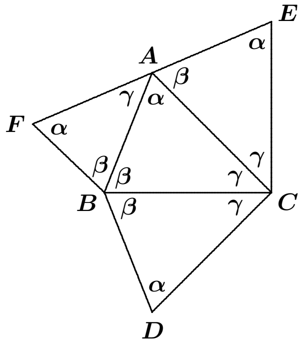
Mayhem Solutions

Some readers pointed out that the solution to M225 which appeared in [2006 : 496–7] *was incorrect. We apologize for this. At this point, we do not have a solution to the problem.*



M251. *Proposed by K.R.S. Sastry, Bangalore, India.*

Let α, β, γ be the angle measures at angles A, B, C, respectively, in $\triangle ABC$. On the sides of $\triangle ABC$, externally, are triangles DBC, EAC, and FBA as in the diagram.

Prove that $AD = EF$ if and only if $\alpha = \pi/2$.

Combination of solutions by Hasan Denker, Istanbul, Turkey; and Jean-David Houle, student, McGill University, Montreal, QC.

It can be seen that triangles ABC and DBC are congruent. From this we conclude that $AB = DB$ and $AC = DC$. Let AG and DG be the altitudes to BC in triangles ABC and DBC, respectively. We can then further conclude that $AG = AB\sin\beta$ and $GD = CD\sin\gamma = AC\sin\gamma$. Consequently,

$$AD \;=\; AG + GD \;=\; AB\sin\beta + AC\sin\gamma\,. \qquad (1)$$

In $\triangle ACE$ we have $AC/\sin\alpha = EA/\sin\gamma$, from which it follows that $EA = AC\sin\gamma/\sin\alpha$. Similarly, in $\triangle AFB$ we have $AF = AB\sin\beta/\sin\alpha$. Since $\alpha+\beta+\gamma=\pi$, we have $EF = EA + AF$, and we obtain

$$EF \;=\; \frac{AC\sin\gamma}{\sin\alpha} + \frac{AB\sin\beta}{\sin\alpha}\,.$$

Using equation (1), we conclude that $EF = AD/\sin\alpha$.

Now, if $\alpha = \pi/2$, then $EF = AD$. Conversely, if $EF = AD$, then $\sin\alpha = 1$, which gives $\alpha = \pi/2$ (since $0 < \alpha < \pi$). Thus, $EF = AD$ if and only if $\alpha = \pi/2$.

Also solved by HOUDA ANOUN, Bordeaux, France; COURTIS G. CHRYSSOSTOMOS and BOTIS A. JIANNHS, Larissa, Greece; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; VEDULA N. MURTY, Dover, PA, USA; and ALEX REMOROV, student, William Lyon Mackenzie Collegiate Institute, Toronto, ON. There was one incorrect solution submitted.

M252. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let x, y, z be positive real numbers. Prove that

$$\left(\frac{x}{y}+\frac{z}{\sqrt[3]{xyz}}\right)^2+\left(\frac{y}{z}+\frac{x}{\sqrt[3]{xyz}}\right)^2+\left(\frac{z}{x}+\frac{y}{\sqrt[3]{xyz}}\right)^2 \;\geq\; 12\,.$$

Essentially the same solution by Mohammed Aassila, Strasbourg, France; Jean-David Houle, student, McGill University, Montreal, QC; D. Kipp Johnson, Beaverton, OR, USA; Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; Phil McCartney, Northern Kentucky University, Highland Heights, KY, USA; Alex Remorov, student, William Lyon Mackenzie Collegiate Institute, Toronto, ON; and Panos E. Tsaoussoglou, Athens, Greece.

