiques et statistique de l'université de Winnipeg, Winnipeg, MB, Canada, a accepté le poste de rédacteur en chef du ***Crux Mathematicorum with Mathematical Mayhem***. Sa nomination entrera en vigueur le 1er juillet 2008. Du 1er Janvier 2008 au 30 Juin 2008, Vazz et moi partagerons les responsabilités de rédacteur en chef du journal.

Voici quelques renseignements à son sujet :

> Vazz est né à Beroun, une ville à l'extérieur de Prague en ancienne Tchécoslovaquie. Ses parents ont émigré au Canada juste après Le Printemps de Prague. Il a grandi à Calgary, Alberta. Il a reçu son B.Sc. de l'université de Calgary, son M.Sc. et son Ph.D. de l'université de Toronto.
>
> Vazz s'est ensuite rendu à l'université de Waterloo grâce à une bourse de recherche postdoctorale du CNRC, il s'est joint ensuite à l'équipe professorale du département de mathématiques et statistique l'université de Winnipeg en 1995, où il est actuellement un professeur agrégé. Il a également un titre de professeur adjoint à l'université de Carleton.
>
> Il a à son actif 20 publications relues par des comités scientifiques dans divers secteurs de combinatoires tels que la théorie des graphes, les structures combinatoires, et la combinatoire des mots. Il a supervisé les études d'un étudiant diplômé qui a reçu une maîtrise de l'université de Carleton. Vazz se considère mathématiquement omnivore, et a une affinité pour toutes les branches des mathématiques.
>
> Il apprécie la nature et le plein air, particulièrement les excursions dans les montagnes rocheuses (Rocky Mountains) quand l'occasion se présente. Ses intérêts incluent l'astronomie, l'histoire, la poésie, la politique, la cuisine, la mycologie, le jardinage, le recyclage, jouer à GO , et jouer du didgeridoo.

Nous vous demandons de faire parvenir à partir de maintenant vos propositions de problèmes et de solutions à Vazz à l'adresse qui figure ailleurs dans cette publication. Vous nous aiderez ainsi à faire la transition tout en douceur. Si vous soumettez vos propositions et solutions par l'intermédiaire des courriers éléctroniques à l'adresse qui figure à l'endos de la couverture intérieure, on les dirigera automatiquement vers lui.

Jim Totten

SKOLIAD No. 106

Robert Bilinski

Please send your solutions to the problems in this edition by ***1 May, 2008***. A copy of **MATHEMATICAL MAYHEM Vol. 8** will be presented to one pre-university reader who sends in solutions before the deadline. The decision of the editor is final.

The editor would like to apologize to Jixuan Wang, student, Don Mills Collegiate Institute, Toronto, ON, who sent in a correct solution to problem 4 (Gilbert's beautiful sum) of Skoliad No. 100 [2007 : 67, 68]. (The official solution was given in [2007 : 394].) Wang's solution was misfiled, thus preventing its timely acknowledgement.

Nos questions proviennent ce mois-ci du Concours de Mathématiques des Maritimes 2006. Nous remercions David Horrocks de l'Université de l'Île du Prince Edouard et John Grant McLoughlin de l'Université du Nouveau-Brunswick.

Concours de Mathématiques des Maritimes 2006

1. À la ligne d'arrivée de la course de 100 m, Alice devance Bob de 10 m et Bob devance Charlie de 20 m. En supposant que chaque coureur court à vitesse constante, de combien Alice a-t-elle devancé Charlie ?

2. Trouver des entiers positifs x et y tels que $\sqrt{x} + \sqrt{y} = \sqrt{2007}$.

3. Une expédition à la planète Bizarro découvre l'énoncé suivant inscrit dans le sable.

$$3x^2 - 25x + 66 = 0 \qquad \Longrightarrow \qquad x = 4 \quad \text{ou} \quad x = 9\,.$$

Quelle est la base pour le système de numération de la planète Bizarro ?

4. Trouver la distance des deux points d'intersection de deux cercles de rayon 1 et 2, respectivement, qui se coupent de sorte que le plus grand passe par le centre du plus petit.

5. On écrit au tableau les entiers positifs de 1 à n. Un des nombres est éffacé. La moyenne des $n-1$ qui restent est $46\frac{20}{23}$. Déterminer la valeur de n ainsi que le nombre éffacé.

6. Les points $P_1(0,1)$, $P_2(0,0)$, $P_3(1,0)$, et $P_4(1,1)$ sont les sommets d'un carré. Pour $n \geq 5$, soit P_n le point défini comme suit, où $r(n)$ dénote le reste de la division de n par 8 :

$$P_n = \begin{cases} \text{point milieu du segment } P_{n-3}P_{n-4} & \text{si } r(n) = 1,\ 2,\ \text{or } 3, \\ \text{point milieu du segment } P_{n-4}P_{n-7} & \text{si } r(n) = 4, \\ \text{point milieu du segment } P_{n-1}P_{n-4} & \text{si } r(n) = 5, \\ \text{point milieu du segment } P_{n-4}P_{n-5} & \text{si } r(n) = 0,\ 6,\ \text{ou } 7. \end{cases}$$

Trouver les coordonnées de P_{2007}.

2006 Maritime Mathematics Competition

1. In a 100-meter race, Alice beat Bob by 10 meters and Bob beat Charlie by 20 meters. Assuming that each runner ran at a constant speed, by how much did Alice beat Charlie?

2. Find positive integers x and y such that $\sqrt{x} + \sqrt{y} = \sqrt{2007}$.

3. An expedition to the planet Bizarro finds the following equation scrawled in the dust.

$$3x^2 - 25x + 66 = 0 \qquad \Longrightarrow \qquad x = 4 \quad \text{or} \quad x = 9\,.$$

What base is used for the number system on Bizarro?

4. Two circles, one of radius 1, the other of radius 2, intersect so that the larger circle passes through the centre of the smaller circle. Find the distance between the two points at which the circles intersect.

5. The positive integers from 1 up to n (inclusive) are written on a blackboard. After one number is erased, the average (arithmetic mean) of the remaining $n-1$ numbers is $46\frac{20}{23}$. Determine n and the number that was erased.

6. Points $P_1(0,1)$, $P_2(0,0)$, $P_3(1,0)$, and $P_4(1,1)$ are the vertices of a square. For $n \geq 5$, let P_n be defined as below, where $r(n)$ is the remainder when n is divided by 8.

$$P_n = \begin{cases} \text{mid-point of } P_{n-3}P_{n-4} & \text{if } r(n) = 1,\ 2,\ \text{or } 3, \\ \text{mid-point of } P_{n-4}P_{n-7} & \text{if } r(n) = 4, \\ \text{mid-point of } P_{n-1}P_{n-4} & \text{if } r(n) = 5, \\ \text{mid-point of } P_{n-4}P_{n-5} & \text{if } r(n) = 0,\ 6,\ \text{or } 7. \end{cases}$$

Find the coordinates of P_{2007}.

Next, we give the solutions to the 6th Annual CNU Regional High School Mathematics Contest (2005) [2007 : 129–131].

1. There are 8 girls and 6 boys at the Math Club at Central High School. The Club needs to send a delegation to a conference, and the delegation must contain exactly two girls and two boys. The number of possible delegations that can be formed from the membership of the club is

(A) 480 (B) 420 (C) 576 (D) 1680

Identical solutions by Natalia Desy, student, Palembang, Indonesia; Jaclyn Chang, student, John Ware Junior High School, Calgary, AB; and Jixuan Wang, student, Don Mills Collegiate Institute, Toronto, ON.

The number of delegations is $C_2^8 \times C_2^6 = \frac{8!}{2!6!} \times \frac{6!}{2!4!} = 420$.

4. The remainder of 7^{100} divided by 9 is

(A) 3 (B) 4 (C) 7 (D) 5

Solution by the editor.

Let us examine $7^k/9$ for the first few positive integer values of k:

k	1	2	3	4
$7^k/9$	$\frac{7}{9}$	$\frac{49}{9}$	$\frac{343}{9}$	$\frac{2401}{9}$
remainder	7	4	1	7

One can check the next few values of k to see that the remainders cycle through the values 7, 4, and 1. In particular, every third remainder is 1. Thus, $7^{99}/9$ has a remainder of 1, and $7^{100}/9$ has a remainder of 7.

The above argument is NOT a proof that 7 is the correct answer, but it does produce the right answer. For the sake of completeness, we provide a proof below.

We shall prove by induction that all numbers of the form 7^{3n} divided by 9 give a remainder of 1, or $7^{3n} \equiv 1 \pmod 9$ for all positive integers n.

First of all, we have $7^3/9 = 343/9 = 38 + \frac{1}{9}$; hence, the statement is true for $n = 1$.

Let us suppose the statement is true for some positive integer n. That is, there exists a number t such that $7^{3n}/9 = t + \frac{1}{9}$. We then have

$$\frac{7^{3(n+1)}}{9} = \frac{7^{3n}}{9} \cdot 7^3 = \left(t + \frac{1}{9}\right) \cdot 7^3 = 7^3 t + \frac{7^3}{9} = 7^3 t + 38 + \frac{1}{9}\,.$$

Hence, the statement is true for $n + 1$ if it is true for n. Since it is true for $n = 1$, it is true for all positive integers n.

There were three incomplete solutions submitted.

7. When $(x^{\frac{1}{2}} - x^{\frac{2}{3}})^7$ is multiplied out and simplified, one of the terms has the form Kx^4 where K is a constant. Find K.

(A) 7 (B) -7 (C) 35 (D) -35

Solution by Natalia Desy, student, Palembang, Indonesia.

We use the Binomial Theorem. The expansion of $(x^{\frac{1}{2}} - x^{\frac{2}{3}})^7$ has terms of the form $C_i^7(x^{\frac{1}{2}})^{7-i}(-x^{\frac{2}{3}})^i$, or $(-1)^i C_i^7 x^{\frac{7}{2}+\frac{i}{6}}$. Thus, the term equal to x^4 is the one with i such that $\frac{7}{2} + \frac{i}{6} = 4$, or $i = 3$. Its coefficient is $(-1)^3 C_3^7 = -35$.

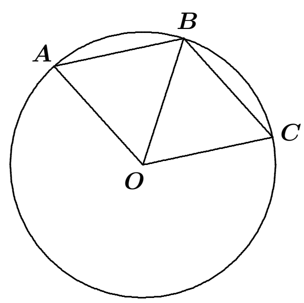
Also solved by JACLYN CHANG, student, John Ware Junior High School, Calgary, AB; and JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON.

8. Two points are picked at random on the unit circle $x^2 + y^2 = 1$. What is the probability the the chord joining the two points has length at least 1?

(A) $\frac{1}{4}$ (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) $\frac{2}{3}$

Solution by Jixuan Wang, student, Don Mills Collegiate Institute, Toronto, ON.

Since we do not need coordinates, we consider only a circle with radius 1. Let the first point be B. Let A and C be the points on the circle which are exactly 1 unit away from B. Now, $AB = BC = OC = OB = OA = 1$. Thus, the triangles AOB and BOC are equilateral, and the arc AC has angle $120°$. This is exactly one-third of the circle. Thus, two-thirds of the points on the circle are at least a distance of 1 from B. This result is independent of the position of B. Therefore, the probability we seek is $2/3$.



Also solved by JACLYN CHANG, student, John Ware Junior High School, Calgary, AB.

11. Let m be a constant. The graphs of the lines $y = x - 2$ and $y = mx + 3$ intersect at a point whose x–coordinate and y–coordinate are both positive if and only if

(A) $m = 1$ (B) $m < 1$ (C) $m > -\frac{3}{2}$ (D) $-\frac{3}{2} < m < 1$

Identical solutions by Natalia Desy, student, Palembang, Indonesia; and Jixuan Wang, student, Don Mills Collegiate Institute, Toronto, ON.

Setting the two equations equal, we get $x - 2 = mx + 3$, which yields $(m - 1)x = -5$; that is, $x = \dfrac{5}{1 - m}$. For x to be positive, $1 - m$ must be positive; that is, $m < 1$. Replacing x in the first equation by $\frac{5}{1-m}$, we see that $y = \frac{5}{1-m} - 2$. In order for y to be positive, we need $m > -3/2$. Hence, $-3/2 < m < 1$.

Also solved by JACLYN CHANG, student, John Ware Junior High School, Calgary, AB.

13. Let $f(x)$ be a function such that $f(x) + 2f(-x) = \sin x$ for every real number x. What is the value of $f\left(\frac{\pi}{2}\right)$?

(A) -1 (B) $-\frac{1}{2}$ (C) $\frac{1}{2}$ (D) 1

Solution by Jixuan Wang, student, Don Mills Collegiate Institute, Toronto, ON.

First, notice that we know nothing of the relationship between $f(x)$ and $f(-x)$. Thus, we will need two equations to solve for $f\left(\frac{\pi}{2}\right)$.

To create two equations, we set x to be $\frac{\pi}{2}$ and $-\frac{\pi}{2}$ in the given equation:

$$\begin{aligned} f\left(\tfrac{\pi}{2}\right) + 2f\left(-\tfrac{\pi}{2}\right) &= \sin\left(\tfrac{\pi}{2}\right) = 1\,, \\ f\left(-\tfrac{\pi}{2}\right) + 2f\left(\tfrac{\pi}{2}\right) &= \sin\left(-\tfrac{\pi}{2}\right) = -1\,. \end{aligned}$$

Combining these two equations gives us $3f\left(\frac{\pi}{2}\right) = -3$ or $f\left(\frac{\pi}{2}\right) = -1$.

Two incorrect solutions were also submitted.

15. $\sqrt{7+4\sqrt{3}} - \sqrt{7-4\sqrt{3}} =$

(A) 4 (B) $2\sqrt{3}$ (C) $\sqrt{6}$ (D) 2

Solution by Jaclyn Chang, student, John Ware Junior High School, Calgary, AB, expanded by the editor.

We have

$$\begin{aligned} &\left(\sqrt{7+4\sqrt{3}} - \sqrt{7-4\sqrt{3}}\right)^2 \\ &\quad = (7+4\sqrt{3}) - 2\sqrt{(7+4\sqrt{3})(7-4\sqrt{3})} + (7-4\sqrt{3}) \\ &\quad = 14 - 2\sqrt{49 - 16\cdot 3} \\ &\quad = 14 - 2\sqrt{1} = 14 - 2 = 12\,. \end{aligned}$$

Hence,

$$\sqrt{7+4\sqrt{3}} - \sqrt{7-4\sqrt{3}} = \sqrt{12} = 2\sqrt{3}\,.$$

Also solved by NATALIA DESY, student, Palembang, Indonesia; and JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON.

29. One root of $mx^2 - 10x + 3 = 0$ is two thirds of the other root. What is the sum of the roots?

(A) $\frac{3}{2}$ (B) $\frac{5}{2}$ (C) $\frac{7}{2}$ (D) $\frac{5}{4}$

Solution by Jaclyn Chang, student, John Ware Junior High School, Calgary, AB, modified by the editor.

Since the equation has two roots, $m \neq 0$. Then we can divide by m to get the equivalent equation $x^2 - (10/m)x + 3/m = 0$. Now we see that the sum of the roots is $10/m$ and the product of the roots is $3/m$. If the smaller root is denoted by z and the larger root by y, we have the equations

$$zy = \frac{3}{m}, \qquad z + y = \frac{10}{m}, \qquad \text{and} \qquad z = \frac{2}{3}y. \tag{1}$$

Using the third equation to eliminate z from the second equation, we get $\frac{2}{3}y + y = \frac{10}{m}$; that is, $\frac{5}{3}y = \frac{10}{m}$, or $y = \frac{6}{m}$. Then

$$z = \frac{2}{3}y = \frac{2}{3}\left(\frac{6}{m}\right) = \frac{4}{m}.$$

Now we can eliminate both y and z from the first equation of (1) to get $\left(\frac{4}{m}\right)\left(\frac{6}{m}\right) = \frac{3}{m}$. Solving, we find that $m = 8$. Then $y = \frac{6}{m} = \frac{3}{4}$ and $z = \frac{4}{m} = \frac{1}{2}$. Finally, $z + y = \frac{1}{2} + \frac{3}{4} = \frac{5}{4}$.

Also solved by NATALIA DESY, student, Palembang, Indonesia; and JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON.

33. Calculate the expression $1 \times 1! + 2 \times 2! + 3 \times 3! + \cdots + n \times n!$.

(A) $(n^2 + n + 1)n!$ (B) $(n + 1)! - 1$

(C) $(n + 2)! - n!$ (D) $(n!)^2 - 1$

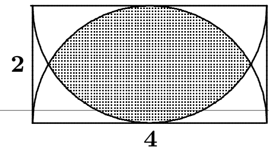
Solution by Natalia Desy, student, Palembang, Indonesia.

Let p be the given expression. Then

$$\begin{aligned}
p + 1 &= 1 + 1 \times 1! + 2 \times 2! + 3 \times 3! + 4 \times 4! + \cdots + n \times n! \\
&= 2! + 2 \times 2! + 3 \times 3! + 4 \times 4! + \cdots + n \times n! \\
&= 3 \times 2! + 3 \times 3! + 4 \times 4! + \cdots + n \times n! \\
&= 3! + 3 \times 3! + 4 \times 4! + \cdots + n \times n! \\
&= 4 \times 3! + 4 \times 4! + \cdots + n \times n! \\
&= 4! + 4 \times 4! + \cdots + n \times n!
\end{aligned}$$

Continuing in this way, we get $p + 1 = (n + 1)!$; thus, $p = (n + 1)! - 1$.

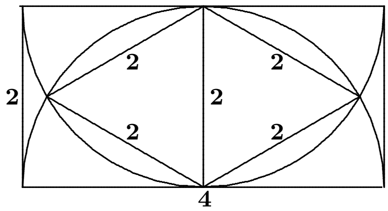
There were two solutions submitted with the correct answer, but without any formal justification.



