The Converse of Schiffler's Theorem

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The Schiffler point of a triangle is named after the proposer of problem 1018 [1985 : 51; 1986 : 150–152]. We have recently observed the twentieth anniversary of this notable discovery, and while the explosion of interest in the topic continues to amaze us, at least one simple aspect seems to have been overlooked. The situation is this: If P is a point in the plane of triangle ABC, but not on any of its side lines, then the Euler lines of the four triangles ABC, PBC, APC, and ABP may or may not concur. Kurt Schiffler discovered that *when P is located at the incentre of $\triangle ABC$, then the Euler lines concur at the point now bearing his name.* Let us use the notation found in Clark Kimberling's *Encyclopedia of Triangle Centers* (ETC) [3], where the incentre is denoted by X_1 and the Schiffler point by X_{21}. Schiffler's Theorem is then

$P = X_1$ implies the four Euler lines concur at X_{21}.

The converse, however, is false since there is a second valid solution that appears in ETC as X_{3065}:

The concurrence of the Euler lines at X_{21} implies that $P = X_1$ or $P = X_{3065}$.

Confirming that X_{3065} is defined by trilinear coordinates $x : y : z$, where

$$\begin{aligned} x &= \frac{1}{1+2(\cos A - \cos B - \cos C)}, \\ y &= \frac{1}{1+2(\cos B - \cos C - \cos A)}, \\ \text{and} \quad z &= \frac{1}{1+2(\cos C - \cos A - \cos B)}. \end{aligned}$$

gives impetus to further investigations. The point X_{3065} turns out to be the isogonal conjugate of X_{484} (the first Evans perspector), which we write as $X_{3065} = X_{484}^{-1}$. Bernard Gibert mentions the point X_{484}^{-1} on his website [2] as $E546$, but not in connection with X_{21} and the concurrence properties that are of interest here.

Although there is an elementary geometric proof of Schiffler's theorem in ***Crux Mathematicorum***, I have yet to find a comparable argument for the converse. However, since we know the trilinear coordinates of X_{3065} and X_{21}, we can find the equations of the Euler lines of ABX_{3065}, etc., and confirm algebraically that these Euler lines do indeed pass through X_{21}.

The point $X_{3065} = X_{484}^{-1}$ has a relatively simple construction, as indicated in Figure 1 (based on the description of X_{484} in ETC):

1. Locate the incentre I and the excentres I_A, I_B, and I_C of $\triangle ABC$.
2. Define A', B', and C' to be the reflected images of the vertices A, B, and C in the opposite sides BC, CA, and AB, respectively.
3. Define D to be the point of concurrence of lines $A'I_A$, $B'I_B$, and $C'I_C$ (concurrent specifically at point X_{484}, the first Evans perspector).
4. Define D_A, D_B, and D_C to be the reflections of D in AI, BI, and CI, respectively.
5. The required point X_{484}^{-1} is the point of concurrence of lines AD_A, BD_B, and CD_C (concurrent specifically at the isogonal conjugate of point X_{484}).