Applying the AM–GM Inequality, we obtain the following inequalities:

$$\begin{aligned} \frac{x}{y}+\frac{z}{\sqrt[3]{xyz}} &\geq 2\sqrt{\frac{xz}{y\sqrt[3]{xyz}}}\,, \\ \frac{y}{z}+\frac{x}{\sqrt[3]{xyz}} &\geq 2\sqrt{\frac{yx}{z\sqrt[3]{xyz}}}\,, \\ \frac{z}{x}+\frac{y}{\sqrt[3]{xyz}} &\geq 2\sqrt{\frac{zy}{x\sqrt[3]{xyz}}}\,. \end{aligned}$$

We can then conclude that

$$\begin{aligned} &\left(\frac{x}{y}+\frac{z}{\sqrt[3]{xyz}}\right)^2+\left(\frac{y}{z}+\frac{x}{\sqrt[3]{xyz}}\right)^2+\left(\frac{z}{x}+\frac{y}{\sqrt[3]{xyz}}\right)^2 \\ &\qquad\geq 4\left(\frac{xz}{y\sqrt[3]{xyz}}+\frac{yx}{z\sqrt[3]{xyz}}+\frac{zy}{x\sqrt[3]{xyz}}\right)\,. \end{aligned}$$

Applying the AM–GM Inequality to the right side of this last inequality, we obtain

$$4\left(\frac{xz}{y\sqrt[3]{xyz}}+\frac{yx}{z\sqrt[3]{xyz}}+\frac{zy}{x\sqrt[3]{xyz}}\right) \geq 4\cdot 3\sqrt[3]{\frac{xz}{y\sqrt[3]{xyz}}\cdot\frac{yx}{z\sqrt[3]{xyz}}\cdot\frac{zy}{x\sqrt[3]{xyz}}} = 12\,.$$

Thus,

$$\left(\frac{x}{y}+\frac{z}{\sqrt[3]{xyz}}\right)^2+\left(\frac{y}{z}+\frac{x}{\sqrt[3]{xyz}}\right)^2+\left(\frac{z}{x}+\frac{y}{\sqrt[3]{xyz}}\right)^2 \geq 12\,.$$

Note that equality holds if and only if $x = y = z$.

Also solved by ZAFAR AHMED, BARC, Mumbai, India; ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; ROY BARBARA, Lebanese University, Fanar, Lebanon; MIHÁLY BENCZE, Brasov, Romania; QUANG CAO MINH, Nguyen Binh Khiem High School, Vinh Long, Vietnam; SHI CHANGWEI, Xi'an City, Shaan Xi Province, China; COURTIS G. CHRYSSOSTOMOS, Larissa, Greece; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie, AB; JOE HOWARD, Portales, NM, USA; KEE-WAI LAU, Hong Kong, China; MATTI LEHTINEN, National Defence College, Helsinki, Finland; VEDULA N. MURTY, Dover, PA, USA; JOEL SCHLOSBERG, Bayside, NY, USA; and KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India.

Bencze actually outlined a solution for a more general problem: If $x_1, x_2, \ldots, x_n$ are positive real numbers, then we have

$$\sum_{\text{cyclic}}\left(\frac{x_1}{x_2}+\frac{x_3}{\sqrt[n]{x_1x_2\cdots x_n}}\right)^{\alpha} \geq 2^{\alpha}n$$

for all $\alpha \in (-\infty,0)\cup(1,\infty)$. Of course, the current problem is the case $n = 3$ and $\alpha = 2$.

M253. *Proposed by Fabio Zucca, Politecnico di Milano, Milano, Italy.*

Consider the set of lattice points $\{(x, y)\}$ where x and y are integers such that $0 \le x \le 7$ and $0 \le y \le 7$. Two points are selected at random from this set. All points have the same probability of being selected and the points need not be distinct. Find the probability that the area of the triangle (possibly degenerate) formed by these two points and the point $(0, 0)$ is an integer (possibly 0).

Solution by Alex Remorov, student, William Lyon Mackenzie Collegiate Institute, Toronto, ON.

Let us denote the area of a triangle XYZ by $[XYZ]$. Recall that for a triangle with vertices $A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$,

$$\begin{aligned}[ABC] &= \frac{1}{2}\left|\det\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1\end{bmatrix}\right| \\ &= \frac{1}{2}\left|x_1y_2 + x_2y_3 + x_3y_1 - x_2y_1 - x_3y_2 - x_1y_3\right| .\end{aligned}$$

For the purpose of our problem, let the two points selected at random be $P(a, b)$ and $Q(c, d)$. If O is the origin, then $[PQO] = \frac{1}{2}|ad - bc|$. In order for $[PQO]$ to be an integer, $|ad - bc|$ must be even, and therefore, ad and bc must have the same parity. This will occur in the following two cases:

Case I. ad and bc are both odd.

