MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as a Mathematical Journal for and by High School and University Students. It continues, with the same emphasis, as an integral part of Crux Mathematicorum with Mathematical Mayhem.

The Mayhem Editor is Jeff Hooper (Acadia University). The Assistant Mayhem Editor is Ian VanderBurgh (University of Waterloo). The other staff members are John Grant McLoughlin (University of New Brunswick), Monika Khbeis (Ascension of Our Lord Secondary School, Mississauga), Eric Robert (Leo Hayes High School, Fredericton), Larry Rice (University of Waterloo), and Ron Lancaster (University of Toronto).

Mayhem Problems

Please send your solutions to the problems in this edition by 1 March 2008. Solutions received after this date will only be considered if there is time before publication of the solutions.

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English.

The editor thanks Jean-Marc Terrier of the University of Montreal for translations of the problems.

We falsely attributed Mayhem proposal M276 [2007 : 8] to Babis Stergiou. The proposal was actually submitted by George Apostolopoulos, Mesologi, Greece, and we apologize to George for this oversight.


Let a, b, and c be real numbers such that both a + b + c and ab + bc + ca are rational numbers, and a + b + c ≠ 0. Show that a^4 + b^4 + c^4 is a rational number if and only if the product abc is a rational number.

M313. Proposed by Babis Stergiou, Chalkida, Greece.

Two circles with centres K and L intersect at points A and B. The tangent at A to the circle centred at L meets segment KB at M and the tangent at A to the circle centred at K meets segment BL at N. Prove that AB ⊥ MN.


Let a be a real number with a > 1. Solve the following equation for x:

\[ a^{1/x}x + a^x/x = 2a. \]

Let $ABC$ be a triangle. Let $D$ be the intersection of $AB$ with the interior bisector of angle $C$, and let $E$ be the mid-point of $AB$. Show that $CD + CE < AC + BC$.

M316. Proposed by Neven Jurić, Zagreb, Croatia.

Determine the value of $\sum_{1 \leq k \leq 99} \frac{1}{k\sqrt{k} + \sqrt{k + (k + 1)\sqrt{k}}}$.

M317. Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John’s, NL.

Square $ABCD$ is inscribed in a sector of a circle of radius 1 so that there is one vertex on each radius and two vertices on the arc. The angle at the centre is $2\theta$. Determine the value of $\theta$ that results in the square of largest area.

M318. Proposed by Houda Anoun, Bordeaux, France.

Are there real numbers $x$ and $y$ such that $x^2 + xy = 3$ and $x - y^2 = 2$?


Soit $a$, $b$ et $c$ trois nombres réels tels que les sommes $a + b + c$ et $ab + bc + ca$ sont des nombres rationnels, et $a + b + c \neq 0$. Montrer qu'alors $a^4 + b^4 + c^4$ est un nombre rationnel si et seulement si le produit $abc$ est un nombre rationnel.

M313. Proposé par Babis Stergiou, Chalkida, Grèce.

Deux cercles de centres $K$ et $L$ se coupent aux points $A$ et $B$. La tangente en $A$ au cercle centré en $L$ coupe le segment $KB$ en $M$ et la tangente en $A$ au cercle centré en $K$ coupe le segment $BL$ en $N$. Montrer que $AB \perp MN$.


Soit $a$ un nombre réel avec $a > 1$. Résoudre l’équation suivante par rapport à $x$:

$$a^{1/2}x + a^2/x = 2a.$$ 


Dans un triangle $ABC$, soit $D$ le point d’intersection de $AB$ avec la bissectrice intérieure de l’angle $C$, et soit $E$ le point milieu de $AB$. Montrer que $CD + CE < AC + BC$. 