36. A rectangle has length 4 and height 2. What is the area of the shaded region, which is the intersection of the two semicircles pictured?

(A) $\frac{4\pi}{3} + 2\sqrt{3}$ (B) $\frac{4\pi}{3} - 2\sqrt{3}$ (C) $\frac{8\pi}{3} - 2\sqrt{3}$ (D) $\frac{8\pi}{3} + 2\sqrt{3}$

Solution by Jaclyn Chang, student, John Ware Junior High School, Calgary, AB.



Split the rectangle into two squares, and draw the lines shown in the diagram above. This creates two equilateral triangles within the diagram. Thus, we can determine the shaded area as the sum of the areas of the two equilateral triangles and the four "slices" (each bounded by an arc and a chord which is the side of one of the triangles). The area of each equilateral triangle is $\sqrt{3}$, and the area of each slice is $\frac{2\pi}{3} - \sqrt{3}$. Four slices plus two equilateral triangles add up to $\frac{8\pi}{3} - 2\sqrt{3}$.

Also solved by NATALIA DESY, student, Palembang, Indonesia; and JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON.

That brings us to the end of another issue. This month's winner of a past Volume of Mayhem is Justin Wang. Congratulations, Justin! Continue sending in your contests and solutions.

In Memoriam

Jordi Dou Mas de Xexas, 1911–2007

We have just learned that Jordi Dou passed away on October 17, 2007. Jordi was featured in our ***CRUX with MAYHEM*** Profiles section [2006 : 65]. He is fondly remembered by many long-time ***Crux Mathematicorum*** and ***CRUX with MAYHEM*** readers and will be sorely missed.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

The Mayhem Editor is Jeff Hooper (Acadia University). The Assistant Mayhem Editor is Ian VanderBurgh (University of Waterloo). The other staff members are John Grant McLoughlin (University of New Brunswick), Monika Khbeis (Ascension of Our Lord Secondary School, Mississauga), Eric Robert (Leo Hayes High School, Fredericton), Larry Rice (University of Waterloo), and Ron Lancaster (University of Toronto).

Mayhem Problems

Veuillez nous transmettre vos solutions aux problèmes du présent numéro avant le **premier avril 2008**. *Les solutions reçues après cette date ne seront prises en compte que s'il nous reste du temps avant la publication des solutions.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais.

La rédaction souhaite remercier Jean-Marc Terrier, de l'Université de Montréal, d'avoir traduit les problèmes.

M319. *Proposé par Dragoljub Milošević, Pranjani, Serbie.*

Si, dans un triangle rectangle, on désigne par h l'hypoténuse et par a la hauteur, montrer que

$$\frac{a}{h}+\frac{h}{a} \;\geq\; \frac{5}{2}\,.$$

Quand y a-t-il égalité ?

M320. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Si p et q forment une paire de nombres premiers jumeaux, montrer que les nombres p^4+4 et q^4+4 ne sont jamais relativement premiers.

M321. *Proposé par José Luis Díaz-Barrero, Université Polytechnique de Catalogne, Barcelone, Espagne.*

Déterminer tous les entiers positifs n et k pour lesquels on a

$$\frac{\binom{n}{n-1}^6+\binom{n-2}{k}^6+\binom{n+3}{n+1}^3}{3\binom{n-2}{k}^2\binom{n+3}{2}} \;=\; n^2\,.$$

M322. *Proposé par Panos E. Tsaoussoglou, Athènes, Grèce.*

Soit a, b et c trois nombres réels positifs. Montrer que

$$\frac{a^3+b^3+c^3}{3abc} + \frac{8abc}{(a+b)(b+c)(c+a)} \;\geq\; 2\,.$$

M323. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Trouver toutes les solutions réelles (x,y) de l'équation

$$20\sin x - 21\cos x \;=\; 81y^2 - 18y + 30\,.$$

M324. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Deux fonctions f, $g : \mathbb{R} \to \mathbb{R}$ sont définies par

$$f(x) \;=\; 3x - 1 + |2x+1| \qquad \text{et} \qquad g(x) \;=\; \tfrac{1}{5}(3x + 5 - |2x+5|)\,.$$

Montrer que $g \circ f = f \circ g$ et $(f \circ f)^{-1} = g \circ g$.

M325. *Proposé par Bruce Shawyer, Université Memorial de Terre-Neuve, St. John's, NL.*

Soit a, b et c trois chiffres différents de zéro. Pour calculer la fraction $\frac{ab}{ca}$, où ab et ca représentent les entiers à deux chiffres $10a+b$ et $10c+a$, un étudiant applique faussement la loi de simplification, simplifiant le a du numérateur avec le a du dénominateur. Par exemple, si $a = 6$, $b = 5$ et $c = 2$, l'étudiant obtiendrait $65/26 = 5/2$ (en "simplifiant" les 6 !).

Déterminer tous les triplets (a,b,c) pour lesquels cet étudiant obtiendrait un résultat juste.

. . . .

M319. *Proposed by Dragoljub Milošević, Pranjani, Serbie.*

If h and a are the hypotenuse and altitude, respectively, of a right-angled triangle, prove that

$$\frac{a}{h} + \frac{h}{a} \;\geq\; \frac{5}{2}\,.$$

When does equality hold?

M320. *Proposed by Mihály Bencze, Brasov, Romania.*

If p and q are any pair of twin primes, show that the numbers p^4+4 and q^4+4 are never relatively prime.

M321. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Determine all positive integers n and k for which we have

$$\frac{\binom{n}{n-1}^6 + \binom{n-2}{k}^6 + \binom{n+3}{n+1}^3}{3\binom{n-2}{k}^2\binom{n+3}{2}} = n^2 .$$

M322. *Proposed by Panos E. Tsaoussoglou, Athens, Greece.*

Let a, b, and c be positive real numbers. Prove that

$$\frac{a^3+b^3+c^3}{3abc} + \frac{8abc}{(a+b)(b+c)(c+a)} \geq 2 .$$

M323. *Proposed by Mihály Bencze, Brasov, Romania.*

Find all real solutions (x, y) to the equation

$$20\sin x - 21\cos x = 81y^2 - 18y + 30 .$$

M324. *Proposed by Mihály Bencze, Brasov, Romania.*

Let functions f, $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = 3x - 1 + |2x+1| \qquad \text{and} \qquad g(x) = \tfrac{1}{5}(3x + 5 - |2x+5|) .$$

Prove that $g \circ f = f \circ g$ and $(f \circ f)^{-1} = g \circ g$.

M325. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Let a, b, and c be non-zero digits. A student takes the fraction $\frac{ab}{ca}$, where ab and ca represent the two-digit integers $10a+b$ and $10c+a$, and applies a (false) cancellation law, cancelling the a from the numerator with the a from the denominator. For example, if $a = 6$, $b = 5$, and $c = 2$, the student would obtain $65/26 = 5/2$ (by 'cancelling' the 6s!).

Determine all triples (a, b, c) for which this student actually obtains the correct answer.

Mayhem Solutions

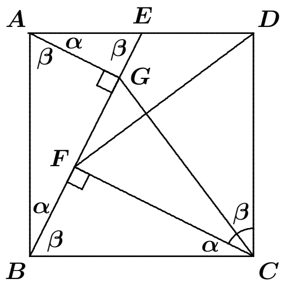
We would like to apologize to RICHARD I. HESS, Rancho Palos Verdes, CA, USA, whose solutions to M265–M268 were misfiled and did not surface until the November issue was being printed.

M269. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Let $ABCD$ be a square. Let E be the mid-point of the side AD, let F be the point on EB such that CF is perpendicular to EB, and let G be the point on EB such that AG is perpendicular to EB. Show that $DF = CG$.

Solution by Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina.

We will prove that the statement is true, more generally, when E is an arbitrary point on AD.



We first have $\angle AEB = \angle FBC = \beta$, since $AD \parallel BC$. Moreover, since $\triangle EGA$, $\triangle AGB$, and $\triangle BFC$ are all right triangles and $ABCD$ is a square, we get $\angle GAB = \angle FBC = \angle FCD = \beta$, and $\angle EAG = \angle ABF = \angle FCB = \alpha = 90^\circ - \beta$.

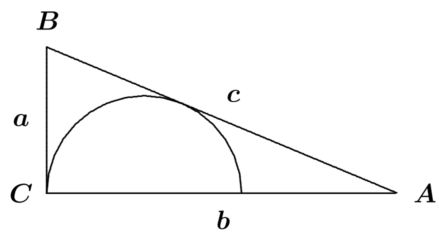
Now, since $\triangle AGB$ and $\triangle BFC$ are right triangles, with $AB = BC$, and $\angle GAB = \angle FBC$, we see that $\triangle AGB$ is congruent to $\triangle BFC$. Hence, $FC = BG$.

Finally, we have $\triangle BGC$ congruent to $\triangle CFD$, because $BG = FC$, $BC = DC$, and $\angle GBC = \angle FCD$. Therefore, $CG = DF$.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; HASAN DENKER, Istanbul, Turkey; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; TAICHI MAEKAWA, Takatsuki City, Osaka, Japan (4 solutions); KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India; DANIEL TSAI, student, Taipei American School, Taipei, Taiwan; and TITU ZVONARU, Comăneşti, Romania.

Amengual Covas, Denker, and Zvonaru also solved this more general case of the problem.

M270. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*



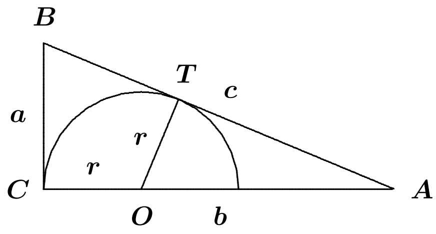
A right triangle has legs of lengths a and b and a hypotenuse of length c. A semicircle has its diameter on the side of length b and is tangent to the other two sides. Determine the radius of the semicircle in terms of a, b, and c.

Solution by Natalia Desy, student, Palembang, Indonesia.

In triangle ABC, let T denote the point of tangency of side AB to the inscribed semicircle with centre O and radius r.

Since $\angle OTA = 90^\circ$, we see that $\sin A = r/(b-r)$; but we also have $\sin A = a/c$. Equating these two expressions for $\sin A$ yields $cr = ab - ar$. Solving for r gives

$$r = \frac{ab}{a+c}.$$



Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; DENISE CORNWELL, student, Angelo State University, San Angelo, TX, USA; HASAN DENKER, Istanbul, Turkey; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; GUSTAVO KRIMKER, Universidad CAECE, Buenos Aires, Argentina; TAICHI MAEKAWA, Takatsuki City, Osaka, Japan (6 solutions); KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India; DANIEL TSAI, student, Taipei American School, Taipei, Taiwan; and TITU ZVONARU, Comănești, Romania.

M271. *Proposé par Yakub N. Aliyev, Université d'Etat de Bakou, Bakou, Azerbaïdjan.*

Sachant que dans un hexagone convexe $ABCDEF$, les côtés BC, DE et FA sont respectivement parallèles aux diagonales AD, CF et EB, on désigne respectivement par K, L et M les intersections des droites AB avec CD, CD avec EF, et EF avec AB ; on désigne enfin par P, Q et R les intersections respectives de CF avec BE, de BE avec AD, et de AD avec CF. Montrer que KP, MR et LQ se coupent en un même point.

Solution par Saturnino Campo Ruiz, "Fray Luis de León" de Salamanca, Salamanca, Espagne, modifié par le rédacteur.

Par la réciproque du Théorème de Pascal, l'hexagone $BCFADE$, dont les côtés opposés sont parallèles, peut être inscrit dans une conique, donc les sommets de l'hexagone $ABEFCD$ sont sur une conique. Par le Théorème de Pascal, les côtés opposés de l'hexagone $ABEFCD$ se coupent en trois points collinéaires. Par le Théorème de Desargues, $\triangle PQR$ et $\triangle KLM$ sont en perspective axiale si et seulement si ils sont en perspective centrale, donc KP, MR, et LQ se coupent en un même point.

Une solution incorrecte a aussi été soumise.

M272. *Proposed by John Grant McLoughlin, University of New Brunswick, Fredericton, NB.*

Let $ABCD$ be a parallelogram, and let P be a point situated on AB. If the ratio of the area of triangle ABC to that of quadrilateral $APCD$ is m/n, determine the ratio of AP to PB.

Composite of solutions submitted by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam; Hasan Denker, Istanbul, Turkey; Richard I. Hess, Rancho Palos Verdes, CA, USA; Kunal Singh, student, Kendriya Vidyalaya School, Shillong, India; and Titu Zvonaru, Comănești, Romania.

Let $[\mathcal{P}]$ denote the area of polygon $\mathcal{P}$. The ratio of the area of triangle ABC to that of quadrilateral $APCD$ is $[ABC]/[APCD] = m/n$. Then

$$\frac{n}{m} = \frac{[APCD]}{[ABC]}.$$

Since $[APCD] = [ACD] + [APC]$ and $[ACD] = \frac{1}{2}[ABCD] = [ABC]$, we have

$$\frac{n}{m} = \frac{[ACD]+[APC]}{[ABC]} = 1 + \frac{[APC]}{[ABC]}.$$

Since triangles APC and ABC share a common height with respect to their bases AP and AB, we see that $[APC]/[ABC] = AP/AB$. Therefore,

$$\frac{n}{m} = 1 + \frac{AP}{AB} = 1 + \frac{AP}{AP+PB}.$$

Then

$$\frac{AP}{AP+PB} = \frac{n-m}{m};$$

that is,

$$m \cdot AP = (n-m)AP + (n-m)PB.$$

Thus,

$$\frac{AP}{PB} = \frac{n-m}{2m-n}.$$

There were two incorrect solutions submitted.

M273. *Proposed by John Grant McLoughlin, University of New Brunswick, Fredericton, NB.*

The letters A, B, C, D, E, F, G, and H represent distinct digits. Determine their values given that the two products shown are true. (Note that the first digit of a number must be non-zero.)

$$\begin{array}{r} ABCD \\ \times E \\ \hline DCBA \end{array} \qquad \begin{array}{r} BFDG \\ \times G \\ \hline GDFB \end{array}$$

Solution by the proposer, modified by the editor.

Consider the second product above. From the thousands digit, we see that $G \cdot B \le G$ and, hence, $B = 1$. Then, since $G \cdot G$ ends in $B = 1$, we have $G = 9$. Now the product is $1FD9 \cdot 9 = 9DF1$. We cannot have $F > 1$, because the product $1FD9 \cdot 9$ would then have 5 digits instead of 4, and we cannot have $F = 1$, since $B = 1$. Thus $F = 0$. From $10D9 \cdot 9 = 9D01$, we find that the only choice for D is $D = 8$.

The first product is now $A1C8 \cdot E = 8C1A$, with A, C, and E having values in $\{2, 3, 4, 5, 6, 7\}$. Since $A1C8 \cdot E$ is even, we see that $8C1A$ is even, which implies that A is even. We then note that the digit 8 in $8C1A$ comes from adding a carried digit to the product $A \cdot E$. The carried digit is at most 1, since it comes from $1C8 \cdot E$; therefore, $A \cdot E$ is either 7 or 8. Since A is even, we must have $A \cdot E = 8$. Now, either $A = 2$ and $E = 4$, or $A = 4$ and $E = 2$. If $A = 4$ and $E = 2$, we have $41C8 \cdot 2 = 8C14$, which is impossible. We conclude that $A = 2$ and $E = 4$, which gives us $21C8 \cdot 4 = 8C12$. Finally, we verify that the only choice for C is $C = 7$.

Thus, $A = 2$, $B = 1$, $C = 7$, $D = 8$, $E = 4$, $F = 0$, and $G = 9$, and the products are

$$\begin{array}{r} 2178 \\ \times 4 \\ \hline 8712 \end{array} \qquad \begin{array}{r} 1089 \\ \times 9 \\ \hline 9801 \end{array}$$

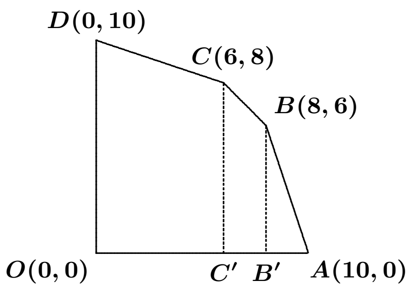
Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; DANIEL TSAI, student, Taipei American School, Taipei, Taiwan; and TITU ZVONARU, Cománeşti, Romania. One incorrect solution was also submitted.

M274. *Proposed by Neven Jurič, Zagreb, Croatia.*

Determine the area of the polygon whose vertices are all the points on the circle $x^2 + y^2 = 100$ where both coordinates are integers.

Solution by Titu Zvonaru, Cománeşti, Romania.

Let O be the centre of the polygon. Considering the vertices of the polygon in the first quadrant only, we obtain only the vertices $A(10, 0)$, $B(8, 6)$, $C(6, 8)$, and $D(10, 0)$. Since the polygon is symmetrical about both the x-axis and y-axis, its area is four times the area of pentagon $OABCD$. To find the area of $OABCD$, we sum the areas of the trapezoids $OC'CD$ and $C'B'BC$ and the triangle $B'AB$, where B' and C' are the orthogonal projections of B and C, respectively, onto the x-axis.



If we denote the area of polygon $\mathcal{P}$ by $[\mathcal{P}]$, then

$$\begin{aligned} [OABCD] &= [OC'CD] + [C'B'BC] + [B'AB]\,, \\ &= \tfrac{1}{2} \cdot 6 \cdot (10 + 8) + \tfrac{1}{2} \cdot 2 \cdot (8 + 6) + \tfrac{1}{2} \cdot 2 \cdot 6 \;=\; 74\,. \end{aligned}$$

Hence, the area of the polygon is $4(74) = 296$.