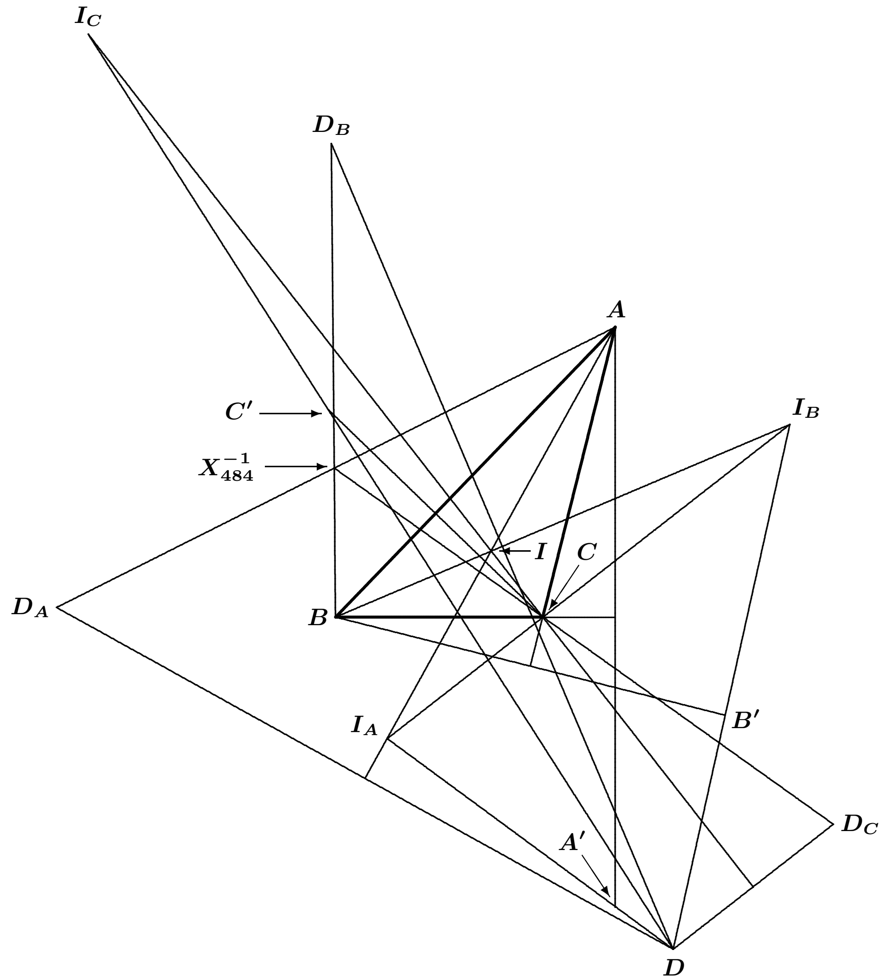

Figure 1: Construction of $X_{3065} = X_{484}^{-1}$

Not unexpectedly, our converse has a more general aspect. The investigation is guided by a theorem that appears as an exercise in [4], page 200; a proof can be found in [1], Theorem 6.1.

The locus of a point P in the plane of triangle ABC (with side lines omitted) such that the Euler lines of the four triangles ABC, PBC, APC, and ABP concur is the union of the Neuberg cubic of $\triangle ABC$ and its circumcircle.

The theorem does not, of course, give any guidance as to the location of the point of concurrence with respect to the given point P, but it does provide a good starting point for a computer investigation, assisted by the established data in ETC. Some numbered assertions follow. There is good computer evidence for them all, but I have verified only those found in the table in (3) below.

Notation. The Parry reflection point, X_{399}, lies on the Neuberg cubic (an established fact). A secant through this point, but not tangent to the cubic, will cut the cubic at two further points Z and Z', and the Euler line of $\triangle ABC$ at T, say. Let O be the circumcentre of $\triangle ABC$ and S the mid-point of OT. Then we have

(1) Conjecture: The Euler lines of the triangles ABC, ZBC, AZC, ABZ, $Z'BC$, $AZ'C$, and ABZ' concur at S.

We refer to Z and Z' as Schiffler conjugates; each position $P = Z$ or $P = Z'$ determines Euler lines that concur at the same point S.

(2) Conjecture: Every pair of Schiffler conjugates lies on a line through X_{399}.

Note that conjectures (1) and (2) combine to assert that, in general, each concurrence point S comes from precisely two positions of P on the Neuberg cubic. (Of course, if P is any point different from a vertex on the circumcircle of $\triangle ABC$ then the four resulting Euler lines all pass through the circumcentre.)

We now list some examples where Schiffler conjugates are found among centres that have been indexed in ETC (and therefore have a ready basis of attestation). Each entry amounts to an individual assertion. See Figure 2. Again we denote the isogonal conjugate of X_N by X_N^{-1}.