\textbf{M316. Proposé par Neven Jurčić Zagreb, Croatie.}

Trouver la valeur de \[ \sum_{1 \leq k \leq 99} \frac{1}{k\sqrt{k} + 1 + (k + 1)\sqrt{k}}. \]

\textbf{M317. Proposé par Bruce Shawyer, Université Memorial de Terre-Neuve, St. John’s, NL.}

On inscrit un carré $ABCD$ dans un secteur d’un cercle de rayon 1 de sorte qu’il ait un sommet sur chacun des rayons frontières et deux sommets sur l’arc frontière. Si l’angle au centre vaut $2\theta$, déterminer la valeur de $\theta$ qui rend l’aire du carré maximale.

\textbf{M318. Proposé par Houda Anoun, Bordeaux, France.}

Existe-t-il des nombres réels $x$ et $y$ tels que $x^2 + xy = 3$ et $x - y^2 = 2$?

\section*{Mayhem Solutions}

\textbf{M263. Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.}

Let $a$, $b$, and $n$ be integers such that $(a^2 + b^2)/5 = n$. Prove that $n = c^2 + d^2$ for some integers $c$ and $d$.

\textit{Solution by Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina, modified by the editor.}

Without loss of generality, we assume that the remainder when $a$ is divided by 5 is at least as large as the remainder when $b$ is divided by 5. Since $5 \mid (a^2 + b^2)$, the possibilities for $(a,b) \pmod{5}$ are then $(0,0)$, $(2,1)$, $(3,1)$, $(4,2)$, and $(4,3)$. We consider each of these cases in turn.

\textbf{Case 1: $(a,b) \equiv (0,0) \pmod{5}$.}

Then $a = 5k$, $b = 5t$, where $k$, $t \in \mathbb{Z}$. Therefore,

\[ n = \frac{a^2 + b^2}{5} = 5k^2 + 5t^2 = (2k + t)^2 + (k - 2t)^2, \]

from which we see that $n = c^2 + d^2$, where $c = 2k + t$ and $d = k - 2t$.

\textbf{Case 2: $(a,b) \equiv (2,1) \pmod{5}$.}

Then $a = 5k + 2$, $b = 5t + 1$, where $k$, $t \in \mathbb{Z}$. Therefore,

\[ n = \frac{a^2 + b^2}{5} = 5k^2 + 4k + 5t^2 + 2t + 1 = (2k + t + 1)^2 + (k - 2t)^2, \]

from which we see that $n = c^2 + d^2$, where $c = 2k + t + 1$ and $d = k - 2t$. 

Case 3: \((a, b) \equiv (3, 1) \pmod{5}\).
Then \(a = 5k + 3\), \(b = 5t + 1\), where \(k, t \in \mathbb{Z}\). Therefore,
\[
n = \frac{a^2 + b^2}{5} = \frac{5k^2 + 6k + 5t^2 + 2t + 2}{5} = (2k - t + 1)^2 + (2t + k + 1)^2,
\]
from which we see that \(n = c^2 + d^2\), where \(c = 2k - t + 1\) and \(d = 2t + k + 1\).

Case 4: \((a, b) \equiv (4, 2) \pmod{5}\).
Then \(a = 5k + 4\), \(b = 5t + 2\), where \(k, t \in \mathbb{Z}\). Therefore,
\[
n = \frac{a^2 + b^2}{5} = \frac{5k^2 + 8k + 5t^2 + 4t + 4}{5} = (2k + t + 2)^2 + (2t - k)^2,
\]
from which we see that \(n = c^2 + d^2\), where \(c = 2k + t + 2\) and \(d = 2t - k\).

Case 5: \((a, b) \equiv (4, 3) \pmod{5}\).
Then \(a = 5k + 4\), \(b = 5t + 3\), where \(k, t \in \mathbb{Z}\). Therefore,
\[
n = \frac{a^2 + b^2}{5} = \frac{5k^2 + 8k + 5t^2 + 6t + 5}{5} = (k + 2t + 2)^2 + (2k - t + 1)^2,
\]
from which we see that \(n = c^2 + d^2\), where \(c = k + 2t + 2\) and \(d = 2k - t + 1\).