Also solved by DENISE CORNWELL, student, Angelo State University, San Angelo, TX, USA; HASAN DENKER, Istanbul, Turkey; NATALIA DESY, student, Palembang, Indonesia; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and DANIEL TSAI, student, Taipei American School, Taipei, Taiwan. There was one incorrect solution submitted.

M275. *Proposed by K.R.S. Sastry, Bangalore, India.*

A primitive Pythagorean triangle (PPT) is a right triangle whose sides have lengths which are integers with a greatest common divisor of 1. Among all pairs of non-congruent PPTs which have congruent incircles with an integer radius, find a pair for which this radius is minimized.

Solution by Titu Zvonaru, Comănești, Romania.

It is known that a PPT has hypotenuse of length $m^2 + n^2$ and legs of length $2mn$ and $m^2 - n^2$, where m and n are integers of different parity with $\gcd(m, n) = 1$.

Let ABC be a PPT, and let r be its inradius. Let $s = \frac{1}{2}(a + b + c)$ (the semiperimeter of $\triangle ABC$). On the one hand, the area of $\triangle ABC$ is $\frac{1}{2}(2mn)(m^2 - n^2) = mn(m^2 - n^2)$; on the other hand, the area is rs. Thus,

$$r = \frac{mn(m^2 - n^2)}{s} = \frac{mn(m^2 - n^2)}{\frac{1}{2}(2mn + m^2 - n^2 + m^2 + n^2)} = \frac{mn(m^2 - n^2)}{mn + m^2} = n(m - n)\,.$$

Searching for a pair of PPTs with minimal r, we find that:

(i) If $r = 1$, then $n(m - n) = 1$, which implies that $n = 1$ and $m = 2$. Therefore, we only have one PPT when $r = 1$.

(ii) If $r = 2$, then $n(m - n) = 2$, which implies that $(m, n) = (3, 1)$ or $(m, n) = (3, 2)$. Since 1 and 3 have same parity, we again have only one PPT when $r = 2$.

(iii) If $r = 3$, then $n(m - n) = 3$, which implies that $(m, n) = (4, 1)$ or $(m, n) = (4, 3)$. These lead to the required pair of PPTs.

Therefore, the pair of PPTs having congruent incircles with minimal integer radius are the triangle with side lengths 8, 15, 17 and the triangle with side lengths 7, 24, 25.

Also solved by HASAN DENKER, Istanbul, Turkey; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; DANIEL TSAI, student, Taipei American School, Taipei, Taiwan; and the proposer.

Problem of the Month

Ian VanderBurgh

As the final Problem of the Month for 2007, we have a mean sort of problem . . .

Problem (2001 American Invitational Mathematics Examination)

A finite set S of distinct real numbers has the following properties: the mean of $S \cup \{1\}$ is 13 less than the mean of S and the mean of $S \cup \{2001\}$ is 27 more than the mean of S. Find the mean of S.

Have you guessed that this was a problem from a 2001 contest? One notational reminder: $S \cup \{1\}$ means the *union* of S and $\{1\}$. In other words, it is the set that we get by adding 1 to the list of numbers already in S. (Actually, there is an additional technicality here that we'll look at quickly after the solution of the problem.)

Problems involving means usually require us to remember the fact that the mean of a list of numbers equals the sum of the numbers divided by the number of numbers in the list. In fact, that's all we really need to know here.

Solution: Let n be the number of numbers in the set S (in mathematical language, n is the *cardinality* of S). Let u be the sum of the numbers in the set S. Then the mean of the numbers in S equals $\frac{u}{n}$. When the number 1 is added to the set S, the new mean is $\frac{u+1}{n+1}$ since the sum of the number in S increases by 1 and the number of numbers in S also increases by 1. Similarly, when the number 2001 is added to the set S, the new mean is $\frac{u+2001}{n+1}$.

Therefore, the given information tells us

$$\begin{aligned} \frac{u+1}{n+1} - \frac{u}{n} &= -13\,, \\ \frac{u+2001}{n+1} - \frac{u}{n} &= 27\,. \end{aligned}$$

We need to find the mean of S, in other words $\frac{u}{n}$. In order to find $\frac{u}{n}$, it looks like we pretty much have to solve this system of two equations in two unknowns for u and n. However, there is a slight wrinkle: the AIME exam from which this problem is taken did not permit the use of a calculator! (For some of us, this may be more of a concern than for others.) So let's try to solve this system of equations in a clever way.

Can you see a manipulation that we can perform that will allow us to solve for n almost immediately? Try fiddling around for a couple of minutes before reading on.

Did you get it? What happens when we subtract the first equation from the second? When we do this, we get

$$\frac{u+2001}{n+1} - \frac{u+1}{n+1} - \frac{u}{n} + \frac{u}{n} \;=\; 27 - (-13)\,,$$

which simplifies to $\frac{2000}{n+1} = 40$; that is, $n+1 = 50$, or $n = 49$.

Now we need to find u. Substituting $n = 49$ into the first equation, we obtain

$$\frac{u+1}{50} - \frac{u}{49} \;=\; -13\,.$$

Being without a calculator, the idea of trying to get a common denominator on the left side seems a bit scary. Remember, though, that we really need $\frac{u}{49}$ (not u):

$$\begin{aligned}
\frac{u}{50} + \frac{1}{50} - \frac{u}{49} &= -13\,,\\
\frac{49}{50}\left(\frac{u}{49}\right) - \frac{50}{50}\left(\frac{u}{49}\right) &= -13 - \frac{1}{50}\,,\\
-\frac{1}{50}\left(\frac{u}{49}\right) &= -13 - \frac{1}{50}\,,\\
\frac{u}{49} &= 50(13) + 1 \;=\; 651\,.
\end{aligned}$$

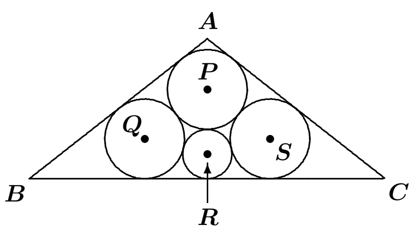
(That worked out pretty well—no ugly calculations!) Therefore, the mean of S is 651.

Those of you who have written the AIME before will be relieved by this answer, as it is an integer between 000 and 999, as per AIME prescription.

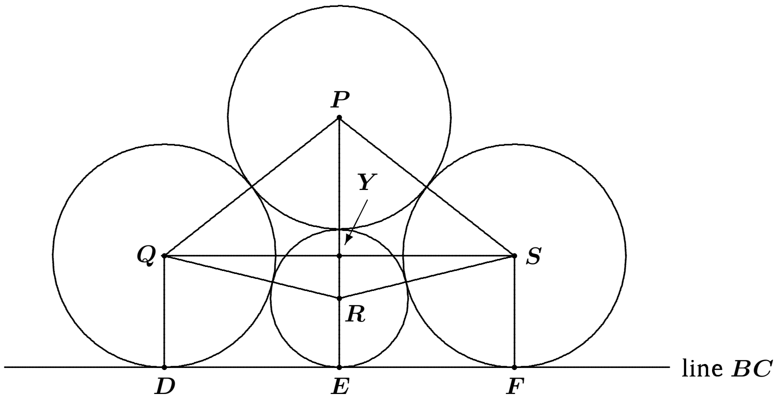
I mentioned a technicality before we launched into the solution. When we look at the union $S \cup \{1\}$, technically we add 1 to the set S only if 1 does not already appear in S. (For example, $\{1,\,2,\,3\} \cup \{1\} = \{1,\,2,\,3\}$.) Do we need to worry about this here? In fact, we don't: since the mean actually changes when we perform each of the two unions, the numbers 1 and 2001 could not have been in S to begin with.

Last month, I left you with a challenge problem, adapted from this year's Hypatia Contest. We repeat the problem, followed by its solution:

In the diagram, the circles with centres P, Q and S all have radius 1. Each is tangent to two sides of the isosceles $\triangle ABC$ and to the circle with centre R; the circle with centre P is tangent to both of the other circles of radius 1. What is the radius of the circle with centre R?



Solution to November's Challenge Problem: Drop perpendiculars to BC from Q, R, and S at D, E, and F, respectively. Since the circles with centres Q, R, and S are tangent to BC, we see that D, E, and F are the points of tangency of these circles to BC. Thus, $QD = SF = 1$. Let $RE = r$.



Join QR, RS, SP, PQ, and PR. Since we are connecting centres of tangent circles, then $PQ = PS = 2$ and $QR = RS = PR = 1 + r$. By symmetry, PRE is a straight line (that is, PE passes through R). Join QS. Since QD and SF are perpendicular to BC, then QS is parallel to BC. Thus, QS is perpendicular to PR, meeting at Y. Since $QD = 1$, then $YE = 1$. Since $RE = r$, then $YR = 1 - r$. Since $QR = 1 + r$, $YR = 1 - r$, and $\triangle QYR$ is right-angled at Y, then, by the Pythagorean Theorem,

$$\begin{aligned} QY^2 \;=\; QR^2 - YR^2 &= (1+r)^2 - (1-r)^2 \\ &= (1 + 2r + r^2) - (1 - 2r + r^2) \;=\; 4r\,. \end{aligned}$$

Since $PR = 1 + r$ and $YR = 1 - r$, then $PY = PR - YR = 2r$. Since $\triangle PYQ$ is right-angled at Y, then

$$\begin{aligned} PY^2 + YQ^2 &= PQ^2\,, \\ (2r)^2 + 4r &= 2^2\,, \\ 4r^2 + 4r &= 4\,, \\ r^2 + r - 1 &= 0\,. \end{aligned}$$

By the quadratic formula, $r = \dfrac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2} = \dfrac{-1 \pm \sqrt{5}}{2}$. Since $r > 0$, then $r = \dfrac{-1 + \sqrt{5}}{2}$ (which is the reciprocal of the famous "golden ratio").

THE OLYMPIAD CORNER

No. 266

R.E. Woodrow

We begin this number with the six problems of the 2004 Chinese Mathematical Olympiad, Macau. Thanks go to Christopher Small, Canadian team Leader to the IMO in Athens, Greece, for collecting them for our use.

2004 CHINESE MATHEMATICAL OLYMPIAD
January 8-9, 2004, Macau

1. The vertices E, F, G, and H of a convex quadrilateral $EFGH$ are points on the sides AB, BC, CD, and DA, respectively, of a convex quadrilateral $ABCD$ satisfying

$$\frac{AE}{EB}\cdot\frac{BF}{FC}\cdot\frac{CG}{GD}\cdot\frac{DH}{HA} \;=\; 1\,.$$

The points A, B, C, and D are on the sides H_1E_1, E_1F_1, F_1G_1, and G_1H_1, respectively, of a convex quadrilateral $E_1F_1G_1H_1$ satisfying $E_1F_1 \parallel EF$, $F_1G_1 \parallel FG$, $G_1H_1 \parallel GH$, and $H_1E_1 \parallel HE$.

Given that $\frac{E_1A}{AH_1} = \lambda$, find the value of $\frac{F_1C}{CG_1}$.

2. Given a positive integer c, let x_1, x_2, ... be a sequence satisfying the following conditions: $x_1 = c$, and for $n = 2, 3, \ldots$,

$$x_n \;=\; x_{n-1} + \left\lfloor \frac{2x_{n-1}-(n+2)}{n} \right\rfloor + 1\,,$$

where $\lfloor x \rfloor$ denotes the largest integer not exceeding x.

Find a formula for x_n in terms of n and c.

3. Let M be a set of n points in a plane satisfying the following conditions:

(i) There are 7 points in M which are the 7 vertices of a convex 7-gon.

(ii) For any 5 points in M, if they are the 5 vertices of a convex 5-gon, then there is at least 1 point in M in the interior of the 5-gon.

Find the minimum value of n for which such a set M exists.

4. For any real number a and positive integer n, prove the following:

(a) There exists a unique sequence of real numbers x_0, x_1, ..., x_n, x_{n+1} satisfying $x_0 = x_{n+1} = 0$ and, for $i = 1, 2, \ldots, n$,

$$\tfrac{1}{2}(x_{i+1}+x_{i-1}) \;=\; x_i + x_i^3 - a^3\,.$$

(b) The sequence in (a) satisfies $|x_i| \le |a|$ for $i = 0, 1, 2, \ldots, n+1$.

5. Given any positive integer $n \geq 2$, let a_i $(i = 1, 2, \ldots, n)$ be positive integers satisfying $a_1 < a_2 < \cdots < a_n$ and $\sum_{i=1}^{n} \frac{1}{a_i} \leq 1$. Prove that for any real number x,

$$\left(\sum_{i=1}^{n} \frac{1}{a_i^2 + x^2}\right)^2 \leq \frac{1}{2} \cdot \frac{1}{a_1(a_1 - 1) + x^2}.$$

6. Prove that all but finitely many positive integers n can be represented as a sum of 2004 positive integers, $n = a_1 + a_2 + \cdots + a_{2004}$, such that $1 \leq a_1 < a_2 < \cdots < a_{2004}$ and $a_i \mid a_{i+1}$ for $i = 1, 2, \ldots, 2003$.

As a second Olympiad set, we give the four problems of the Singapore Mathematical Olympiad 2004 (Open Section, Special Round). Thanks again go to Christopher Small for collecting them.

SINGAPORE MATHEMATICAL OLYMPIAD 2004
Open Section, Special Round
June 26, 2004

1. Let m and n be integers such that $m \geq n > 1$. Let $F_1, \ldots, F_k$ be a collection of n-element subsets of $\{1, \ldots, m\}$ such that $F_i \cap F_j$ contains at most 1 element, $1 \leq i < j \leq k$. Show that

$$k \leq \frac{m(m-1)}{n(n-1)}.$$

2. Find the number of ordered pairs (a, b), where a and b are integers and $1 \leq a, b \leq 2004$, such that the equation $x^2 + ax + b = 167y$ has integer solutions in x and y. Justify your answer.

3. Let AD be the common chord of two circles Γ_1 and Γ_2. A line through D intersects Γ_1 at B and Γ_2 at C. Let E be a point on the segment AD different from A and D. The line CE intersects Γ_1 at P and Q. The line BE intersects Γ_2 at M and N.

(i) Prove that P, Q, M, and N lie on the circumference of a circle Γ_3.

(ii) If the centre of Γ_3 is O, prove that OD is perpendicular to BC.

4. If $0 < x_1, x_2, \ldots, x_n \leq 1$, where $n \geq 1$, show that

$$\frac{x_1}{1+(n-1)x_1} + \frac{x_2}{1+(n-1)x_2} + \cdots + \frac{x_n}{1+(n-1)x_n} \leq 1.$$

Next, we give the four problems of the 18th Nordic Mathematical Contest 2004. Thanks again go to Christopher Small for collecting them.

18th NORDIC MATHEMATICAL CONTEST 2004
April 1, 2004

1. Let 27 balls be numbered from 1 to 27 and distributed into three bowls: one red, one blue, and one yellow. How many balls can there be in the red bowl, given that the means of the numbers on the balls in the red, blue, and yellow bowls are 15, 3, and 18, respectively?

2. Let $f_1 = 0$, $f_2 = 1$, and $f_{n+2} = f_{n+1} + f_n$ for $n = 1, 2, \ldots$, be the sequence of Fibonacci numbers. Show that there exists a (strictly) increasing infinite arithmetic sequence of integers which has no numbers in common with the Fibonacci sequence.

3. Let $x_{11}, x_{21}, \ldots, x_{n1}$, for $n > 2$, be a sequence of integers which are not all equal. For $k = 1, 2, 3, \ldots$, let $x_{i,k+1} = \frac{1}{2}(x_{ik} + x_{i+1,k})$ for $i = 1, 2, \ldots, n-1$, and $x_{n,k+1} = \frac{1}{2}(x_{nk} + x_{1k})$. Show that if n is odd, there exist indices j and k such that x_{jk} is not an integer.

4. Let a, b, c, and R be the side lengths and the circumradius of a triangle. Show that

$$\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} \geq \frac{1}{R^2}.$$

Finally, to provide extra problems for the (Canadian) winter break, we give the problems proposed to the jury but not used at the 2004 International Mathematical Olympiad (IMO) in Athens. Thanks go to Christopher Small, Canadian Team Leader, for obtaining them for us.

2004 IMO (ATHENS)
Problems Proposed but not Used

Algebra

A1. An infinite sequence $a_0, a_1, a_2, \ldots$ of real numbers satisfies the condition $a_n = |a_{n+1} - a_{n+2}|$ for every $n \geq 0$, with a_0 and a_1 positive and distinct. Can this sequence be bounded?

A2. Does there exist a function $s : \mathbb{Q} \to \{-1, 1\}$ such that, if x and y are distinct rational numbers satisfying $xy = 1$ or $x + y \in \{0, 1\}$, then $s(x)s(y) = -1$? Justify your answer.

A3. Let a, b, $c > 0$ and $ab + bc + ca = 1$. Prove the inequality

$$\sqrt[3]{\frac{1}{a} + 6b} + \sqrt[3]{\frac{1}{b} + 6c} + \sqrt[3]{\frac{1}{c} + 6a} \leq \frac{1}{abc}.$$

A4. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the equation

$$f\left(x^2 + y^2 + 2f(xy)\right) = \left(f(x+y)\right)^2$$

for all x, $y \in \mathbb{R}$.

A5. Let $n > 1$, and let a_1, a_2, ..., a_n be positive real numbers. Denote by g_n their geometric mean, and by A_1, A_2, ..., A_n the sequence of arithmetic means defined by

$$A_k = \frac{a_1 + a_2 + \cdots + a_k}{k},$$

for $k = 1, 2, \ldots, n$. Let G_n be the geometric mean of A_1, A_2, ..., A_n. Prove the inequality

$$n \sqrt[n]{\frac{G_n}{A_n}} + \frac{g_n}{G_n} \leq n + 1,$$

and establish the cases of equality.

