Problem of the Month

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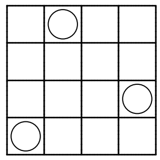
This month's problem involves probability.



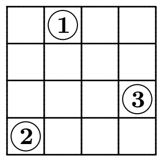
Problem (2007 Euclid Contest)

In the 4×4 grid shown, three coins are randomly placed in different squares. Determine the probability that no two coins lie in the same row or column.

As someone once suggested to me, many probability problems are just combinatorial (counting) problems where you divide by the size of the sample space whenever you want to get a probability. This problem, in particular, boils down to counting the possibilities correctly.

Before actually solving the problem, let's consider how to count the ways of placing the coins on the grid. For the moment, we will require only that no two coins be placed on the same square, without worrying about whether they are in different rows or columns. The number of ways of placing the coins depends on whether the coins are considered to be *distinguishable*. That is, can we tell them apart or are they identical?

First suppose the coins are distinguishable. We will use numbers to refer to them. Coin 1 is the one that is placed first on the grid, followed by coin 2, then coin 3. Coin 1 may be placed anywhere, which means there are 16 possible squares for it. For each of these placements of coin 1, there are 15 open squares remaining in which coin 2 may be placed, giving $16 \cdot 15$ ways of placing the first two coins. For each of these ways, there are 14 squares in which coin 3 may be placed, giving $16 \cdot 15 \cdot 14$ ways of placing all three coins. The figure at right shows one way.

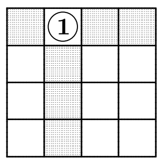


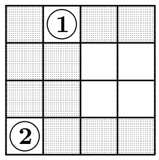
Now suppose the coins are indistinguishable. In this case, they have no numbers. What we want to count now is the number of *configurations* of the coins once they have all been placed on the grid, without regard for the order in which they are placed. Since there are 3 coins and 16 squares, the number of possible configurations is $\binom{16}{3} = \frac{16(15)(14)}{3!} = 560$. This is just our answer for the case where the coins are distinguishable divided by $3!$, the number of ways of rearranging the coins among themselves.

Now let's solve the problem.

Solution 1: Assume the coins are distinguishable. In how many ways can they be placed on the grid so that no two coins are in the same row or column?

There are 16 possible squares for coin 1. Once it has been placed, there are 3 rows and 3 columns remaining that do not contain coin 1, giving $3 \cdot 3 = 9$ squares where coin 2 may be put (the 9 white squares in the figure at right). Thus, there are $16 \cdot 9$ ways of placing the first two coins in different rows and columns.





Once the first two coins have been placed, there are 2 rows and 2 columns remaining that do not contain a coin. Thus, there are $2 \cdot 2 = 4$ squares in which coin 3 may be placed (these are the 4 white squares in the figure at right.) Altogether, there are $16 \cdot 9 \cdot 4$ ways of putting the three coins in different rows and columns.

Since the total number of ways of placing the three coins on the grid is $16 \cdot 15 \cdot 14$ (as we saw earlier), the probability that the three coins are placed in different rows and columns is $\frac{16 \cdot 9 \cdot 4}{16 \cdot 15 \cdot 14} = \frac{6}{35}$.

We can actually compute probabilities at each stage of the calculation instead of waiting until the end. The probability of placing the first two coins in different rows and columns is $\frac{16 \cdot 9}{16 \cdot 15} = \frac{3}{5}$. Given that the first two coins are in different rows and columns, the probability of placing the third coin in a different row and column from each of the first two is $\frac{4}{14} = \frac{2}{7}$ (since there are 4 acceptable squares out of 14 open squares). The probability of placing all three coins in different rows and columns is then $\frac{3}{5} \cdot \frac{2}{7} = \frac{6}{35}$.

Solution 2: This time, we regard the coins as indistinguishable. We will find the number of configurations for the coins in which the coins are in different rows and columns. This could be done by using our counting method for the case where the coins are distinguishable and then dividing by $3!$, but here is a way to count the configurations directly instead.

First, pick the 3 rows in which the coins will be put. There are $\binom{4}{3} = 4$ ways to do this. In the topmost row of these 3 rows, there are 4 possible squares for a coin. In the middle row of these rows, there are 3 possible squares for a coin (since it can't be in the same column as the coin in the topmost row). In the bottom row of these rows, there are 2 possible squares for a coin (since it can't be in the same column as either of the other two coins). Thus, there are $4 \cdot 4 \cdot 3 \cdot 2 = 96$ configurations in which no two coins are in the same row or column.

Finally, the required probability is $\frac{96}{560} = \frac{6}{35}$.

This problem has an interesting history. The initial version asked the same question for 3 coins on a 5×5 grid. (Can you solve this version?) During the development of the 2007 Euclid Contest, the problem was changed to the following problem before being changed back to its original form:

Three *different* numbers are chosen from the set

$$\{11, 12, 13, 14, 21, 22, 23, 24, 31, 32, 33, 34, 41, 42, 43, 44\}.$$

What is the probability that no two of these numbers have the same units digit or the same tens digit?

This problem seemed quite a lot harder than the problem with the coins, which is strange, as it is actually the same problem! Can you see why?

You might like to try solving a more general problem where k coins are placed on an $n \times n$ grid (with $k \leq n$, of course).

Note. The author wishes to acknowledge the contributions of Bruce Crofoot in the preparation of this column.