This is true if and only if a, b, c, and d are all odd. Since a, b, c, and d belong to the set $\{0, 1, 2, 3, 4, 5, 6, 7\}$, the probability that they are all odd is $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{16}$.

Case II. ad and bc are both even.

This is true if and only if a and d are not both odd and b and c are not both odd. The probability that a and d are not both odd is $\frac{3}{4}$, and the same is true for b and c. Therefore, the probability that ad and bc are both even is $\frac{3}{4} \cdot \frac{3}{4} = \frac{9}{16}$.

Since Case I and Case II are mutually exclusive, the probability that the area of the triangle is an integer is $\frac{1}{16} + \frac{9}{16} = \frac{5}{8}$.

Also solved by HASAN DENKER, Istanbul, Turkey; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and D. KIPP JOHNSON, Beaverton, OR, USA. One incorrect solution was also submitted.

M254. *Proposé par Edward T.H. Wang, Université Wilfrid Laurier, Waterloo, ON.*

Evaluer la somme $S_{2006} = \sum_{k=1}^{2006} (-1)^k \frac{k^2 - 3}{(k + 1)!}$. [On rappelle que $n! = n \cdot (n - 1) \cdots 3 \cdot 2 \cdot 1$; par exemple, $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$.]

Solution par Jean-David Houle, étudiant, Université McGill, Montréal, QC.

Pour $k \geq 1$, nous savons que

$$\begin{aligned} \frac{k^2-3}{(k+1)!} &= \frac{k(k+1)}{(k+1)!} - \frac{k+1}{(k+1)!} - \frac{2}{(k+1)!} = \frac{1}{(k-1)!} - \frac{1}{k!} - \frac{2}{(k+1)!} \\ &= \left(\frac{1}{(k-1)!} - \frac{1}{(k+1)!}\right) - \left(\frac{1}{k!} + \frac{1}{(k+1)!}\right). \end{aligned}$$

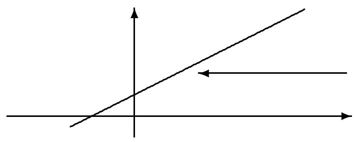
Par manipulations algébriques, on obtient :

$$\begin{aligned} S_{2006} &= \sum_{k=1}^{2006} (-1)^k \frac{k^2-3}{(k+1)!} \\ &= \left(\sum_{k=1}^{2006} \frac{(-1)^k}{(k-1)!} - \sum_{k=1}^{2006} \frac{(-1)^k}{(k+1)!}\right) - \left(\sum_{k=1}^{2006} \frac{(-1)^k}{k!} + \sum_{k=1}^{2006} \frac{(-1)^k}{(k+1)!}\right) \\ &= \left(-1+1+\sum_{k=3}^{2006} \frac{(-1)^k}{(k-1)!} - \sum_{k=1}^{2004} \frac{(-1)^k}{(k+1)!} + \frac{1}{2006!} - \frac{1}{2007!}\right) \\ &\qquad - \left(-1+\sum_{k=2}^{2006} \frac{(-1)^k}{k!} - \sum_{k=1}^{2005} \frac{(-1)^{k+1}}{(k+1)!} + \frac{1}{2007!}\right) \\ &= \left(\frac{1}{2006!} - \frac{1}{2007!}\right) - \left(-1 + \frac{1}{2007!}\right) \\ &= \frac{1}{2006!} - \frac{2}{2007!} + 1 = \frac{2007-2}{2007!} + 1 = \frac{2005}{2007!} + 1\,. \end{aligned}$$

Autres solutions soumises par RICHARD I. HESS, Rancho Palos Verdes, CA, USA ; et VEDULA N. MURTY, Dover, PA, USA. Deux solutions incorrectes ont aussi été soumises.