Combinatorics

C1. There are 10001 students at a university. Some students join together to form several clubs (a student may belong to different clubs). Some clubs join together to form several societies (a club may belong to different societies). There are a total of k societies. Suppose that the following conditions hold:

(i) Each pair of students are in exactly one club.

(ii) For each student and each society, the student is in exactly one club of the society.

(iii) Each club has an odd number of students. In addition, a club with $2m + 1$ students (m is a positive integer) is in exactly m societies.

Find all possible values of k.

C2. Let n and k be positive integers. There are given n circles in the plane. Every two of them intersect at two distinct points, and all points of intersection they determine are pairwise distinct. Each intersection point must be coloured with one of n distinct colours so that each colour is used at least once and exactly k distinct colours occur on each circle. Find all values of $n \geq 2$ and k for which such a colouring is possible.

C3. The following operation is allowed on a finite graph: Choose an arbitrary cycle of length 4 (if there is any), choose an arbitrary edge in that cycle, and delete it from the graph. For a fixed integer $n \geq 4$, find the least number of edges of a graph that can be obtained by repeated applications of this operation from the complete graph on n vertices (where each pair of vertices are joined by an edge).

C4. Consider a matrix of size $n \times n$ whose entries are real numbers of absolute value not exceeding 1, such that the sum of all entries is 0. Let n be an even positive integer. Determine the least number C such that every such matrix necessarily has a row or a column with the sum of its entries not exceeding C in absolute value.

C5. Let N be a positive integer. Two players A and B, taking turns, write numbers from the set $\{1, \ldots, N\}$ on a blackboard. Player A begins the game by writing 1 on his first move. Then, if a player has written n on a certain move, his adversary is allowed to write $n+1$ or $2n$ (provided the number he writes does not exceed N). The player who writes N wins. We say that N is of type A or of type B according as A or B has a winning strategy.

(a) Determine whether $N = 2004$ is of type A or of type B.

(b) Find the least $N > 2004$ whose type is different from the one of 2004.

C6. For an $n \times n$ matrix A, let X_i be the set of entries in row i, and Y_j the set of entries in column j, for $1 \leq i, j \leq n$. We say that A is *golden* if $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ are distinct sets. Find the least integer n such that there exists a 2004×2004 golden matrix with entries in the set $\{1, 2, \ldots, n\}$.

C7. For a finite graph G, let $f(G)$ be the number of triangles and $g(G)$ the number of tetrahedra formed by edges of G. Find the least constant c such that $g(G)^3 \leq c \cdot f(G)^4$ for every graph G.

Geometry

G1. The circle Γ and the line ℓ do not intersect. Let AB be the diameter of Γ perpendicular to ℓ, with B closer to ℓ than A. An arbitrary point C different from both A and B is chosen on Γ. The line AC intersects ℓ at D. The line DE is tangent to Γ at E, with B and E on the same side of AC. Let BE intersect ℓ at F, and let AF intersect Γ at $G \neq A$. Prove that the reflection of G in AB lies on the line CF.

G2. Let O be the circumcentre of an acute-angled triangle ABC with $\angle B < \angle C$. The line AO meets the side BC at D. The circumcentres of the triangles ABD and ACD are E and F, respectively. Extend the sides BA and CA beyond A, and choose on the respective extensions points G and H such that $AG = AC$ and $AH = AB$. Prove that the quadrilateral $EFGH$ is a rectangle if and only if $\angle ACB - \angle ABC = 60°$.

G3. Let $A_1A_2\dots A_n$ be a regular n-gon. The points $B_1, \dots, B_{n-1}$ are defined as follows:

(i) If $i = 1$ or $i = n-1$, then B_i is the mid-point of the side A_iA_{i+1};

(ii) If $i \neq 1$, $i \neq n-1$, and S is the intersection point of A_1A_{i+1} and A_nA_i, then B_i is the intersection point of the bisector of the angle A_iSA_{i+1} with A_iA_{i+1}.

Prove that $\angle A_1B_1A_n + \angle A_1B_2A_n + \dots + \angle A_1B_{n-1}A_n = 180^\circ$.

G4. Let $\mathcal{P}$ be a convex polygon. Prove that there is a convex hexagon which is contained in $\mathcal{P}$ and which occupies at least 75 percent of the area of $\mathcal{P}$.

G5. For a given triangle ABC, let X be a variable point on the line BC such that C lies between B and X and the incircles of the triangles ABX and ACX intersect at two distinct points P and Q. Prove that the line PQ passes through a point independent of X.

G6. A cyclic quadrilateral $ABCD$ is given. The lines AD and BC intersect at E, with C between B and E; the diagonals AC and BD intersect at F. Let M be the mid-point of the side CD, and let $N \neq M$ be a point on the circumcircle of the triangle ABM such that $AN/BN = AM/BM$. Prove that the points E, F, and N are collinear.

Number Theory

N1. Let $\tau(n)$ denote the number of positive divisors of the positive integer n. Prove that there exist infinitely many positive integers a such that the equation $\tau(an) = n$ does not have a positive integer solution n.

N2. The function ψ from the set $\mathbb{N}$ of positive integers into itself is defined by the equality

$$\psi(n) = \sum_{k=1}^{n} (k, n), \qquad \text{for } n \in \mathbb{N},$$

where (k, n) denotes the greatest common divisor of k and n.

(a) Prove that $\psi(mn) = \psi(m)\psi(n)$ for every two relatively prime m, $n \in \mathbb{N}$.

(b) Prove that, for each $a \in \mathbb{N}$, the equation $\psi(x) = ax$ has a solution.

(c) Find all $a \in \mathbb{N}$ such that the equation $\psi(x) = ax$ has a unique solution.

N3. A function f from the set of positive integers $\mathbb{N}$ into itself is such that, for all m, $n \in \mathbb{N}$, the number $(m^2 + n)^2$ is divisible by $\left(f(m)\right)^2 + f(n)$. Prove that $f(n) = n$ for each $n \in \mathbb{N}$.

N4. Let k be a fixed integer greater than 1, and let $m = 4k^2 - 5$. Show that there exist positive integers a and b such that the sequence $\{x_n\}$ defined by $x_0 = 1$, $x_1 = b$, and $x_{n+2} = x_{n+1} + x_n$ for $n = 0, 1, 2, \ldots$ has all of its terms relatively prime to m.

N5. Given an integer $n > 1$, denote by P_n the product of all positive integers x less than n and such that n divides $x^2 - 1$. For each $n > 1$, find the remainder of P_n on division by n.

N6. Let p be an odd prime and n a positive integer. In the coordinate plane, eight distinct points with integer coordinates lie on a circle with diameter of length p^n. Prove that there exists a triangle with vertices at three of the given points such that the squares of its side lengths are integers divisible by p^{n+1}.

Next we given an alternate solution for problem 3 of the Italy Team Selection Contest 1999 [2002 : 356–357], for which a solution was given earlier [2004 : 492–494].

3. (a) Determine all the strictly monotone functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x + f(y)) = f(x) + y\,, \quad \forall\, x, y \in \mathbb{R}\,. \tag{1}$$

(b) Prove that for every integer $n > 1$, there do not exist strictly monotone functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x + f(y)) = f(x) + y^n\,, \quad \forall\, x, y \in \mathbb{R}\,. \tag{2}$$

Alternate solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

(a) Adding z to both sides of (1) and taking f, we get

$$f\Big(z + f(x + f(y))\Big) = f(y + z + f(x))\,. \tag{3}$$

Applying (1) to each side of (3), we have $f(z) + x + f(y) = f(y + z) + x$, or $f(y) + f(z) = f(y + z)$, for all $y, z \in \mathbb{R}$. This is the well-known Cauchy Equation, for which the monotone solutions are $f(x) = cx$, where $c = f(1)$. Substituting $f(x) = cx$ into (1), we find that $c(x + cy) = cx + y$. Hence, $c = \pm 1$ and $f(x) = \pm x$.

(b) Taking f of both sides of (2), we get

$$f\Big(0 + f(x + f(y))\Big) = f(y^n + f(x))\,. \tag{4}$$

Applying (2) to each side of (4), we have $f(0) + \big(x + f(y)\big)^n = f(y^n) + x^n$, or

$$\sum_{k=1}^{n} \binom{n}{k} x^{n-k} f(y)^k \;=\; f(y^n) - f(0)\,, \tag{5}$$

for all x, $y \in \mathbb{R}$. Note that the right side of (5) does not depend on x. Therefore, as the coefficient of nx^{n-1} on the left side of (5), $f(y) = 0$ for all $y \in \mathbb{R}$. But $f \equiv 0$ is clearly not a solution of (2).

Note that the monotonicity assumption was not needed here (part (b)).

We turn to readers' solutions to problems from the Thai Mathematical Olympiad 2002 given in the November 2006 *Corner* [2006 : 440–442].

3. Find the maximum real number K such that

$$\frac{1}{ka+b} + \frac{1}{kb+a} \geq \frac{K}{a+b}$$

for all a, $b > 0$ and all $k \in [0, \pi]$.

Solved by Michel Bataille, Rouen, France; and Pierre Bornsztein, Maisons-Laffitte, France. We give Bataille's write-up.

The maximal value of K is $\frac{4}{\pi+1}$.

First, if $a = b > 0$ and $k = \pi$, then

$$(a+b) \cdot \left(\frac{1}{ka+b} + \frac{1}{kb+a} \right) \;=\; \frac{4}{\pi+1}\,.$$

Hence, we have $K \leq 4/(\pi+1)$ for any K satisfying the conditions.

To complete the proof, we show that

$$\frac{1}{ka+b} + \frac{1}{kb+a} \;\geq\; \frac{4}{\pi+1} \cdot \frac{1}{a+b}$$

for all a, $b > 0$ and $k \in [0, \pi]$. Using the HM–GM Inequality, we get

$$\begin{aligned} \frac{1}{2}\left(\frac{1}{ka+b} + \frac{1}{kb+a} \right) \;&\geq\; \frac{2}{ka+b+kb+a} \\ &=\; \frac{2}{k+1} \cdot \frac{1}{a+b} \;\geq\; \frac{2}{\pi+1} \cdot \frac{1}{a+b}\,, \end{aligned}$$

for all a, $b > 0$ and $k \in [0, \pi]$. The result follows.

4. Let x_1 and x_2 be consecutive integers (that is, $x_2 = x_1 + 1$). For each integer $n \geq 3$, let x_n be the remainder when $x_{n-1}^2 + x_{n-2}^2$ is divided by 7. If $x_{2545} = 1$, determine the value of x_4.

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; Geoffrey A. Kandall, Hamden, CT, USA; and Ioannis Katsikis, Athens, Greece. We give Kandall's write-up.

For each positive integer n, let $\overline{x}_n$ be the residue of x_n modulo 7 (so that $0 \leq \overline{x}_n \leq 6$). It is evident from the table below that $x_n = x_{n+3}$ for all $n \geq 6$. Since $2545 \equiv 7 \pmod{3}$, we have $x_{2545} = x_7$. There are only two sequences with $x_7 = 1$, and in each case $x_4 = 1$.

$\overline{x}_1$	$\overline{x}_2$	$\overline{x}_3$	$\overline{x}_4$	$\overline{x}_5$	$\overline{x}_6$	$\overline{x}_7$	$\overline{x}_8$	$\overline{x}_9$	$\overline{x}_{10}$	$\overline{x}_{11}$
0	1	1	2	5	1	5	5	1	5	5
1	2	5	1	5	5	1	5	5	1	5
2	3	6	3	3	4	4	4	4	4	4
3	4	4	4	4	4	4	4	4	4	4
4	5	6	5	5	1	5	5	1	5	5
5	6	5	5	1	5	5	1	5	5	1
6	0	1	1	2	5	1	5	5	1	5

6. Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a non-constant polynomial with integer coefficients. Assume that $p(-1) = 0$ and $p(\sqrt{2})$ is an integer. Show that there is an integer k such that $p(k) + a_k$ is even.

Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Ioannis Katsikis, Athens, Greece. We present Bornsztein's approach.

We prove that $k = 1$ works.

We have $p(1) = \sum a_{2i} + \sum a_{2i+1}$. But $\sum a_{2i} = \sum a_{2i+1}$, because $p(-1) = 0$. Thus, $p(1)$ is even. Since $\sqrt{2}$ is irrational and $p(\sqrt{2})$ is an integer, we have $\sum a_{2i+1} 2^i = 0$. It follows that a_1 is even. Then $p(1) + a_1$ is even, as claimed.

7. Find all integers n with the property that both $n + 2002$ and $n - 2002$ are perfect squares.

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; Geoffrey A. Kandall, Hamden, CT, USA; and Ioannis Katsikis, Athens, Greece. We give the solution of Bornsztein.

Assume that $n + 2002 = a^2$ and $n - 2002 = b^2$ for some non-negative integers a and b. Then $n = \frac{1}{2}(a^2 + b^2)$. Also, $a^2 - b^2 = 4004$, which gives $(a - b)(a + b) = 2^2 \times 7 \times 11 \times 13$.

Note that $0 < a - b \leq a + b$ and that $a - b$ and $a + b$ have the same parity. Then, since their product is even, they are even. This means that $(a-b, a+b)$ is one of the pairs $(2, 2002)$, $(14, 286)$, $(22, 182)$, and $(26, 154)$. The corresponding pairs (a, b) are $(1002, 1000)$, $(150, 136)$, $(102, 80)$, and $(90, 64)$. The desired values of n are then 1002002, 20498, 8402, and 6098.

9. Find the greatest integer which divides

$$(a-b)(b-c)(c-d)(d-a)(a-c)(b-d)$$

for any integers a, b, c, d.

Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Ioannis Katsikis, Athens, Greece. We give Bornsztein's approach.

We prove that 12 is the desired maximum.

Let $f(a,b,c,d) = (a-b)(b-c)(c-d)(d-a)(a-c)(b-d)$. The greatest common divisor of the products $f(a,b,c,d)$ is a divisor of 12, because $f(0,1,2,3) = -12$.

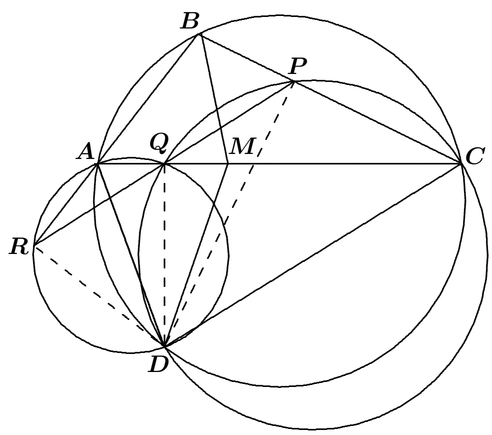
On the other hand, among any four integers at least two have the same residue modulo 3, which ensures that $f(a,b,c,d) \equiv 0 \pmod{3}$. Moreover, among four integers, either two are even and two are odd, or at least three have the same parity. In both cases, $f(a,b,c,d)$ is a multiple of 4. Therefore, the product $f(a,b,c,d)$ is always a multiple of 12, and we are done.

Next we turn to the solutions sent in by readers to the short-listed problems of the 44^{th} International Mathematical Olympiad, which appear in [2006 : 501–503].

G1. Let $ABCD$ be a cyclic quadrilateral. Let P, Q, R be the feet of the perpendiculars from D to the lines BC, CA, AB, respectively. Show that $PQ = QR$ if and only if the bisectors of $\angle ABC$ and $\angle ADC$ are concurrent with AC.

Solution by Michel Bataille, Rouen, France.

Let A, B, C denote the angles of $\triangle ABC$. Let M be the point of intersection of the internal bisector of $\angle ABC$ with AC. Recall that M is the point of the line segment AC characterized by $\dfrac{MA}{MC} = \dfrac{BA}{BC}$.



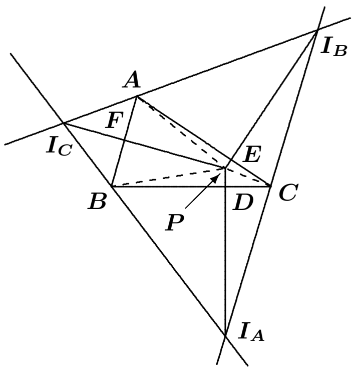
Points Q and R lie on the circle with diameter AD; hence, by the Law of Sines, $QR = DA \sin\angle RAQ = DA \sin A$. Similarly, $QP = DC \sin C$. Then

$$\frac{QR}{QP} = \frac{DA\sin A}{DC\sin C} = \frac{DA}{DC}\cdot\frac{BC}{BA} = \frac{DA}{DC}\cdot\frac{MC}{MA}.$$

Now $PQ = QR$ if and only if $\frac{MA}{MC} = \frac{DA}{DC}$, which is equivalent to DM being the internal bisector of $\angle ADC$.

G3. Let ABC be a triangle, and let P be a point in its interior. Denote by D, E, F the feet of the perpendiculars from P to the lines BC, CA, AB, respectively. Suppose that $AP^2 + PD^2 = BP^2 + PE^2 = CP^2 + PF^2$. Denote by I_A, I_B, I_C the excentres of the triangle ABC. Prove that P is the circumcentre of the triangle $I_AI_BI_C$.

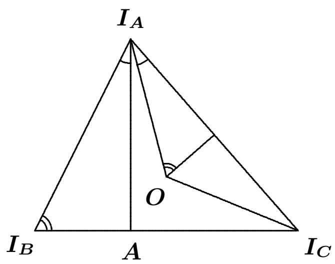
Solution by Michel Bataille, Rouen, France.



Let $a = BC$, $b = CA$, $c = AB$, and $s = \frac{1}{2}(a+b+c)$. From the hypothesis, $BD^2 = BP^2 - PD^2 = AP^2 - PE^2 = AE^2$; hence, $BD = AE$. Similarly, $CE = BF$ and $AF = CD$. Since $BD + DC = a$, $CE + EA = b$, and $AF + FB = c$, we find that $BD = AE = s-c$, $CE = BF = s-a$, and $AF = CD = s-b$, which implies that D, E, and F are the points of tangency of the excircles with the sides BC, CA, and AB, respectively. In particular, D is on the line PI_A, and similar results hold for E and F.