Therefore, if \((a^2 + b^2)/5 = n\), where \(a, b, n \in \mathbb{Z}\), then \(n = c^2 + d^2\) for some \(c, d \in \mathbb{Z}\).

Also solved by ARKADY ALT, San Jose, CA, USA; HASAN DENKER, Istanbul, Turkey; and D. KIPP JOHNSON, Beaverton, OR, USA. One incomplete solution was also submitted.

Johnson’s argument involved the fact that \(n\) can be written as the sum of two squares if and only if its prime factorization contains no odd powers of primes \(q \equiv 3 \pmod{4}\). Since \(5n = a^2 + b^2\), it must be possible to write \(n\) as a sum of two squares.

**M264.** Proposed by Yakub N. Aliyev, Baku State University, Baku, Azerbaijan.

Given 1001 real numbers placed around a circle such that each number is the arithmetic mean of its neighbours or else its two neighbours are equal, prove that all the numbers are equal.

Solution by D. Kipp Johnson. Beaverton, OR, USA, modified by the editor.

Let \(x_1, x_2, \ldots, x_{1001}\) denote 1001 real numbers placed around a circle in a fixed direction. For convenience, we regard the indices 1, 2, \ldots, 1001 as integers modulo 1001 (in particular, \(x_0 = x_{1001}\) and \(x_{1002} = x_1\)).

For each \(i\), either \(2x_i = x_{i-1} + x_{i+1}\) (if \(x_i\) is the arithmetic mean of its neighbours) or else \(x_{i-1} + x_{i+1}\); in either case, \(|x_{i+1} - x_i| = |x_i - x_{i-1}|\). Consequently, if we let \(\delta = |x_2 - x_1|\), then we have \(|x_{i+1} - x_i| = \delta\) for all \(i\).

Note that \(\sum_{i=1}^{1001} (x_{i+1} - x_i) = x_{1002} - x_1 = 0\), since \(x_{1002} = x_1\). Each term of this sum is either \(\delta\) or \(-\delta\). If \(\delta \neq 0\), then the number of positive terms in the sum must equal the number of negative terms (since the sum is 0). But this is impossible, because the sum has an odd number of terms. We conclude that \(\delta = 0\).
Therefore, any two neighbouring numbers are equal, and thus, all of the numbers $x_1$, $x_2$, \ldots, $x_{1001}$ must be equal.

There were two incorrect solutions submitted.

**M265. Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.**

Given triangle $ABC$ and $DE \parallel BC$, with $D \in AB$ and $E \in AC$. Drop perpendiculars from $D$ and $E$ to $BC$, meeting $BC$ at $F$ and $K$, respectively. If $\frac{[ABC]}{[DEKF]} = \frac{32}{7}$, determine the ratio $\frac{AD}{DB}$.

*Solution by Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina.*

Let $AH$ be the altitude from $A$ to $BC$, meeting $DE$ at $L$. From $\frac{[ABC]}{[DEKF]} = \frac{32}{7}$, it follows that $\frac{1}{2} AH \cdot BC = \frac{32}{7} DE \cdot LH$; that is,

$$\frac{AH}{LH} = \frac{64}{7} \cdot \frac{DE}{BC}. \quad (1)$$

Since $DE \parallel BC$, triangles $ABC$ and $ADE$ are similar. Thus,

$$\frac{AD}{AB} = \frac{DE}{BC}. \quad (2)$$

The condition $DE \parallel BC$ also implies that

$$\frac{AH}{LH} = \frac{AB}{DB}. \quad (3)$$

Substituting (2) and (3) into (1), we obtain $\frac{AB}{DB} = \frac{64}{7} \cdot \frac{AD}{AB}$. Continuing to solve for $\frac{AB}{DB}$ leads to the following equivalent statements:

$$\frac{AB^2}{DB^2} = \frac{64}{7} \cdot \frac{AD}{DB}, \quad \frac{AD}{DB} \cdot \frac{AB}{DB} = \frac{64}{7} \cdot \frac{AD}{DB}, \quad \left(\frac{AD}{DB} \cdot \frac{AB}{DB}\right)^2 = \frac{64}{7} \cdot \frac{AD}{DB},$$

$$\left(\frac{AD}{DB} + 1\right)^2 = \frac{64}{7} \cdot \frac{AD}{DB}, \quad \left(\frac{AD}{DB}\right)^2 - \frac{50}{7} \frac{AD}{DB} + 1 = 0,$$

resulting in $\frac{AD}{DB} = \frac{1}{7}$ or $\frac{AD}{DB} = 7$.