(3) Examples of Schiffler conjugates:

We refer to a point X_R in the table below. Its trilinear coordinates are $\alpha : \beta : \gamma$ where

$$\alpha = \frac{2U_2}{\cos B + \cos C} - U_1 \cos A\,, \qquad \beta = \frac{2U_2}{\cos C + \cos A} - U_1 \cos B\,,$$
$$\text{and} \qquad \gamma = \frac{2U_2}{\cos A + \cos B} - U_1 \cos C\,,$$

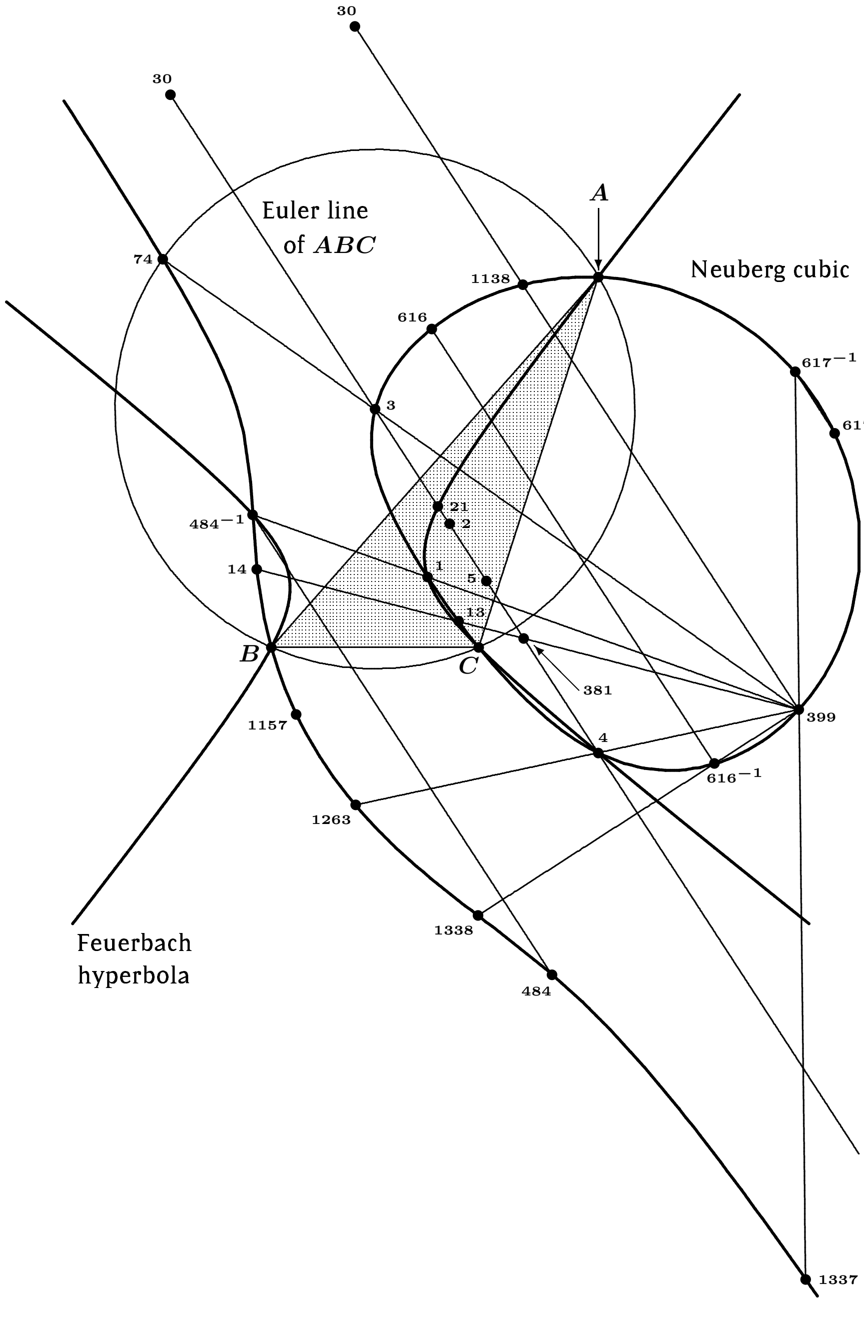

Figure 2: Neuberg cubic, Feuerbach hyperbola,
and examples of Schiffler conjugates

with

$$\begin{aligned} U_1 &= \frac{a}{\cos B + \cos C} + \frac{b}{\cos C + \cos A} + \frac{c}{\cos A + \cos B} \\ \text{and} \quad U_2 &= a\cos A + b\cos B + c\cos C\,. \end{aligned}$$