Now, $\angle BCI_A = \frac{1}{2}(\pi - C)$, since I_AI_B is the external bisector of $\angle ACB$. Using this and similar equalities, we obtain the angles of $\triangle I_AI_BI_C$:

$$\angle I_A = \tfrac{1}{2}(\pi - A), \quad \angle I_B = \tfrac{1}{2}(\pi - B), \quad \angle I_C = \tfrac{1}{2}(\pi - C).$$



The internal bisector I_AA is perpendicular to the external bisector I_BI_C; whence, I_AA is the altitude from I_A in $\triangle I_AI_BI_C$. Moreover,

$$I_CI_AA = \tfrac{\pi}{2} - \angle AI_CI_A = \tfrac{1}{2}C$$

and

$$\angle DI_AC = \tfrac{\pi}{2} - \angle BCI_A = \tfrac{1}{2}C.$$

Therefore, I_AD is the image of the altitude I_AA in the internal bisector of $\angle I_BI_AI_C$. It follows that I_AD passes through the circumcentre of $\triangle I_AI_BI_C$ (see the figure for a proof without words). Similarly, I_BE and I_CF pass through this circumcentre. The required result follows.

To round out this number of the *Corner*, we finish our file of solutions from our readers to the remaining short-listed problems of the 44th International Mathematical Olympiad given in [2007 : 19–21].

A4. Let n be a positive integer, and let $x_1 \leq x_2 \leq \cdots \leq x_n$ be real numbers.

(a) Prove that $\left(\sum_{i,j=1}^{n} |x_i - x_j|\right)^2 \leq \frac{2(n^2-1)}{3} \sum_{i,j=1}^{n} (x_i - x_j)^2$.

(b) Show that the equality holds if and only if $x_1, \ldots, x_n$ is an arithmetic sequence.

Solution by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain, modified by the editor.

(a) The Cauchy-Schwarz Inequality yields

$$\left(\sum_{i=1}^{n}\sum_{j=1}^{n} |i-j||x_i - x_j|\right)^2 \leq \left(\sum_{i=1}^{n}\sum_{j=1}^{n}(i-j)^2\right)\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(x_i - x_j)^2\right). \quad (1)$$

Evaluating the first term on the right side of (1), we get

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(i-j)^2 = 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(i-j)^2 = 2\sum_{k=1}^{n-1}(n-k)k^2 = \frac{n^2(n^2-1)}{6}.$$

We also get

$$\sum_{i=1}^{n}\sum_{j=1}^{n} |i-j||x_i - x_j| = \frac{n}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}|x_i - x_j|. \quad (2)$$

[Ed. To obtain (2), we proceed as follows:

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{n} |i-j||x_i - x_j| &= 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(j-i)(x_j - x_i)\\ &= 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(j-i)x_j - 2\sum_{i=1}^{n-1}x_i\sum_{j=i+1}^{n}(j-i)\\ &= 2\sum_{j=2}^{n}x_j\sum_{i=1}^{j-1}(j-i) - 2\sum_{i=1}^{n-1}x_i\sum_{j=i+1}^{n}(j-i).\end{aligned}$$

In the last step, we changed the order of summation in the first sum on the right side of the equation. Now we change the variable of summation from i to $k = j - i$ in the first sum and from j to $k = j - i$ in the second sum:

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{n}|i-j||x_i-x_j| &= 2\sum_{j=2}^{n}x_j\sum_{k=1}^{j-1}k \;-\; 2\sum_{i=1}^{n-1}x_i\sum_{k=1}^{n-i}k \\ &= 2\sum_{j=2}^{n}x_j\frac{j(j-1)}{2} \;-\; 2\sum_{i=1}^{n-1}x_i\frac{(n-i)(n-i+1)}{2} \\ &= \sum_{i=1}^{n}x_i\Big(i(i-1)-(n-i)(n-i+1)\Big) \\ &= n\sum_{i=1}^{n}x_i(2i-1-n)\,.\end{aligned}$$

Similarly, $\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n}|x_i-x_j| = 2\sum\limits_{i=1}^{n}x_i(2i-1-n)$. Thus, we obtain (2).]

Therefore, (1) becomes

$$\frac{n^2}{4}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}|x_i-x_j|\right)^2 \;\le\; \frac{n^2(n^2-1)}{6}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(x_i-x_j)^2\right)\,,$$

from which the proposed inequality follows.

(b) Equality holds when it holds in (1). Therefore, equality holds when there is some constant $d\in\mathbb{R}$ such that $d|i-j| = |x_i-x_j|$ for all i and j. But $d|i-j| = |x_i-x_j|$ for all i and j if and only if $x_{i+1}-x_i = d$ for all i. Thus, equality holds if and only if x_1, x_2, ..., x_n is an arithmetic sequence.

C4. Let x_1, ..., x_n and y_1, ..., y_n be real numbers. Let $A = (a_{ij})_{1\le i,j\le n}$ be the matrix with entries

$$a_{ij} \;=\; \begin{cases}1 & \text{if } x_i+y_j\ge 0;\\ 0 & \text{if } x_i+y_j<0.\end{cases}$$

Suppose that B is an $n\times n$ matrix with entries 0, 1 such that the sum of the elements in each row and each column of B is equal to the corresponding sum for the matrix A. Prove that $A = B$.

Solution by Michel Bataille, Rouen, France.

We will say that A is associated with the n-tuples $(x_1,x_2,\dots,x_n)$ and $(y_1,y_2,\dots,y_n)$, and that B is sum-related to A. Let σ and τ be permutations of $\{1, 2, \dots, n\}$ such that $x_{\sigma(1)}\le x_{\sigma(2)}\le\cdots\le x_{\sigma(n)}$ and $y_{\tau(1)}\le y_{\tau(2)}\le\cdots\le y_{\tau(n)}$. Then the matrix A' associated with $(x_{\sigma(1)},x_{\sigma(2)},\dots,x_{\sigma(n)})$ and $(y_{\tau(1)},y_{\tau(2)},\dots,y_{\tau(n)})$ is given by $a'_{i,j} = a_{\sigma(i),\tau(j)}$. Moreover, if $B = (b_{k,j})_{1\le i,j\le n}$, the matrix B' given by $b'_{i,j} = b_{\sigma(i),\tau(j)}$ is sum-related to A'. If we can prove that $B' = A'$, it will immediately follow that $B = A$. Thus, without loss of generality, we may as well show the property in the case $x_1\le x_2\le\cdots\le x_n$ and $y_1\le y_2\le\cdots\le y_n$.

The proof is by induction on n. The equality $A = B$ is obvious for $n = 1$. Let $n \geq 2$, and assume that the property is true for $(n-1) \times (n-1)$ matrices. Let A be the $n \times n$ matrix associated with $x_1 \leq x_2 \leq \cdots \leq x_n$ and $y_1 \leq y_2 \leq \cdots \leq y_n$, and let B be a matrix that is sum-related to A. If $A = 0$, then we must have $B = 0$. If $A \neq 0$, then certainly $a_{n,n} = 1$ (since $x_n + y_n = \max\{x_i + y_j\}$). We observe the following facts:

- if $i < n$ and $a_{i,n} = 1$, then $a_{i+1,n} = 1$ (since $x_{i+1} + y_n \geq x_i + y_n \geq 0$); and similarly, if $j < n$ and $a_{n,j} = 1$, then $a_{n,j+1} = 1$.
- if $k < n$ and $a_{k,n} = 0$, then the k^{th} row of A is null (since, for $j \leq n$, we have $x_k + y_j \leq x_k + y_n < 0$). Similarly, the ℓ^{th} column is null if $a_{n,\ell} = 0$ (where $\ell < n$). Because B is sum-related to A, it follows that any 1 situated in the n^{th} column of B cannot be in row k in case $a_{k,n} = 0$. Since the n^{th} columns of A and B have the same number of 1s, and these 1s are all situated below all possible 0s, the n^{th} columns of A and B must be equal. In the same way, we can prove that the n^{th} rows of A and B are equal. Now, delete the n^{th} row and column both in A and B and call the resulting matrices A_1 and B_1. Clearly, A_1 is associated with $x_1 \leq x_2 \leq \cdots \leq x_{n-1}$ and $y_1 \leq y_2 \leq \cdots \leq y_{n-1}$, and B_1 is sum-related to A_1. From the induction hypothesis, $B_1 = A_1$. Then $B = A$ follows.

G5. Let ABC be an isosceles triangle with $AC = BC$, whose incentre is I. Let P be a point on the circumcircle of the triangle AIB lying inside the triangle ABC. The lines through P parallel to CA and CB meet AB at D and E, respectively. The line through P parallel to AB meets CA and CB at F and G, respectively. Prove that the lines DF and EG intersect on the circumcircle of the triangle ABC.

Solved by Ricardo Barroso Campos, University of Seville, Seville, Spain; Michel Bataille, Rouen, France; and Francisco Javier García Capitán, IES Álvarez Cubero, Priego de Córdoba, Spain. We give the solution of Barroso Campos, modified by the editor.

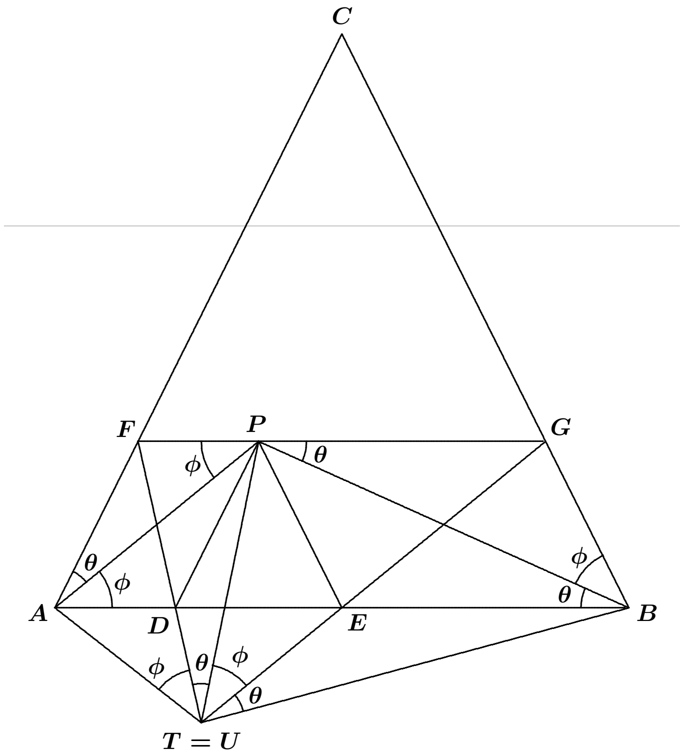
Set $\alpha = \angle CAB = \angle CBA$. Then $\angle ACB = 180^\circ - 2\alpha$. Since P lies on the circumcircle of triangle AIB, we have $\angle APB = \angle AIB = 180^\circ - \alpha$.

Set $\theta = \angle FAP$ and $\phi = \angle PAB$. Then $\phi = \alpha - \theta$,

$$\angle PBA = 180^\circ - \angle PAB - \angle APB = 180^\circ - (\alpha - \theta) - (180^\circ - \alpha) = \theta,$$

and $\angle PBG = \angle CBA - \angle PBA = \alpha - \theta = \phi$. Also, since $FG \parallel AB$, we have $\angle APF = \angle PAB = \phi$ and $\angle BPG = \angle PBA = \theta$.

Since $\angle FPE = \angle FGB = 180^\circ - \alpha$ and $\angle FAE = \angle CAB = \alpha$, we see that $\angle FPE + \angle FAE = 180^\circ$; hence, quadrilateral $FPEA$ is cyclic. Let GE produced intersect the circle $FPEA$ at T. Then $\angle FTP = \angle FAP = \theta$, $\angle ATF = \angle APF = \phi$, and $\angle PTG = \angle PTE = \angle PAE = \phi$.



Likewise, quadrilateral $DPGB$ is cyclic. Let FD produced intersect the circle $DPGB$ at U. Then $\angle PUG = \angle PBG = \phi$, $\angle BUG = \angle BPG = \theta$, and $\angle FUP = \angle DUP = \angle DBP = \angle PBA = \theta$.

Now we have $\angle FTP = \angle FUP = \theta$ and $\angle PTG = \angle PUG = \phi$, meaning that segments FP and PG are seen from both U and T with angles θ and ϕ, respectively; hence, $U = T$. Therefore,

$$\begin{aligned} \angle ATB &= \angle ATF + \angle FTP + \angle PTG + \angle GTB \\ &= (\alpha - \theta) + \theta + (\alpha - \theta) + \theta \;=\; 2\alpha\,. \end{aligned}$$

Finally, $\angle ACB + \angle ATB = (180^\circ - 2\alpha) + 2\alpha = 180^\circ$, which implies that T is on the circumcircle of triangle ABC.

That completes the *Corner* for this issue and this volume. As you may have noticed, we are now publishing solutions slightly less than one year after the corresponding problems appeared. We need you to send in your nice solutions and generalizations.

BOOK REVIEW

John Grant McLoughlin

The IMO Compendium (A Collection of Problems Suggested for the International Mathematical Olympiads: 1959–2004)
By Dušan Djukić, Vladimir Janković, Ivan Matić, and Nikola Petrović, Springer, 2006
ISBN 978-0-387-24299-6, hardcover, 740+xiv pages, US$79.95
Reviewed by ***Larry Rice*** *and* ***Ian VanderBurgh****, University of Waterloo, Waterloo, ON*

This book contains all of the problems from the International Mathematical Olympiads (IMOs) from 1959 to 2004, as well as all of the short-listed problems from most of these years, and the long-listed problems from many of these years. In total, it contains a whopping 1900 IMO level problems! Solutions are given for all of the short-listed problems and for the problems that actually appeared on the IMOs. Also, there are 20 pages of briefly stated "useful results". The sheer size of the task of collecting, solving and typesetting this amount of material is truly mind-boggling, and the authors are to be commended.

While many of the problems do have solutions (about 900 of them), the solutions take up 300 pages of the book, while the 1900 problems take up 300 pages as well. As a result, we found the solutions to be quite terse and difficult to sort through. (In our opinion, this is not unusual for this level of book.) The long-listed problems do not have solutions, but some problems have notes to indicate potential difficulties.

Is this book useful? Yes and no. The IMO problems themselves are available online with some searching. The short-listed and long-listed problems are likely much less widely available, so this book is an excellent resource for these problems. The solutions given are perhaps useful starting places for looking at these problems, but would need a fair amount of extra work to sort out. The book would likely be useful only to a handful of very advanced secondary school students in Canada in any given year. It might be of more use, though, to teachers or professors leading mathematics camps, math circles' programs for very talented students, or even undergraduate level competitions' programs. The list of results presented is again probably not that useful to any student for whom the book itself is useful, but might be a good source of suggested topics to an instructor.

An added side benefit of this book, though, is the historical timeline that it gives. This happens in two ways: first, through the trend that can be seen in the increase in difficulty of the problems over the years, and second, through the political history demonstrated by the countries involved over the years, including those that no longer exist and those new countries that have emerged over this period.

PROBLEMS

Toutes solutions aux problèmes dans ce numéro doivent nous parvenir au plus tard le **1er juin 2008**. *Une étoile* ($\star$) *après le numéro indique que le problème a été soumis sans solution.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais. Dans la section des solutions, le problème sera publié dans la langue de la principale solution présentée.

La rédaction souhaite remercier Jean-Marc Terrier, de l'Université de Montréal, d'avoir traduit les problèmes.

3289. *Proposé par Virgil Nicula, Bucarest, Roumanie.*

Soit ABC un triangle à l'intérieur duquel il existe un point D tel que $\angle DAB = \angle DCA$ et $\angle DBA = \angle DAC$. Soit respectivement les points E et F sur les droites AB et CA tels que $AB = BE$ et $CA = AF$. Montrer que les points A, E, D et F sont sur un même cercle.

3290. *Proposé par Virgil Nicula, Bucarest, Roumanie.*

Soit $ABCD$ un trapèze avec $AD \parallel BC$. Désignons respectivement par a et b les longueurs de AD et BC. Soit respectivement M, P et Q les points milieu de CD, AM et BM. Si N est l'intersection de DP et CQ, montrer que N appartient à l'intérieur du triangle ABM si et seulement si $\frac{1}{3} < \frac{a}{b} < 3$.

3291. *Proposé par Virgil Nicula, Bucarest, Roumanie.*

Soit ABC un triangle isocèle avec $AB = AC$. Trouver tout les points P tels que la somme des carrés des distances des points A, B et C à une droite quelconque par P est constante.

3292. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit a, b, c et d des nombres réels arbitraires. Montrer que

$$\begin{aligned} 11a^2 + 11b^2 + 221c^2 + 131d^2 + 22ab + 202cd + 48c + 6 \\ \geq\ 98ac + 98bc + 38ad + 38bd + 12a + 12b + 12d\,. \end{aligned}$$

3293. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Montrer que

$$\prod_{k=1}^{n} \frac{\arcsin\left(\dfrac{9k+2}{\sqrt{27k^3+54k^2+36k+8}}\right)}{\arctan\left(\dfrac{1}{\sqrt{3k+1}}\right)} \;=\; 3^n\,.$$

3294. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Pour tous les entiers positifs m et n, montrer que

$$m(m+1)n^2(n+1)^2(2n^2+2n-1) - n(n+1)m^2(m+1)^2(2m^2+2m-1)$$

est divisible par 720.