*Also solved by COURTIS G. CHRYSSOSTOMOS, Larissa, Greece; and SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina. There were three incomplete solutions submitted.*
\textbf{M266.} Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.

A pair of two-digit numbers has the following properties:

1. The sum of the four digits is 17.
2. The sum of the two numbers is 89.
3. The product of the four digits is 210.
4. The product of the two numbers is 1924.

Determine the two numbers.

\textit{Solution by Geoffrey A. Kandall, Hamden, CT, USA.}

Since $1924 = 2^2 \cdot 13 \cdot 37$, there are exactly 12 positive divisors of 1924, namely 1, 2, 4, 13, 26, 37, 52, 74, 148, 481, 962, and 1924. Thus, the only factorizations of 1924 as the product of a pair of two-digit numbers are $26 \cdot 74$ and $37 \cdot 52$. It follows from any one of the first three properties that the numbers we seek are 37 and 52.

\textit{Remark:} If we assume only properties 2 and 4, we need not make any assumption about the digits of the numbers: if two numbers $x$ and $y$ satisfy $x + y = 89$ and $xy = 1924$, it follows that the numbers are 37 and 52.

Also solved by COURTIS G. CHRYSSOSTOMOS, Larissa, Greece; HASAN DENKER, Istanbul, Turkey; JOSE LUIS DIAZ-BARRERO, Universitat Politècnica de Catalunya, Barcelona, Spain; D. KIPP, JOHNSTON, Beaverton, OR, USA; SALEM MALIKIC, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; VEDULA N. MURALI, Dover, PA, USA; and KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India.

\textbf{M267.} Proposed by the Mayhem Staff.

Find a quintic polynomial $f(x)$ such that, if $n$ is a positive integer consisting of the digit 7 repeated $k$ times, then $f(n)$ consists of the digit 7 repeated $5k + 3$ times. (For example, $f(77) = 7777777777777$.) Compare with M256 [2006 : 265].

\textit{Solution by Arkady Alt, San Jose, CA, USA, modified by the editor.}

Let $f(x)$ be a polynomial with the desired property. If $n$ is a positive integer consisting of the digit 7 repeated $k$ times, then $n = \frac{7}{9}(10^k - 1)$. We require $f(n)$ to consist of the digit 7 repeated $5k + 3$ times; that is, $f(n) = \frac{7}{9}(10^{5k+3} - 1)$.

Since $n = \frac{7}{9}(10^k - 1)$, we have $10^k - 1 = \frac{9}{7}n$, and thus $10^k = \frac{9}{7}n + 1$. Then

$$f(n) = \frac{7}{9}(1000 \cdot (10^k)^5 - 1) = \frac{7}{9} \left( 1000 \cdot \left( \frac{9}{7}n + 1 \right)^5 - 1 \right).$$

Thus, $f(x) = \frac{7}{9} \left( 1000 \cdot \left( \frac{9}{7}x + 1 \right)^5 - 1 \right)$. 

We can write \( f(x) \) in a somewhat nicer form:

\[
\begin{align*}
 f(x) &= \frac{7000}{9} \left( \frac{9}{7} x + 1 \right)^5 - \frac{7}{9} = \frac{7000}{9} \left( \left( \frac{9}{7} x + 1 \right)^5 - 1 \right) + \frac{7000}{9} - \frac{7}{9} \\
 &= 777 + \frac{7000}{9} \left( \left( \frac{9}{7} x + 1 \right)^5 - 1 \right).
\end{align*}
\]

Also solved by CURTIS G. CHRYSSOSTOMOS, Larissa, Greece; and D. KIPP JOHNSON, Beaverton, OR, USA.