Z	Z'	S = Concurrence point on Euler line of $\triangle ABC$	T = intersection of ZZ' with Euler line of $\triangle ABC$
X_1 (incentre)	$X_{3065} = X_{484}^{-1}$	X_{21} (Schiffler pt)	X_R
X_{13} (1st Fermat pt)	X_{14} (2nd Fermat pt)	X_2 (centroid)	X_{381} (mid-point of X_2 and X_4)
X_4 (orthocentre)	X_{1263}	X_5 (9-pt centre)	X_4 (orthocentre)
X_3 (circumcentre)	$X_{74} = X_{30}^{-1}$	X_3 (circumcentre)	X_3 (circumcentre)
$X_{1138} = X_{399}^{-1}$	X_{30} (Euler infinity pt)	X_{30} (Euler infinity pt)	X_{30} (Euler infinity pt)
X_{1337} (1st Wernau pt)	X_{617}^{-1}	S_1 (Not indexed in ETC)	X_3 reflected in S_1
X_{1338} (2nd Wernau pt)	X_{616}^{-1}	S_2 (Not indexed in ETC)	X_3 reflected in S_2

Let H be the orthocentre of $\triangle ABC$. Then, with reference to claim (2) above, we now have seven notable points (A, B, C, H, Z, Z', and S). Taking three points at a time (as vertices) we can, in general, form 35 distinct triangles. Each of these has a nine-point circle, four of which will be identical (for ABC, HBC, AHC, and ABH). This leaves up to 32 distinct circles in the plane of ABC, which I will refer to as a *Schiffler Set*.

(4) Conjecture: All circles of a Schiffler Set concur at a fixed point.

(5) Some examples (again, individual assertions):

1. For the Schiffler Set $(A, B, C, H, X_1, X_{3065}, X_{21})$, the point of concurrence is X_{11}.
2. For the Schiffler Set $(A, B, C, H, X_{13}, X_{14}, X_2)$, the point of concurrence is X_{115}.
3. For the Schiffler Set $(A, B, C, H, X_{1263}, X_4, X_5)$, the point of concurrence is X_{137}. (Note that $X_4 = H$.)
4. For the Schiffler Set $(A, B, C, H, X_3, X_{74}, X_3)$, the point of concurrence is X_{125}.

We have observed that in each case all seven points lie on a conic, the centre of which, call it X_C, happens to be the point of concurrence of the Schiffler Set that they define. For example, the conic through A, B, C, H, X_1, X_{3065}, and X_{21} is the Feuerbach hyperbola with centre $X_C = X_{11}$. This suggests a more comprehensive theorem:

(6) Conjecture: If point Z lies on the Neuberg cubic of ABC, then the hyperbola through A, B, C, H, and Z also passes through Z', on the Neuberg cubic, and S, the point of concurrence of the Euler lines of triangles ABC, ZBC, AZC, ABZ, $Z'BC$, $AZ'C$, and ABZ'. The centre of the hyperbola, X_C, is the point of concurrence of the Schiffler set.

Remark. Bernard Gibert, in his study of the Neuberg cubic [2] mentions the isogonal conjugates $X_{616}^{-1} = E558$ and $X_{617}^{-1} = E559$, but not in the above context.

Finally, there are elementary proofs for some of the entries in the table in (3). Here is a proof that $P = X_{13}$ and $P = X_{14}$ both determine Euler lines that intersect in the centroid.