3295. *Proposé par Michel Bataille, Rouen, France.*

Soit $u : \mathbb{R} \to \mathbb{R}$ une fonction bornée. Pour $x > 0$, soit

$$\begin{aligned} f(x) &= \sup\{u(t) : t > \ln(1/x)\} \\ \text{et} \quad g(x) &= \sup\{u(t) - xe^{-t} : t \in \mathbb{R}\}\,. \end{aligned}$$

Montrer que $\lim_{x\to 0} f(x) = \lim_{x\to\infty} g(x)$.

3296. *Proposé par Michel Bataille, Rouen, France.*

Trouver la plus grande constante K telle que

$$\frac{b^2c^2}{a^2(a-b)(a-c)} + \frac{c^2a^2}{b^2(b-c)(b-a)} + \frac{a^2b^2}{c^2(c-a)(c-b)} > K$$

pour tous les nombres réels positifs distincts a, b et c.

3297. *Proposé par Stanley Rabinowitz, MathPro Press, Chelmsford, MA, USA.*

Si A, B et C sont les angles d'un triangle, montrer que

$$\sin A + \sin B \sin C \leq \frac{1+\sqrt{5}}{2}\,.$$

Quand y a-t-il égalité ?

3298. *Proposé par Stanley Rabinowitz, MathPro Press, Chelmsford, MA, USA.*

Soit ABC un triangle d'aire $\frac{1}{2}$ où a est la longueur du côté opposé au sommet A. Montrer que

$$a^2 + \csc A \geq \sqrt{5}\,.$$

[*Ed. :* Le proposeur n'a qu'une démonstration par ordinateur de ce fait. Il espère que quelqu'un parmi les lecteurs de ***CRUX with MAYHEM*** trouvera une solution plus simple.]

3299. *Proposé par Victor Oxman, Western Galilee College, Israël.*

Etant donné trois nombres réels a, b et w_b, montrer que

(a) s'il existe un triangle ABC avec $BC = a$, $CA = b$, et si la longueur de la bissectrice intérieure de l'angle B est égale à w_b, alors il est unique à un isomorphisme près ;

(b) pour que l'existence d'un triangle comme dans (a) soit assurée, il est nécessaire et suffisant que

$$b > \frac{2a\,|a - w_b|}{2a - w_b} \geq 0\,;$$

(c) si h_a est la longueur de la hauteur abaissée sur le côté BC d'un triangle comme dans (a), on a $b > |a - w_b| + \frac{1}{2}h_a$.

3300. *Proposé par Arkady Alt, San José, CA, É-U.*

Soit a, b et c trois nombres réels positifs. Pour tout entier positif n, on définit

$$F_n = \left(\frac{3(a^n + b^n + c^n)}{a+b+c} - \sum_{\text{cyclic}} \frac{b^n + c^n}{b+c}\right).$$

(a) Montrer que $F_n \geq 0$ pour $n \leq 5$.

(b)★ Montrer si oui ou non, $F_n \geq 0$ pour $n \geq 6$.

. . . .

3289. *Proposed by Virgil Nicula, Bucharest, Romania.*

Let ABC be a triangle for which there exists a point D in its interior such that $\angle DAB = \angle DCA$ and $\angle DBA = \angle DAC$. Let E and F be points on the lines AB and CA, respectively, such that $AB = BE$ and $CA = AF$. Prove that the points A, E, D, and F are concyclic.

3290. *Proposed by Virgil Nicula, Bucharest, Romania.*

Let $ABCD$ be a trapezoid with $AD \parallel BC$. Denote the lengths of AD and BC by a and b, respectively. Let M be the mid-point of CD, and let P and Q be the mid-points of AM and BM, respectively. If N is the intersection of DP and CQ, prove that N belongs to the interior of $\triangle ABM$ if and only if $\frac{1}{3} < \frac{a}{b} < 3$.

3291. *Proposed by Virgil Nicula, Bucharest, Romania.*

Let ABC be an isosceles triangle with $AB = AC$. Find all points P such that the sum of the squares of the distances of the points A, B, and C from any line through P is constant.

3292. *Proposed by Mihály Bencze, Brasov, Romania.*

Let a, b, c, and d be arbitrary real numbers. Show that

$$\begin{aligned} 11a^2 + 11b^2 + 221c^2 + 131d^2 + 22ab + 202cd + 48c + 6 \\ \geq \; 98ac + 98bc + 38ad + 38bd + 12a + 12b + 12d\,. \end{aligned}$$

3293. *Proposed by Mihály Bencze, Brasov, Romania.*

Prove that

$$\prod_{k=1}^{n} \frac{\arcsin\left(\dfrac{9k+2}{\sqrt{27k^3+54k^2+36k+8}}\right)}{\arctan\left(\dfrac{1}{\sqrt{3k+1}}\right)} \;=\; 3^n\,.$$

3294. *Proposed by Mihály Bencze, Brasov, Romania.*

For all positive integers m and n, show that

$$m(m+1)n^2(n+1)^2(2n^2+2n-1) - n(n+1)m^2(m+1)^2(2m^2+2m-1)$$

is divisible by 720.

3295. *Proposed by Michel Bataille, Rouen, France.*

Let $u : \mathbb{R} \to \mathbb{R}$ be a bounded function. For $x > 0$, let

$$\begin{aligned} f(x) &= \sup\{u(t) : t > \ln(1/x)\} \\ \text{and} \quad g(x) &= \sup\{u(t) - xe^{-t} : t \in \mathbb{R}\}\,. \end{aligned}$$

Prove that $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to \infty} g(x)$.

3296. *Proposed by Michel Bataille, Rouen, France.*

Find the greatest constant K such that

$$\frac{b^2c^2}{a^2(a-b)(a-c)} + \frac{c^2a^2}{b^2(b-c)(b-a)} + \frac{a^2b^2}{c^2(c-a)(c-b)} \;>\; K$$

for all distinct positive real numbers a, b, and c.

3297. *Proposed by Stanley Rabinowitz, MathPro Press, Chelmsford, MA, USA.*

If A, B, and C are the angles of a triangle, prove that

$$\sin A + \sin B \sin C \;\leq\; \frac{1+\sqrt{5}}{2}\,.$$

When does equality hold?

3298. *Proposed by Stanley Rabinowitz, MathPro Press, Chelmsford, MA, USA.*

Let ABC be a triangle of area $\frac{1}{2}$ in which a is the length of the side opposite vertex A. Prove that

$$a^2 + \csc A \ \geq \ \sqrt{5}\,.$$

[*Ed.:* The proposer's only proof of this is by computer. He is hoping that some ***CRUX with MAYHEM*** reader will find a simpler solution.]

3299. *Proposed by Victor Oxman, Western Galilee College, Israel.*

Given positive real numbers a, b, and w_b, show that

(a) if a triangle ABC exists with $BC = a$, $CA = b$, and the length of the interior bisector of angle B equal to w_b, then it is unique up to isomorphism;

(b) for the existence of such a triangle in (a), it is necessary and sufficient that
$$b \ > \ \frac{2a\,|a - w_b|}{2a - w_b} \ \geq \ 0\,;$$

(c) if h_a is the length of the altitude to side BC in such a triangle in (a), we have $b > |a - w_b| + \frac{1}{2}h_a$.

3300. *Proposed by Arkady Alt, San Jose, CA, USA.*

Let a, b, and c be positive real numbers. For any positive integer n define
$$F_n \ = \ \left(\frac{3(a^n + b^n + c^n)}{a + b + c} - \sum_{\text{cyclic}} \frac{b^n + c^n}{b + c} \right)\,.$$

(a) Prove that $F_n \geq 0$ for $n \leq 5$.

(b)★ Prove or disprove that $F_n \geq 0$ for $n \geq 6$.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

3189. [2006 : 514, 517] *Proposed by K.R.S. Sastry, Bangalore, India.*

In $\triangle ABC$, let A be the largest of the three angles. Let α denote the measure of angle A, and let h, w, and m denote the lengths of the altitude, the internal angle bisector, and the median, all measured from A to the side BC.

(a) Determine the area of $\triangle ABC$ in terms of α, h, and w.

(b) Determine the area of $\triangle ABC$ in terms of α, m, and w.

[*Ed:* The reader may wish to look at Mayhem problem M63 and the accompanying solution [2003 : 427–428].]

Solution by Geoffrey A. Kandall, Hamden, CT, USA.

Since $[ABC] = \frac{1}{2}bc\sin\alpha$, in both parts of the problem we want to express bc in terms of the given data.

(a) Let T be the foot of the bisector of angle A. We use the standard formula $bc = w^2 + BT \cdot TC$. By the Law of Sines,

$$\frac{BT}{\sin\left(\frac{1}{2}\alpha\right)} = \frac{c}{\sin\angle ATB} = \frac{cw}{h}.$$

Similarly,

$$\frac{TC}{\sin\left(\frac{1}{2}\alpha\right)} = \frac{b}{\sin\angle CTA} = \frac{bw}{h}.$$

Thus, $bc = w^2 + w^2\sin^2\left(\frac{1}{2}\alpha\right) \cdot bc/h^2$; whence,

$$bc = \frac{h^2w^2}{h^2 - w^2\sin^2\left(\frac{1}{2}\alpha\right)}.$$

Consequently,

$$[ABC] = \frac{h^2w^2\sin\alpha}{2\left(h^2 - w^2\sin^2\left(\frac{1}{2}\alpha\right)\right)}.$$

(b) Here we use the standard formulas $4m^2 = b^2 + c^2 + 2bc\cos\alpha$ and $b + c = 2bc\cos\left(\frac{1}{2}\alpha\right)/w$. We modify the former to

$$4m^2 = (b+c)^2 - 2bc(1-\cos\alpha) = (b+c)^2 - 4bc\sin^2\left(\tfrac{1}{2}\alpha\right)$$

and substitute the value of $b + c$ from the latter to obtain

$$4m^2 \;=\; \frac{4b^2c^2}{w^2}\cos^2\left(\tfrac{1}{2}\alpha\right) - 4bc\sin^2\left(\tfrac{1}{2}\alpha\right);$$

that is,

$$(bc)^2 - \left[w^2\tan^2\left(\tfrac{1}{2}\alpha\right)\right](bc) - m^2w^2\sec^2\left(\tfrac{1}{2}\alpha\right) \;=\; 0\,.$$

Consequently,

$$bc \;=\; \frac{w}{2}\left(w\tan^2\left(\tfrac{1}{2}\alpha\right) + \sqrt{w^2\tan^4\left(\tfrac{1}{2}\alpha\right) + 4m^2\sec^2\left(\tfrac{1}{2}\alpha\right)}\,\right)\;;$$

hence,

$$[ABC] \;=\; \frac{w\sin\alpha}{4}\left(w\tan^2\left(\tfrac{1}{2}\alpha\right) + \sqrt{w^2\tan^4\left(\tfrac{1}{2}\alpha\right) + 4m^2\sec^2\left(\tfrac{1}{2}\alpha\right)}\,\right)\,.$$

Also solved by ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam (part (a) only); CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA (part (a) only); WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; JOEL SCHLOSBERG, Bayside, NY, USA; TITU ZVONARU, Comăneşti, Romania; and the proposer.

The idea for the problem was suggested by M63 [2003 : 427–428], which dealt with a particular right triangle. The proposer believes that there might even be an interesting formula for the area of $\triangle ABC$ given α together with the length of two cevians from A; presumably, the cevians should be described by the ratio into which their feet divide BC.

3190. [2006 : 514, 517] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Let A be a point on the circle Γ, and let P be a point outside Γ. Construct a line ℓ through P which intersects Γ at B and C such that

$$2(BC) \;=\; AB + AC\,.$$

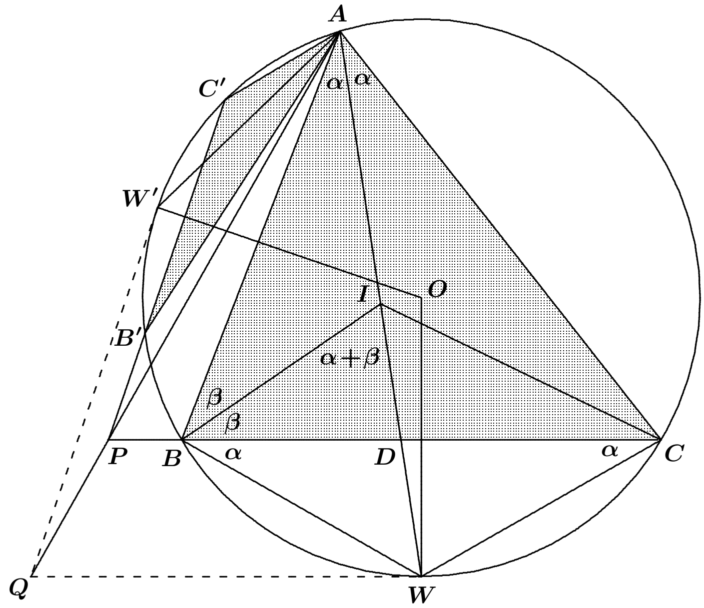
Solution by Claudio Arconcher, Jundiaí, Brazil.

Analysis. Suppose there is a point D on side BC of $\triangle ABC$ such that $BD = \frac{1}{2}AB$ and $DC = \frac{1}{2}AC$. We then would have $2(BC) = AB + AC$, which is the desired condition. With this choice of D, we have

$$\frac{AB}{BD} \;=\; \frac{AC}{DC} \;=\; \frac{2}{1}\,;$$

whence, AD is the bisector of $\angle BAC$. The incentre I of $\triangle ABC$ therefore lies on AD; whence (since BI bisects $\angle CBA$), $AB : BD = AI : ID = 2 : 1$, or

$$AI \;=\; 2ID\,.$$



Let W be the point of intersection of the circumcircle Γ of $\triangle ABC$ with the interior angle bisector of $\angle BAC$. Because $\triangle ABW \sim \triangle BDW$ [they share the angle at W and $\angle WBD = \angle WBC = \angle WAC = \angle WAB$, marked α in the figure], we have $AB : BD = BW : DW = 2 : 1$, so that $BW = 2DW$. Furthermore, since $BW = IW$ [$\triangle WBI$ is isosceles because the angles at B and at I both equal $\alpha + \beta$, as shown in the figure], we have

$$IW = 2DW = 2ID .$$

It follows that

$$AW = \tfrac{4}{3}AD .$$

Finally, note that the radius OW of Γ is perpendicular to side BC (because W is the mid-point of arc BC), which implies that the tangent to Γ at W is parallel to the line BC. That tangent, therefore, meets the line AP in a point, say Q, for which $AQ = \frac{4}{3}AP$.

Construction. Construct the point Q on the extension of AP beyond P so that $PQ = \frac{1}{3}PA$. Draw either tangent from Q to the circle Γ, calling W the point of tangency. The line through P parallel to QW is then a line that meets Γ in points B and C for which $2(BC) = AB + AC$, as was to be constructed. Since there are two lines tangent to the circumcircle from Q, there are two solution lines—the second is denoted by $B'C'$ in the figure.

Also solved by MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; SOTIRIS LOURIDAS, Aegaleo, Greece; and the proposer.

Triangles for which $2(BC) = AB + AC$ have many interesting properties. Note that problem 3197 (later in this issue) deals with such a triangle. Some references are given there.

3191. [2006 : 515, 517] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

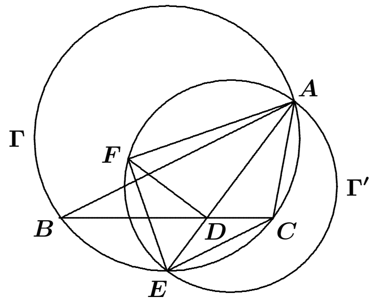
Let Γ be the circumcircle of $\triangle ABC$, let AD be the internal angle bisector of $\angle BAC$ with D on BC, and let E be the point where AD meets Γ for the second time. Let Γ' be the circle with AE as diameter, and let F be a point of Γ' such that $DF \perp AE$. Prove that $EF = EC$.

Solution by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam.

Since F is on the circle Γ', we have $\angle AFE = 90^\circ$. Since $DF \perp AE$, we obtain $EF^2 = ED \cdot EA$. Hence, $\triangle ACE \sim \triangle CDE$. Thus, $\dfrac{AE}{CE} = \dfrac{CE}{DE}$; that is, $CE^2 = AE \cdot DE$. Therefore,

$$EF^2 = ED \cdot EA = CE^2,$$

which gives $EF = EC$.



Also solved by MOHAMMED AASSILA, Strasbourg, France; MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; CLAUDIO ARCONCHER, Jundiaí, Brazil; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GEOFFREY A. KANDALL, Hamden, CT, USA; TAICHI MAEKAWA, Takatsuki City, Osaka, Japan; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; JOEL SCHLOSBERG, Bayside, NY, USA; BOB SERKEY, Leonia, NJ, USA; TITU ZVONARU, Comăneşti, Romania; and the proposer.

Many of the submitted solutions were equally concise!

3192. [2006 : 515, 517] *Proposed by Mihály Bencze, Brasov, Romania.*

Let $k \in (0,1)$, and let the sequence $\{B_n\}_{n=0}^{\infty}$ be defined by $B_0 = k$, $B_1 = k^2$, and $B_{n+2} = kB_{n+1} + k^2 B_n$ for integers $n \geq 0$. Find $\sum\limits_{n=0}^{\infty} \dfrac{B_n}{n+1}$.

Solution by Michel Bataille, Rouen, France, modified by the editor.

We denote by $\{f_n\}$ the Fibonacci sequence, defined by $f_0 = 0$, $f_1 = 1$, and $f_{n+1} = f_n + f_{n-1}$ for $n \geq 1$. One can show by induction that $B_n = k^{n+1} f_{n+1}$ for all non-negative integers n. Thus, we need to find $\sum\limits_{n=0}^{\infty} f_{n+1}\dfrac{k^{n+1}}{n+1} = \sum\limits_{n=1}^{\infty} f_n \dfrac{k^n}{n}$.