**M268. Proposed by the Mayhem Staff.**

Rectangle \( ABCD \) is inscribed in a circle \( \Gamma \) and \( P \) is a point on \( \Gamma \). Lines parallel to the sides of the rectangle are drawn through \( P \) and meet one pair of sides at points \( W \) and \( X \) and the extensions of the other pair of sides at \( Y \) and \( Z \). Prove that the line through \( W \) and \( Y \) is perpendicular to the line through \( X \) and \( Z \).

**Similar solutions by Curtis G. Chrysostomos, Larissa, Greece; Hasan Denker, Istanbul, Turkey; and Vedula N. Murty, Dover, PA, USA.**

If \( P \) is coincident with any of the four points \( A, B, C, \) or \( D \), then the statement is true.

Suppose now that \( P \notin \{ A, B, C, D \} \). Let the centre of circle \( \Gamma \) be at \((0, 0)\), and let the coordinates of the vertices of the rectangle inscribed in \( \Gamma \) be \( A(-a, b), B(a, b), C(a, -b), \) and \( D(-a, -b) \), for some \( a > 0 \) and \( b > 0 \). Let \( P \) have coordinates \((x_0, y_0)\). The coordinates of points \( W, X, Y, \) and \( Z \) are then \( W(x_0, b), X(x_0, -b), Y(-a, y_0), \) and \( Z(a, y_0) \).

The equation of \( \Gamma \) is \( x^2 + y^2 = a^2 + b^2 \). Since \( P(x_0, y_0) \) is on \( \Gamma \), we have \( x_0^2 + y_0^2 = a^2 + b^2 \); that is,

\[
\frac{x_0^2 - a^2}{y_0^2 - b^2} = -1. \tag{1}
\]

The slope of the line through \( W \) and \( Y \) is \( \frac{y_0 - b}{-a - x_0} \), and the slope of the line through \( X \) and \( Z \) is \( \frac{y_0 + b}{a - x_0} \). The product of these two slopes is

\[
\left( \frac{y_0 - b}{-a - x_0} \right) \cdot \left( \frac{y_0 + b}{a - x_0} \right) = \frac{y_0^2 - b^2}{x_0^2 - a^2} = -1,
\]

where the last step uses (1). Thus, the lines \( WY \) and \( XZ \) are perpendicular.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; CURTIS G. CHRYSSOSTOMOS, Larissa, Greece; and SALEH MALIKIC, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina.
Problem of the Month

Ian VanderBurgh

Rectangles in circles or circles in rectangles? This month, we look at both and review some geometry.

Problem 1 (2005 Fermat Contest)
In the diagram, a semicircle has diameter XY. Rectangle PQRS is inscribed in the semicircle with PQ = 12 and QR = 28. Square STUV has T on RS, U on the semicircle, and V on XY. What is the area of STUV?

Problem 2
In the diagram, the rectangle has height 4 and width 5, the circle with centre A has radius r, and the circle with centre B has radius 1. Each of the circles is tangent to two sides of the rectangle and to the other circle. Determine the value of r.

What are the important facts about circles that we need to know? What strategies should we use? These are good questions to ask (about whatever figures are in the problem) whenever attacking a problem in geometry.

There are really only three facts that we need to be able to attack either of these problems:

(i) The distance from the centre of a circle to any point on its circumference equals the radius.

(ii) If a circle is tangent to a line, the radius to the point of tangency forms a right angle with the line.

(iii) If two circles are tangent, the straight line joining their centres passes through the point of tangency and so its length is the sum of the radii of the circles.

These are good facts, and relatively intuitive. But what strategies should we use? Since it would make sense to try to take advantage of these facts, three useful things to try are:

(a) Joining the centre of a circle to useful points on the circumference.