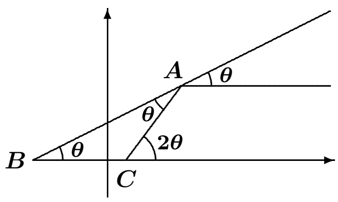
M255. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

The line with slope $\lambda > 0$ acts like a mirror to a ray of light coming along a line parallel to the x–axis. Determine the slope of the reflected ray.



Solution by D. Kipp Johnson, Beaverton, OR, USA.

Let the ray of light hit the mirror at point A, let the x-intercept of the mirror be B, and let the reflected ray of light hit the x-axis at C. If the acute angle formed by the mirror and a horizontal line is θ, then $\lambda = \tan\theta$. Since the angle of incidence and the angle of reflection are equal, we have



$\angle BAC = \angle ABC = \theta$. Then the exterior angle of $\triangle ABC$ at vertex C has measure 2θ, and the slope of the reflected ray is thus

$$\tan 2\theta \;=\; \frac{2\tan\theta}{1-\tan^2\theta} \;=\; \frac{2\lambda}{1-\lambda^2}\,.$$

(If the slope of the mirror is 1, then the reflected ray has undefined slope since it is vertical.)

Also solved by COURTIS G. CHRYSSOSTOMOS, Larissa, Greece; HASAN DENKER, Istanbul, Turkey; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JEAN-DAVID HOULE, student, McGill University, Montreal, QC; and ALEX REMOROV, student, William Lyon Mackenzie Collegiate Institute, Toronto, ON.

M256. *Proposed by the Mayhem Staff.*

Find a quadratic polynomial $f(x)$ such that, if n is a positive integer consisting of the digit 5 repeated k times, then $f(n)$ consists of the digit 5 repeated $2k$ times. (For example, $f(555) = 555555$.)

Solution by Morgan Lynch and Lacey K. Moore, Angelo State University, San Angelo, Texas, USA.

Let $f(x) = ax^2 + bx + c$. Using the given information we obtain the following system of linear equations:

$$\begin{aligned} f(5) &= 25a + 5b + c &= 55\,, \\ f(55) &= 3025a + 55b + c &= 5555\,, \\ f(555) &= 308025a + 555b + c &= 555555\,. \end{aligned}$$

Solving this system, we determine that, $a = \frac{9}{5}$, $b = 2$, and $c = 0$. Thus,

$$f(x) \;=\; \tfrac{9}{5}x^2 + 2x \;=\; x\big(\tfrac{9}{5}x + 2\big)\,.$$

Note that:

$$\begin{aligned} \underbrace{55\cdots55}_{k \text{ times}} &= 5\big(1 + 10^1 + 10^2 + \cdots + 10^{k-1}\big) \;=\; 5\left(\frac{10^k - 1}{9}\right) \\ &= \tfrac{5}{9}\big(10^k - 1\big)\,. \end{aligned}$$

We can now verify our equation:

$$\begin{aligned} f\left(\tfrac{5}{9}(10^k - 1)\right) &= \left(\tfrac{5}{9}(10^k - 1)\right)\left[\tfrac{9}{5}\left(\tfrac{5}{9}(10^k - 1)\right) + 2\right] \\ &= \tfrac{5}{9}(10^k - 1)(10^k + 1) \;=\; \tfrac{5}{9}(10^{2k} - 1) \;=\; \underbrace{55\cdots55}_{2k \text{ times}}\,. \end{aligned}$$

Also solved by HOUDA ANOUN, Bordeaux, France; COURTIS G. CHRYSSOSTOMOS, Larissa, Greece; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; AMY HOLLINGER and CORRIE MEYER, Southeast Missouri State University in Cape Girardeau, Missouri, USA; JEAN-DAVID HOULE, student, McGill University, Montreal, QC; D. KIPP JOHNSON, Beaverton, OR, USA; GUSTAVO KRIMKER, Universidad CAECE, Buenos Aires, Argentina; and ALEX REMOROV, student, William Lyon Mackenzie Collegiate Institute, Toronto, ON. One incorrect solution was also submitted.