Let $\alpha = \frac{1}{2}(1+\sqrt{5})$ and $\beta = \frac{1}{2}(1-\sqrt{5})$. Dropping the condition $k \in (0,1)$, we will prove that the series above converges if and only if $\beta \leq k < -\beta$, and that, in this case, it converges to $\dfrac{1}{\sqrt{5}} \ln\left(\dfrac{1-\beta k}{1-\alpha k}\right)$.

It is well known that the power series $\sum_{n=0}^{\infty} f_n x^x$ converges if and only if $|x| < |\beta|$, in which case $\sum_{n=0}^{\infty} f_n x^n = \frac{x}{1-x-x^2}$, or equivalently,

$$\sum_{n=1}^{\infty} f_n x^{n-1} \;=\; \frac{1}{1-x-x^2}\,.$$

By integration, it follows that the radius of convergence of the power series $\sum_{n=1}^{\infty} f_n \frac{x^n}{n}$ is $|\beta|$, and that, for $|x| < |\beta|$,

$$\begin{aligned}\sum_{n=1}^{\infty} f_n \frac{x^n}{n} \;&=\; \int_0^x \frac{1}{1-t-t^2}\,dt \;=\; \frac{1}{\sqrt{5}}\int_0^x \left(\frac{1}{t+\alpha} - \frac{1}{t+\beta}\right) dt \\ &=\; \frac{1}{\sqrt{5}} \ln\left(\frac{1-\beta x}{1-\alpha x}\right)\,.\end{aligned}$$

As a result, $\sum_{n=0}^{\infty} \frac{B_n}{n+1}$ diverges for $|k| > |\beta|$ and converges for $|k| < |\beta|$, and in the case of convergence, we have

$$\sum_{n=0}^{\infty} \frac{B_n}{n+1} \;=\; \sum_{n=1}^{\infty} f_n \frac{k^n}{n} \;=\; \frac{1}{\sqrt{5}} \ln\left(\frac{1-\beta k}{1-\alpha k}\right)\,.$$

It remains to examine the cases $k = \beta$ and $k = -\beta$. From $k = -\beta$, since

$$\lim_{k\to-\beta^-} \ln\left(\frac{1-\beta k}{1-\alpha k}\right) \;=\; \infty\,,$$

it follows from Abel's Limit Theorem that $\sum_{n=1}^{\infty} f_n \frac{(-\beta)^n}{n}$ diverges. Now consider $k = \beta$. Using $f_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n)$, we obtain

$$f_n \frac{\beta^n}{n} \;=\; \frac{1}{\sqrt{5}}\left(\frac{(-1)^n}{n} - \frac{(\beta^2)^n}{n}\right)\,.$$

Since $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ and $\sum_{n=1}^{\infty} \frac{(\beta^2)^n}{n}$ both converge, the same is true for $\sum_{n=1}^{\infty} f_n \frac{\beta^n}{n}$, and (using Abel's Limit Theorem again)

$$\sum_{n=1}^{\infty} f_n \frac{\beta^n}{n} \;=\; \lim_{k\to\beta+} \ln\left(\frac{1-\beta k}{1-\alpha k}\right) \;=\; \ln\left(\frac{1-\beta^2}{2}\right)\,.$$

The proof is complete.

Also solved by BRIAN D. BEASLEY, Presbyterian College, Clinton, SC, USA; JOHN HAWKINS and DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and JOE HOWARD, Portales, NM, USA.

There were also two partially incorrect answers and three incomplete solutions which did not consider the values of k for which the given series is convergent.

3193. [2006 : 517] *Proposed by Mihály Bencze, Brasov, Romania.*

Let ABC be a triangle, and let A_1, B_1, C_1 be on sides BC, CA, AB, respectively, such that

$$\frac{AC_1}{C_1B} = \frac{BA_1}{A_1C} = \frac{CB_1}{B_1A} = k\,,$$

where k is a positive constant. Let H and H_1 be the orthocentres of $\triangle ABC$ and $\triangle A_1B_1C_1$, respectively, and let O and O_1 be their respective circumcentres. Prove that $OO_1 \parallel HH_1$.

I. *Solution by Michel Bataille, Rouen, France.*

From the definition of A_1, we have $\overrightarrow{A_1B} + k\overrightarrow{A_1C} = \overrightarrow{0}$; whence,

$$(1+k)\overrightarrow{O_1A_1} = \overrightarrow{O_1B} + k\overrightarrow{O_1C}\,.$$

Similar relations hold for $\overrightarrow{O_1B_1}$ and $\overrightarrow{O_1C_1}$.

Now, from the well-known relations $\overrightarrow{O_1H_1} = \overrightarrow{O_1A_1} + \overrightarrow{O_1B_1} + \overrightarrow{O_1C_1}$ and $\overrightarrow{OH} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$, we deduce that

$$(1+k)\overrightarrow{O_1H_1} = \overrightarrow{O_1B} + k\overrightarrow{O_1C} + \overrightarrow{O_1C} + k\overrightarrow{O_1A} + \overrightarrow{O_1A} + k\overrightarrow{O_1B}\,;$$

that is,

$$\overrightarrow{O_1H_1} = \overrightarrow{O_1A} + \overrightarrow{O_1B} + \overrightarrow{O_1C} = 3\overrightarrow{O_1O} + \overrightarrow{OH}\,.$$

Thus, $\overrightarrow{HH_1} = 2\overrightarrow{O_1O}$, which implies the desired result.

II. *Solution by Chip Curtis, Missouri Southern State University, Joplin, MO, USA.*

By Theorem 276 [1, p.175], the proportionality condition implies that triangles $A_1B_1C_1$ and ABC have the same centroid.

On the other hand, by Theorem 257 [1, p.165], in any triangle, the circumcentre O, the centroid G, and the orthocentre H are collinear, with $2OG = GH$. Hence, triangle OGO_1 is similar to triangle HGH_1, so that OO_1 is parallel to HH_1.

References

[1] R. Johnson, *Modern Geometry*, Houghton-Mifflin, 1929.

Also solved by APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; TAICHI MAEKAWA, Takatsuki City, Osaka, Japan; JOEL SCHLOSBERG, Bayside, NY, USA; SKIDMORE COLLEGE PROBLEM SOLVING GROUP, Skidmore College Saratoga Springs, NY, USA; D.J. SMEENK, Zaltbommel, the Netherlands; TITU ZVONARU, Comăneşti, Romania; and the proposer.

3194. [2005 : 515, 517] *Proposed by Mihály Bencze, Brasov, Romania.*

Let n be any positive integer, and let x_k, $y_k \in \mathbb{R}$ for $k = 1, 2, \ldots, n$. Prove that

$$\min\left\{\sum_{k=1}^{n} x_k^2,\ \sum_{k=1}^{n} y_k^2\right\} \cdot \sum_{k=1}^{n}(x_k - y_k)^2 \ \geq \sum_{1\leq i<j\leq n} (x_i y_j - x_j y_i)^2 .$$

Solution by Michel Bataille, Rouen, France, modified by the editor.

Without loss of generality, we suppose that $\sum\limits_{k=1}^{n} x_k^2 \leq \sum\limits_{k=1}^{n} y_k^2$. The proposed inequality then reduces to

$$\left(\sum_{k=1}^{n} x_k^2\right)\left(\sum_{k=1}^{n}(x_k - y_k)^2\right) \ \geq \sum_{1\leq i<j\leq n} (x_i y_j - x_j y_i)^2 .$$

From the well-known identity

$$\left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) = \sum_{1\leq i<j\leq n} (a_i b_j - a_j b_i)^2 + \left(\sum_{k=1}^{n} a_k b_k\right)^2 ,$$

we obtain the inequality

$$\left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) \ \geq \sum_{1\leq i<j\leq n} (a_i b_j - a_j b_i)^2$$

(for all real numbers $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$). Setting $a_k = x_k$ and $b_k = x_k - y_k$ for $k = 1, 2, \ldots, n$ yields

$$\begin{aligned}\left(\sum_{k=1}^{n} x_k^2\right)\left(\sum_{k=1}^{n}(x_k - y_k)^2\right) &\geq \sum_{1\leq i<j\leq n} \big(x_i(x_j - y_j) - x_j(x_i - y_i)\big)^2 \\ &= \sum_{1\leq i<j\leq n} (x_i y_j - x_j y_i)^2 .\end{aligned}$$

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposer.

3195. [2006 : 515, 518] *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

(a) Let n be a natural number, $n \geq 3$. Prove that there is a real number $q_n > 1$ such that for any real numbers $a_1, a_2, \ldots, a_n \in [1/q_n, q_n]$,

$$\frac{a_1}{a_2 + a_3} + \frac{a_2}{a_3 + a_4} + \cdots + \frac{a_n}{a_n + a_1} \ \geq \ \frac{n}{2} .$$

(b)★ Does there exist a real number $q > 1$ such that the inequality in (a) holds for any natural number $n \geq 3$ and for any real numbers $a_1, a_2, \ldots, a_n \in [1/q, q]$?

Solution to part (a) *by the proposer, modified by the editor.*

It is easy to see that the inequality in (a) is equivalent to

$$\sum_{\text{cyclic}} \frac{2q_n^2 a_1 - a_2 - a_3}{a_2 + a_3} \;\geq\; n(q_n^2 - 1)\,. \tag{1}$$

Since $2q_n^2 a_1 - a_2 - a_3 = (q_n^2 a_1 - a_2) + (q_n^2 a_1 - a_3) \geq 0$, the Cauchy-Schwarz Inequality may be applied to get

$$\begin{aligned}&\Bigg(\sum_{\text{cyclic}} (2q_n^2 a_1 - a_2 - a_3)\Bigg)^2 \\ &\qquad \leq \;\Bigg(\sum_{\text{cyclic}} (a_2 + a_3)(2q_n^2 a_1 - a_2 - a_3)\Bigg)\Bigg(\sum_{\text{cyclic}} \frac{2q_n^2 a_1 - a_2 - a_3}{a_2 + a_3}\Bigg)\,.\end{aligned}$$

Thus, to obtain (1), it is sufficient to prove the inequality

$$\begin{aligned}&\Bigg(\sum_{\text{cyclic}} (2q_n^2 a_1 - a_2 - a_3)\Bigg)^2 \\ &\qquad \geq \; n(q_n^2 - 1) \sum_{\text{cyclic}} (a_2 + a_3)(2q_n^2 a_1 - a_2 - a_3)\,.\end{aligned} \tag{2}$$

Since

$$\sum_{\text{cyclic}} (2q_n^2 a_1 - a_2 - a_3) \;=\; \sum_{\text{cyclic}} 2(q_n^2 - 1) a_1 \;=\; 2(q_n^2 - 1) \sum_{\text{cyclic}} a_1$$

and

$$\sum_{\text{cyclic}} (a_2 + a_3)(2q_n^2 a_1 - a_2 - a_3) \;=\; 2q_n^2 \sum_{\text{cyclic}} a_1(a_2 + a_3) - \sum_{\text{cyclic}} (a_1 + a_2)^2\,,$$

the following inequality is equivalent to (2):

$$\frac{4}{n}(q_n^2 - 1)\Bigg(\sum_{\text{cyclic}} a_1\Bigg)^2 \;\geq\; 2q_n^2 \sum_{\text{cyclic}} a_1(a_2 + a_3) - \sum_{\text{cyclic}} (a_1 + a_2)^2\,. \tag{3}$$

We have

$$2 \sum_{\text{cyclic}} a_1(a_2 + a_3) \;=\; 2 \sum_{\text{cyclic}} (a_1 + a_2)(a_2 + a_3) - \sum_{\text{cyclic}} (a_1 + a_2)^2\,;$$

hence, inequality (3) can be written in the form

$$\frac{4}{n}(q_n^2-1)\Big(\sum_{\text{cyclic}} a_1\Big)^2 \geq 2q_n^2\sum_{\text{cyclic}}(a_1+a_2)(a_2+a_3)-(q_n^2+1)\sum_{\text{cyclic}}(a_1+a_2)^2\,. \quad (4)$$

Using the substitution $b_i = a_i + a_{i+1}$ for $i = 1, 2, \ldots, n$, inequality (4) reduces to

$$\frac{1}{n}(q_n^2-1)\Big(\sum_{\text{cyclic}} b_1\Big)^2 \geq 2q_n^2\sum_{\text{cyclic}} b_1b_2-(q_n^2+1)\sum_{\text{cyclic}} b_1^2\,. \quad (5)$$

Since

$$\Big(\sum_{\text{cyclic}} b_1\Big)^2 = n\sum_{\text{cyclic}} b_1^2-\sum_{j<k}(b_j-b_k)^2\,,$$

inequality (5) is equivalent to

$$n\sum_{\text{cyclic}}(b_1-b_2)^2 \geq \left(1-\frac{1}{q_n^2}\right)\sum_{j<k}(b_j-b_k)^2\,. \quad (6)$$

But, for $j<k$, we have

$$\sum_{\text{cyclic}}(b_1-b_2)^2 \geq \sum_{i=j}^{k-1}(b_i-b_{i+1})^2 \geq \frac{1}{k-j}\Big(\sum_{i=j}^{k-1}(b_i-b_{i+1})\Big)^2 \geq \frac{1}{n-1}(b_j-b_k)^2$$

(where we have used the Cauchy-Schwarz Inequality). Summing over j and k with $j<k$ yields

$$\frac{n(n-1)}{2}\sum_{\text{cyclic}}(b_1-b_2)^2 \geq \frac{1}{n-1}\sum_{j<k}(b_j-b_k)^2\,.$$

Comparing this inequality with (6), we see that we may obtain (6) by choosing $1-\frac{1}{q_n^2}=\frac{2}{(n-1)^2}$; that is,

$$q_n = \frac{1}{\sqrt{1-2/(n-1)^2}} = \frac{n-1}{\sqrt{n^2-2n-1}}\,.$$

Since $q_n > \frac{1}{\sqrt{1-2/n^2}} = \frac{n}{\sqrt{n^2-2}} > 1$, we can also choose $q_n = \frac{n}{\sqrt{n^2-2}}$.

The proposer also offered the following remarks.

1. Using the substitution $b_i = x_i - b$ for all $i = 1, 2, \ldots, n$, where $b = \dfrac{b_1 + b_2 + \cdots + b_n}{n}$ and $x_1 + x_2 + \cdots + x_n = 0$, inequality (5) becomes

$$\frac{\sum\limits_{\text{cyclic}} x_1x_2}{\sum\limits_{\text{cyclic}} x_1^2} \le \frac{q_n^2+1}{2q_n^2}.$$

According to Fan's Inequality [1]

$$\frac{\sum\limits_{\text{cyclic}} x_1x_2}{\sum\limits_{\text{cyclic}} x_1^2} \le \cos\frac{2\pi}{n};$$

thus, we may choose $q_n = \dfrac{1}{\sqrt{2\cos\frac{2\pi}{n} - 1}}$. This is the largest value of q_n such that (5) holds for all positive b_i.

2. The inequality

$$\frac{a_1}{a_2+a_3} + \frac{a_2}{a_3+a_4} + \cdots + \frac{a_n}{a_1+a_n} \ge \frac{n}{2}$$

is the well-known Shapiro Inequality. It is valid for any positive real numbers $a_1, a_2, \ldots, a_n$, for even $n \le 12$ and odd $n \le 23$.

3. The previous results are not enough to solve part (b) of the problem because $\lim\limits_{n\to\infty} q_n = 1$.

References

[1] K. Fan, O. Taussky, and J. Todd, *Discrete Analogs of Inequalities of Wirtinger*, Monatsh. Math. 59 (1955), 73–79.

Part (a) *also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria. Part* (b) *remains open.*

3196. [2006 : 515, 518] *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

Let $x_1, x_2, \ldots, x_n$ be positive real numbers. Prove that

$$x_1^n + x_2^n + \cdots + x_n^n + n(n-1)x_1x_2\cdots x_n \ge x_1x_2\cdots x_n(x_1+x_2+\cdots+x_n)\left(\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}\right).$$

Solution by Gabriel Dospinescu, University of Bucharest, Romania.

We will use induction. For $n = 2$, equality holds. Assuming that the inequality is true for $n - 1$ (where $n \geq 3$), we will show that it is true for n.

Using the induction hypothesis, we obtain, for each $k \in \{1, 2, \ldots, n\}$,

$$x_k \sum_{i \neq k} x_i^{n-1} + (n-1)(n-2) x_1 x_2 \cdots x_n \;\geq\; x_1 x_2 \cdots x_n \left(\sum_{i \neq k} x_i \right) \left(\sum_{i \neq k} \frac{1}{x_i} \right) .$$

We write this in a more useful form:

$$\begin{aligned} x_k \sum_{i=1}^{n} x_i^{n-1} \;-\; x_k^n \;+\; (n-1)(n-2) x_1 x_2 \cdots x_n & \\ \geq\; x_1 x_2 \cdots x_n \left[\left(\sum_{i=1}^{n} x_i \right) \left(\sum_{i=1}^{n} \frac{1}{x_i} \right) - x_k \sum_{i=1}^{n} \frac{1}{x_i} - \frac{1}{x_k} \sum_{i=1}^{n} x_i + 1 \right] . & \end{aligned}$$

Summing over k, we find that

$$\begin{aligned} \left(\sum_{i=1}^{n} x_i \right) \cdot \left(\sum_{i=1}^{n} x_i^{n-1} \right) \;-\; \sum_{i=1}^{n} x_i^n \;+\; n(n-1)(n-2) x_1 x_2 \cdots x_n & \\ \geq\; x_1 x_2 \cdots x_n \left[(n-2) \left(\sum_{i=1}^{n} x_i \right) \left(\sum_{i=1}^{n} \frac{1}{x_i} \right) + n \right] . & \end{aligned}$$

Using Suranyi's Inequality (see [1]),

$$\left(\sum_{i=1}^{n} x_i \right) \left(\sum_{i=1}^{n} x_i^{n-1} \right) \;\leq\; (n-1) \sum_{i=1}^{n} x_i^n \;+\; n x_1 x_2 \cdots x_n ,$$

we get

$$\begin{aligned} (n-2) \sum_{i=1}^{n} x_i^n \;+\; n x_1 x_2 \cdots x_n \;+\; n(n-1)(n-2) x_1 x_2 \cdots x_n & \\ \geq\; x_1 x_2 \cdots x_n \left[(n-2) \left(\sum_{i=1}^{n} x_i \right) \left(\sum_{i=1}^{n} \frac{1}{x_i} \right) + n \right] , & \end{aligned}$$

which simplifies to give the desired result.