(b) Joining the centre of a circle to points of tangency with lines.

(c) Joining the centres of mutually tangent circles.

As it turns out, the only mathematical machinery that we will need is the Pythagorean Theorem! This tends to be the case in these types of problems, even in problems that look a fair bit more complicated.
Solution to Problem 1: First, we mark the centre of the semi-circle, O, then we join O to Q, R and U. Let r be the radius of the semi-circle and s the side length of square STUV. Now, O appears to be half-way between P and S. Why is this true? Well, PQ = SR, OQ = OR and \( \angle QPO = \angle RSO = 90^\circ \); hence, \( \triangle QPO \) and \( \triangle RSO \) are congruent. Thus, \( OP = OS = \frac{1}{2} PS = 14 \).

By applying the Theorem of Pythagoras to \( \triangle QPO \), we obtain
\[
r^2 = 12^2 + 14^2 = 340.
\]
(We could at this stage calculate \( r \) itself, but as it turns out, we won’t need to.)

Next, we look at \( \triangle OUV \). This triangle is right-angled at \( V \) with \( OU = r \), \( UV = s \), and \( OV = OS + SV = 14 + s \). Therefore, by the Theorem of Pythagoras,
\[
OU^2 = UV^2 + OV^2,
\]
\[
r^2 = s^2 + (s + 14)^2,
\]
\[
340 = s^2 + s^2 + 28s + 196,
\]
\[
0 = 2s^2 + 28s - 144,
\]
\[
0 = s^2 + 14s - 72,
\]
\[
0 = (s - 4)(s + 18);
\]
thus, \( s = 4 \) or \( s = -18 \). We reject the second solution, since \( s \) must be positive. Therefore, the area of square STUV is 16.

For a complicated problem, we didn’t have to use much machinery. Often, this is the case—a couple of judicious applications of the Pythagorean Theorem and the solution of a quadratic equation or two often does the trick.

Of course, this problem was originally a multiple choice problem, and the proposers demonstrated a malicious streak by asking “The area of square STUV is closest to . . .” and giving the choices 12, 13, 14, 15, and 16. Since the area was exactly 16 (instead of “close” to 16), one could be forgiven for being somewhat concerned! (At the same time, the “niceness” of the answer might lead one to believe that the answer was actually correct.)

Let’s apply what we’ve learned to Problem 2.

Solution to Problem 2: Following our suggestions above, let’s join A and B to points of tangency and to each other, and mark in right angles where we can. At this stage, it looks like we’re stuck. But we had the hint above of trying to use the Pythagorean Theorem. To do that, we need a right-angled triangle. Do you see one? I don’t, so let’s build one, and mark in a few lengths too.
Where does this get us? First, we know that the length of $AB$ is the sum of the radii of the circles, or $r + 1$. Now let’s look at the height of the rectangle, which is 4. Breaking this into three pieces, it equals $r + PB + 1$; hence, $r + PB + 1 = 4$ or $PB = 3 - r$. (Extending $BP$ to the bottom edge of the rectangle might help you see this.) Similarly, looking at the width, $r + AP + 1 = 5$, or $AP = 4 - r$.

Using the Pythagorean Theorem,

\[
AB^2 = AP^2 + PB^2,
\]

\[
(r + 1)^2 = (4 - r)^2 + (3 - r)^2,
\]

\[
r^2 + 2r + 1 = r^2 - 8r + 16 + r^2 - 6r + 9,
\]

\[
0 = r^2 - 16r + 24.
\]

Using the quadratic formula,

\[
r = \frac{16 \pm \sqrt{16^2 - 4(1)(24)}}{2} = \frac{16 \pm \sqrt{160}}{2} = 8 \pm 2\sqrt{10}.
\]

Since the circle with radius $r$ is contained in the rectangle, the radius $r$ cannot equal $8 + 2\sqrt{10}$ (which is larger than 14). Thus, $r = 8 - 2\sqrt{10}$.