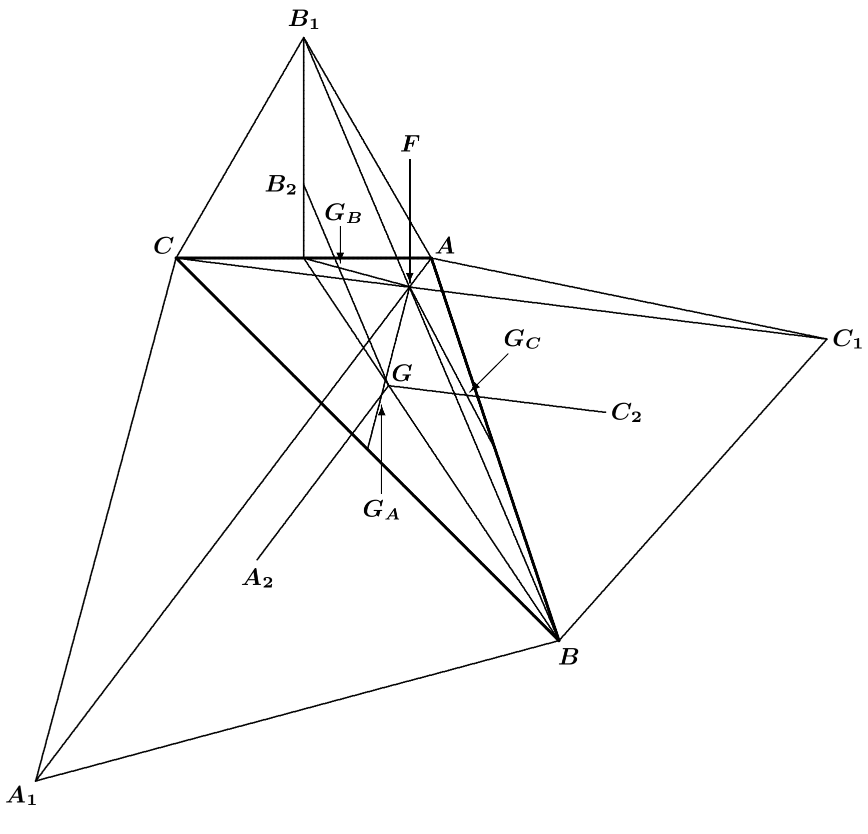

Figure 3: $F = X_{13}$ determines the Euler lines that meet at the centroid G

The first Fermat point $F = X_{13}$ is the intersection of the lines joining the vertices of $\triangle ABC$ to the remote vertices A_1, B_1, and C_1 of equilateral triangles erected externally on the opposite sides (see Figure 3). The centroids B_2 of $\triangle AB_1C$, G_B of $\triangle FAC$, and G of $\triangle BAC$ are each one third of the way from the mid-point of AC to the opposite vertices, which implies that B_2G_B and B_2G are parallel to B_1B (which contains F by definition). That is, G_B lies on B_2G. But B_2 is the circumcentre of $\triangle AFC$; thus, B_2G is the Euler line of $\triangle AFC$. Similarly, C_2G and A_2G are the Euler lines of triangles ABF and BCF. Therefore, point F effects the concurrence of the Euler lines at the centroid of $\triangle ABC$.

The same argument shows that the second Fermat point X_{14}, constructed using equilateral triangles erected internally on the sides of $\triangle ABC$, likewise determines Euler lines that meet at G.

The proof that $P = X_3$ and $P = X_{74}$ both determine Euler lines that intersect in the circumcentre X_3 is left to the reader. It is less complicated than the above proof.

References

[1] Zvonco Čerin, The Neuberg cubic in locus problems, *Math. Pannonica* **11** (2000), 109–124.

[2] Bernard Gibert, Neuberg cubic web pages

http://perso.orange.fr/bernard.gibert/Exemples/k001.html
http://perso.orange.fr/bernard.gibert/Tables/table19.html.

[3] Clark Kimberling, *Encyclopedia of Triangle Centers*.

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[4] F. Morley and F.V. Morley, *Inversive Geometry*, Oxford University Press (1931).

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