For $n \geq 3$, equality holds if and only if $x_1 = x_2 = \cdots = x_n$.

References

[1] T. Andreescu, V. Cîrtoaje, G. Dospinescu, M. Lascu, *Old and New Inequalities*, GIL Publishing House, Zalau, Romania, 2004, pp. 110–111.

No other solutions were submitted. (The above solution was the one that accompanied the problem proposal.)

Walther Janous, Ursulinengymnasium, Innsbruck, Austria noted that this problem is Proposition 3.7 in the proposer's paper entitled The Equal Variable Method, which was published in the electonically distributed journal JIPAM (Journal of Inequalities in Pure and Applied

Mathematics, `http://jipam.vu.edu.au`), *Volume 8 (2007), Issue 1, Article 15, p. 28. That paper gives a reference to* `http://www.mathlinks.ro/Forum/viewtopic.php?t=14906`, *where the proposer had previously posted the inequality. Solutions to the problem are given at both of these locations.*

3197. [2006 : 516, 518] *Proposed by Paul Deiermann, Southeast Missouri State University, Cape Girardeau, MO, USA.*

If AB is a fixed line segment, find the triangle ABC which has maximum area among those which satisfy $\angle AIO = \pi/2$, where I is the incentre of $\triangle ABC$ and O is its circumcentre. What is this maximum area?

Solution by Michel Bataille, Rouen, France.

Let $a = BC$, $b = CA$, and $c = AB$, as usual. It has been shown more than once in this journal that $\angle AIO = \pi/2$ if and only if $2a = b + c$. (See the references below.) It follows that the area F of a triangle satisfying $\angle AIO = \pi/2$ is given by

$$16F^2 \;=\; (a+b+c)(a+b-c)(b+c-a)(c+a-b) \;=\; 3a^2(3a-2c)(2c-a)\,.$$

Moreover, a must satisfy $a > |b - c|$, which is equivalent to requiring that $2a - 2c < a$ and $2c - 2a < a$. It follows that

$$\tfrac{2}{3}c \;<\; a \;<\; 2c\,.$$

Without loss of generality, we can set $c = 1$. Let $f(x) = x^2(3x - 2)(2 - x)$. The derivative of f is then

$$f'(x) \;=\; -12x\left(x - \tfrac{3+\sqrt{3}}{3}\right)\left(x - \tfrac{3-\sqrt{3}}{3}\right)\,.$$

Since $0 < \frac{3-\sqrt{3}}{3} < \frac{2}{3} < \frac{3+\sqrt{3}}{3} < 2$, we see that f reaches its maximum on $\left(\frac{2}{3}, 2\right)$ when $x = \frac{3+\sqrt{3}}{3}$. As a result, $F \le F_m$ where

$$F_m \;=\; \tfrac{\sqrt{3}}{4}\left(f\left(\tfrac{3+\sqrt{3}}{3}\right)\right)^{1/2} \;=\; \left(\tfrac{3+\sqrt{3}}{3}\right)\tfrac{\sqrt[4]{3}}{2\sqrt{2}} \;=\; \tfrac{\sqrt{9+6\sqrt{3}}}{6}\,.$$

To complete our discussion, we show that this value F_m is the area of an actual triangle ABC constructed on the side AB and satisfying $\angle AIO = \pi/2$. With $AB = 1$, let $a = \frac{3+\sqrt{3}}{3}$ and $b = 2a - 1$. Clearly, $a < b + 1$, but also $a > |b - c|$ (because $a \in \left(\frac{2}{3}, 2\right)$. Thus, we can construct a triangle ABC with sides $AB = 1$, $BC = a$, and $CA = b$. In such a triangle, $\angle AIO = \pi/2$ (because $2a = b + c$) and $F = F_m$ (because $a = \frac{3+\sqrt{3}}{3}$). This triangle therefore achieves the maximal area F_m while satisfying all the required constraints.

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; and the proposer.

A proof that $\angle AIO = \pi/2$ if and only if $2a = b+c$ (for a triangle that is not equilateral, of course) is given in [2005 : 520–521]. *Many other properties of these triangles (that is, triangles with sides in arithmetic progression) are discussed in* [2004 : 382–383] *and in the references given there. They are also the subject of a morsel in Ross Honsberger's* Mathematical Morsels *(Dolciani Mathematical Expositions No. 3, 1978, pages 209–210), which is based on problem E411* (Amer. Math. Monthly, *47:10, December, 1940, pages 708–709), where yet other properties and references are provided.*

3198. Replacement. [2007 : 41, 44] *Proposed by Michel Bataille, Rouen, France.*

Let $p = 2n+1$ be a prime, and let s be any integer such that $1 \le s \le n$. Prove that:

(a) $4^s \sum\limits_{k=0}^{n-s} \binom{2s+2k-1}{2s-1} \equiv 1 \pmod{p}$,

(b) $4^s \sum\limits_{k=0}^{n-s} \binom{2s+2k}{2s-1} \equiv -1 \pmod{p}$.

A composite of similar solutions by Richard J. McIntosh, University of Regina, Regina, SK; Joel Schlosberg, Bayside, NY, USA; and the proposer.

Motivation. Our argument comes down to proving a simple property of binomial coefficients: for a given integer r and prime p, with $0 \le r < p$,

$$\binom{r+i}{i} \equiv (-1)^i \binom{p-(r+1)}{i} \pmod{p} ,$$

for $i = 0, 1, \ldots, p-(r+1)$; in words, the i^{th} entry in the r^{th} diagonal of the Pascal Triangle is congruent to the i^{th} entry, or its negative, in the $p-(r+1)^{\text{st}}$ row. We shall use the symbol '$\equiv$' for congruences modulo p and shall assume as known the fact that, for $N \ge 1$,

$$\sum_{r \text{ even}} \binom{N}{r} = \sum_{r \text{ odd}} \binom{N}{r} = 2^{N-1} . \qquad (1)$$

(a) Let $t = \dfrac{p-(2s-1)}{2}$; that is, $2s-1 = p-2t$. Then the proposed identity becomes

$$2^{p-2t+1} \sum_{k=0}^{t-1} \binom{p-2t+2k}{p-2t} \equiv 1 .$$

Since $2^{p+1} \equiv 2^2$ and $\binom{p-2t+2k}{p-2t} = \binom{p-2t+2k}{2k}$, it suffices to prove that

$$\sum_{k=0}^{t-1} \binom{p-2t+2k}{2k} \equiv 2^{2t-2} .$$

Observe that, because $2k < p$,

$$\begin{aligned} \binom{p-2t+2k}{2k} &= \frac{(p-2t+2k)(p-2t+2k-1)\cdots(p-2t+1)}{(2k)!} \\ &\equiv \frac{(-2t+2k)(-2t+2k-1)\cdots(-2t+1)}{(2k)!} \\ &= \frac{(-1)^{2k}(2t-2k)(2t-2k+1)\cdots(2t-1)}{(2k)!} \\ &= \binom{2t-1}{2k}. \end{aligned}$$

Hence,

$$\sum_{k=0}^{t-1}\binom{p-2t+2k}{2k} \equiv \sum_{k=0}^{t-1}\binom{2t-1}{2k} = 2^{2t-2}$$

by (1), as claimed.

(b) We must prove that

$$\sum_{k=0}^{t-1}\binom{p-2t+2k+1}{2k+1} \equiv -2^{2t-2}.$$

We therefore go through the same steps as in part (a), here using the odd terms. We find that

$$\binom{p-2t+2k+1}{2k+1} \equiv -\binom{2t-1}{2k+1};$$

whence the sum turns out to be negative.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and JOSÉ H. NIETO, Universidad del Zulia, Maracaibo, Venezuela.

3199. [2006 : 516, 518] *Proposed by Michel Bataille, Rouen, France.*

Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(xy) = f\big(f(x) + f(y)\big)$ for all real numbers x and y.

Solution by Joel Schlosberg, Bayside, NY, USA.

Constant functions satisfy the given functional equation trivially. We will show that, conversely, any function which satisfies the equation is a constant.

If $f(z) = f(0)$ for some $z \neq 0$, then for any real number x, we have

$$\begin{aligned} f(x) &= f\left(z\cdot\frac{x}{z}\right) = f\left(f(z) + f\left(\frac{x}{z}\right)\right) \\ &= f\left(f(0) + f\left(\frac{x}{z}\right)\right) = f\left(0\cdot\frac{x}{z}\right) = f(0); \end{aligned}$$

thus, f is constant.

On the other hand, if we assume that $f(z) = f(0)$ only for $z = 0$, then, since $f(0) = f(0 \cdot 0) = f\big(2f(0)\big)$, we have $2f(0) = 0$ and, therefore, $f(0) = 0$. For any $x \neq 0$,

$$f(0) \;=\; f(0 \cdot x) \;=\; f\big(f(0) + f(x)\big) \;=\; f\big(f(x)\big)\,.$$

Then $f(x) = 0 = f(0)$, a contradiction.

Also solved by ROY BARBARA, Lebanese University, Fanar, Lebanon; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposer. All the solutions submitted were similar to the featured solution.

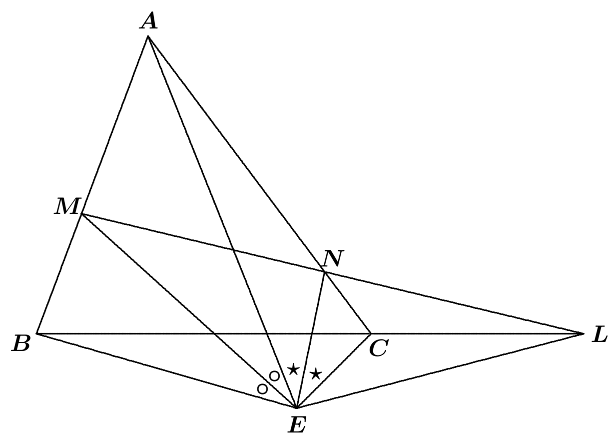
3200. [2006 : 518] *Proposed by Christopher J. Bradley, Bristol, UK.*

Let ABC be a triangle with $\angle B > \angle C$, and let E be the centre of the excircle opposite angle A. Let M and N be points on AB and AC, respectively, such that EM is the internal bisector of $\angle AEB$ and EN is the internal bisector of $\angle AEC$. If MN is extended to meet BC at L, prove that $\angle BEL + \angle CEL = 180^\circ$.

A combined solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Apostolis K. Demis, Varvakeio High School, Athens, Greece; and Joel Schlosberg, Bayside, NY, USA; modified by the editor.

We show that the claim is true under a weaker condition than stated: We only assume that the point E is on none of

1. the perpendicular bisector of the segment BC;
2. the extensions of the segment BC beyond the points B and C;
3. the segments CA and AB and their extensions.



Since EM and EN bisect angles AEB and AEC, respectively, we have

$$\frac{AM}{BM} = \frac{AE}{BE} \quad \text{and} \quad \frac{CN}{AN} = \frac{CE}{AE}.$$

By Menelaus' Theorem,

$$\frac{AM}{BM} \cdot \frac{BL}{CL} \cdot \frac{CN}{AN} = 1,$$

so that

$$\frac{AE}{BE} \cdot \frac{BL}{CL} \cdot \frac{CE}{AE} = 1,$$

or

$$\frac{BL}{CL} = \frac{BE}{CE}.$$

The Law of Sines applied to triangles BEL and CEL yields

$$\frac{BL}{\sin\angle BEL} = \frac{BE}{\sin\angle ELB} \quad \text{and} \quad \frac{CL}{\sin\angle CEL} = \frac{CE}{\sin\angle ELC}.$$

Since $\angle ELB = \angle ELC$, we obtain

$$\sin\angle BEL = \sin\angle CEL,$$

so that

$$\angle BEL = \angle CEL \quad \text{or} \quad \angle BEL + \angle CEL = 180^\circ.$$

Since point L is on the extension of BC, we see that $\angle BEL \neq \angle CEL$ and, therefore,

$$\angle BEL + \angle CEL = 180^\circ,$$

as claimed.

It is easy to show that if point E is on the perpendicular bisector of the segment BC, then the line MN is parallel to the side BC. If point E is on the extensions of the segment BC beyond the points B and C, then point L coincides with the point E.

Also solved by MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer.

YEAR END FINALE

This issue brings to a close my fifth and final year as Editor-in-Chief of ***CRUX with MAYHEM***. It's hard to believe that my term is already up! However, it is the time to bid adieu, since I have just recently retired, and my wife and I wish to spend a lot of time travelling. I will remain involved for the first 6 months of 2008 as a Co-Editor, together with VÁCLAV LINEK of the University of Winnipeg. Then Václav will become the Editor-in-Chief. (See the article at the beginning of this issue.)

Now I want to thank the many individuals who contribute so much to ***CRUX with MAYHEM***, without whose contributions the journal would certainly suffer. The first person to thank, as always, is BRUCE CROFOOT, my Associate Editor. Bruce, it has been a real honour and pleasure to have worked with you on ***CRUX with MAYHEM***! I know that the product we have put forth would not have been as good without all your input. Bruce has decided that it is time to move on to other mathematical pursuits and to spend more quality time with his young family. I wish him well, and I know that his careful scrutiny of every page of every draft will be missed.

Secondly, I wish to thank all the members of the Editorial Board, current and former, with whom I have worked over the last 5 years: JEFF HOOPER, the Mayhem Editor, who oversees all the material in the Mayhem section of each issue; IAN VANDERBURGH, who has not only been the Assistant Mayhem Editor for the past year, but has been writing one of our regular Mayhem features, the Problem of the Month; ROBERT BILINSKI, the Skoliad Editor, who is continually on the prowl for interesting high school mathematics contests; ROBERT WOODROW, for organizing the Olympiad Corner (which he has been doing since 1987); JOHN GRANT McLOUGHLIN, for ensuring that ***CRUX with MAYHEM*** has book reviews that are appropriate to our readership and for collecting those reviews in a timely manner; and BRUCE GILLIGAN, for ensuring that we have quality articles that are appropriate for a problem-solving journal. We are also blessed with a group of Problems Editors (most of whom have been long-serving) who have been a real pleasure to work with: ILIYA BLUSKOV, CHRIS FISHER, MARIA TORRES, and EDWARD WANG, each of whom have primary interests in different fields of mathematics, and who together cover a broad spectrum of the discipline. I also want to thank Maria for her translations from Spanish, as well. Since I became Editor-in-Chief, my predecessor, BRUCE SHAWYER, has remained on the ***CRUX with MAYHEM*** Editorial Board. He has helped considerably with problem moderation and has been an incredible mentor for me. I also want to thank GRAHAM WRIGHT, our Managing Editor, who keeps me meeting (almost) my deadlines for each issue. He has helped me to understand better the inner workings of the Canadian Mathematical Society and where ***CRUX with MAYHEM*** fits into that picture.

There many others who are not members of the Editorial Board, whose time and efforts are really appreciated. I want to thank MONIKA KHBEIS and ERIC ROBERT, with whom I have working closely in their roles of Mayhem problem moderators. Their work has always been well done and on time. The task of providing us with timely translations of our Problems has rested on the shoulders of JEAN-MARC TERRIER and MARTIN GOLDSTEIN. Martin has now retired and all the work is now done by Jean-Marc. I want to thank them for their efforts, and for always coming through even when I have given them very little time for turn-around. The translations are so good that we often reword the English problem after they have translated it into French, as their translation often improves the wording of the original problem. That kind of attention to detail is more than I have a right to ask for, and I am most grateful. I want to thank all the proofreaders. MOHAMMED AASSILA and BILL SANDS have been assisting the editors with this task. The quality of the work of all these people is a vital part of what makes ***CRUX with MAYHEM*** what it is. Thank you, one and all.

Thanks also go to Thompson Rivers University and my colleagues in the Department of Mathematics and Statistics for their continued understanding and support. Special thanks go to SUSAN HOWIE, secretary to our department, for all that she does to give me more time to edit. I can honestly say that I feel quite privileged to have worked for all these years among a group of individuals who truly define the word "colleague". Also, the LaTeX expertise of JOANNE CANAPE at the University of Calgary and TAO GONG at Wilfrid Laurier University is much appreciated.

Thanks to the University of Toronto Press and to Thistle Printing, and TAMI EHRLICH in particular, who continue to print a high-quality product.

The online version of ***CRUX with MAYHEM*** continues to grow, thanks in large part to JUDI BORWEIN at Dalhousie University.

Last but not least, I send my thanks to you, the readers. Without you, ***CRUX with MAYHEM*** would not be possible. I also want to tell you how much I have appreciated working with such an esteemed audience as we have for ***CRUX with MAYHEM***. I have enjoyed the relationship that we have established on both a professional level and (with many of you) on a personal level. I hope to be able to maintain the friendships established through ***CRUX with MAYHEM*** and carry them into my retired life.

Once again, I would like to remind you that, when submitting problem proposals or solutions to problems, your name and address should be on EVERY solution or proposal, and that each solution and proposal should start on a fresh sheet of paper. Otherwise, there may be filing errors, resulting in a submitted solution or proposal being lost or not being properly credited.

I wish everyone the compliments of the season and a very happy, peaceful, and prosperous 2008.

Jim Totten

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with Mathematical Mayhem

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