So the same types of ideas worked in both problems, which is kind of nice. These techniques are relatively straightforward and do not require a lot of sophisticated mathematical knowledge. They do often require some insight (as in the construction of the right-angled triangle in the second problem), but that’s where our problem-solving experience comes in most handy.

I leave you with a challenge problem, adapted from this year’s Hypatia Contest. We’ll look at the solution in next month’s column. While this problem seems more complicated than the ones above, it can be solved using nothing more than the ideas we’ve seen so far.

**Problem** (2007 Hypatia Contest)

In the diagram, the circles with centres $P$, $Q$ and $S$ all have radius 1. Each is tangent to two sides of the isosceles $\triangle ABC$ and to the circle with centre $R$; the circle with centre $P$ is tangent to both of the other circles of radius 1. What is the radius of the circle with centre $R$?
Pólya’s Paragon

Now You See It, Now You Don’t

Jeff Hooper

I’m sure you learned the knack of cancelling at some point. In fact, cancelling has probably become so second-nature that you do it quite without thinking about it. For instance, in a sum like \( x^2 - 3x + 3x - 1 \) or a fraction like \( \frac{49}{35} \), eliminating the \( 3x \) terms from the sum or the factor \( 7 \) from the numerator and denominator of the fraction is almost automatic.

Cancelling can be of great benefit in solving problems, but sometimes it can hide some of the structure of a problem from us. No doubt you have already encountered one way this can happen, namely, the technique of completing the square.

For example, suppose we are asked to show that the expression \( x^2 - x + \frac{1}{2} \) is positive for all real \( x \). Completing the square undoes some simplifying to show that

\[
x^2 - x + \frac{1}{2} = x^2 - 2 \left( \frac{1}{2} x \right) + \frac{1}{4} + \frac{1}{4} = \left( x - \frac{1}{2} \right)^2 + \frac{1}{4}.
\]

The right side can never be less than \( 1/4 \), since the square is non-negative. In this case, inserting some additional terms allows us to rewrite the expression in a way which is more appropriate to the problem.

**Problem 1.** Show that for any positive real numbers \( x \) and \( y \),

\[
4(x^3 + y^3) \geq (x + y)^3.
\]

**Solution 1:** It’s tempting to expand the right side here and work with the resulting expression:

\[
4(x^3 + y^3) \geq x^3 + 3x^2y + 3xy^2 + y^3.
\]

This can work, but again the resulting cancelling can sometimes eliminate too much.

We’ll take another approach. The important idea here is that the expression \( x^3 + y^3 \) actually factors into \((x + y)(x^2 - xy + y^2)\). (Check that!) Using this factorization and cancelling a factor \((x + y)\) from each side (so we’ve temporarily assumed that \( x \neq -y \)), we can rewrite the inequality as

\[
4(x^2 - xy + y^2) \geq (x + y)^2,
\]

or, after simplifying,

\[
3x^2 - 6xy + 3y^2 \geq 0.
\]

Since this is equivalent to \( 3(x - y)^2 \geq 0 \), we have reduced the inequality to something that must always hold. Thus, as in the first example, in a sense
we have unsimplified part of the expression in order to obtain our solution.
[Strictly speaking of course, to finish up this problem, we need to show that
these steps are all reversible, and take care of the assumption we introduced,
but I will leave that for you.]

Sometimes there can be a great deal of potential cancelling in an
expression, even if at first it is not obvious.

For example, consider the following sum:

$$\sum_{k=1}^{1000} \left( \frac{1}{k} - \frac{1}{k+1} \right).$$

It is tempting to simplify the expression in brackets by combining the two
terms, but that gets us nowhere quickly; although each expression reduces
to a term which is not complicated, we still need to add 1000 such terms! In
this case, it is far easier to write out the sum:

$$\left( \frac{1}{1} - \frac{1}{2} \right) + \left( \frac{1}{2} - \frac{1}{3} \right) + \left( \frac{1}{3} - \frac{1}{4} \right) + \cdots + \left( \frac{1}{999} - \frac{1}{1000} \right) + \left( \frac{1}{1000} - \frac{1}{1001} \right).$$

Now you can see that the two 1/2s cancel, as do the two 1/3s, the two 1/4s,
and so on, up to and including the two 1/1000s. The entire sum collapses
leaving only the first and last terms. So we see that

$$\sum_{k=1}^{1000} \left( \frac{1}{k} - \frac{1}{k+1} \right) = 1 - \frac{1}{1001} = \frac{1000}{1001}. $$

A sum such as this is called a telescoping sum, since this collapsing is a
little like the way the sections of a small telescope collapse into one another.
We will have a similar collapse with any sum that has the form

$$\sum_{k=1}^{n} (F(k) - F(k+1)) \quad \text{or} \quad \sum_{k=1}^{n} (F(k+1) - F(k)),$$

where $F(k)$ is some function of $k$. If we write this out in the longer form,
the negative term in one bracket cancels with the positive term in the next.

The real power of this kind of cancellation shows itself when a more
complicated expression can be rearranged into a telescoping form, as in the
next example.

**Problem 2.** Compute the sum

$$\sum_{k=1}^{2007} \frac{1}{k(k+1)}.$$

This is again a large sum, but its terms certainly do not have the form
$F(k) - F(k+1)$. Or do they? We need to put our simplification hat on
backwards here (like we did earlier) and pull this term apart. We get

$$\frac{1}{k(k+1)} = \frac{(k+1) - k}{k(k+1)} = \frac{k+1}{k(k+1)} - \frac{k}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}.$$
Now, proceeding as in our example above, we get
\[
\sum_{k=1}^{2007} \frac{1}{k(k+1)} = \sum_{k=1}^{2007} \left( \frac{1}{k} - \frac{1}{k+1} \right) = 1 - \frac{1}{2008} = \frac{2007}{2008}.
\]

A similar idea applies to products. The standard notation for products in mathematics is \( \prod_{k=1}^{n} a_k \), which represents the product \( a_1 a_2 a_3 \cdots a_n \). An expression like \( \prod_{k=1}^{20} \frac{k+1}{k} \) can be written out as
\[
\prod_{k=1}^{20} \frac{k+1}{k} = \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdots \frac{20}{19} \cdot \frac{21}{20}.
\]
(Note that the last factor in the product has \( k = 20 \) in the denominator.) This time the cancelling is even easier to see, and we wind up with the answer 21.

We will have a similar collapse with any product that has the form
\[
\prod_{k=1}^{n} \frac{F(k+1)}{F(k)} \quad \text{or} \quad \prod_{k=1}^{n} \frac{F(k)}{F(k+1)},
\]
where, as before, \( F(k) \) is some function of \( k \). If we write out the product in the longer form, the numerator in one bracket cancels with the denominator in the next (or vice versa).

In dealing with such sums and products, the main difficulty is often rearranging everything into the correct form.

I’ll close with a few problems for you to try yourself. (The last one will require at least one trigonometric identity.) You might even look through this month’s Mayhem problems too!

1. Show that for any positive integer \( n \), the value of \( n^7 - n \) must always be a multiple of 7.

2. Find the sum
\[
\frac{1}{\sqrt{1} + \sqrt{3}} + \frac{1}{\sqrt{3} + \sqrt{5}} + \frac{1}{\sqrt{5} + \sqrt{7}} + \cdots + \frac{1}{\sqrt{2005} + \sqrt{2007}}.
\]

3. Find the product
\[
\prod_{k=1}^{100} \frac{k^2 + 4k + 4}{k^2 + 3k + 2}.
\]

4. Simplify the expression
\[
\frac{\tan 1}{\cos 2} + \frac{\tan 2}{\cos 4} + \frac{\tan 4}{\cos 8} + \cdots + \frac{\tan 128}{\cos 256}.
\]