MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

Mayhem Problems

Please send your solutions to the problems in this edition by **1 January 2008**. *Solutions received after this date will only be considered if there is time before publication of the solutions.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English.

The editor thanks Jean-Marc Terrier and Martin Goldstein of the University of Montreal for translations of the problems.

M301. *Proposed by D.E. Prithwijit, University College Cork, Republic of Ireland.*

The general term of a sequence is $t_n = n^2 + 20$, for $n \geq 1$. Show that for all $n \geq 1$, the greatest common divisor of t_n and t_{n+1} must be a divisor of 81.

M302. *Proposed by Babis Stergiou, Chalkida, Greece.*

A triangle ABC has $\angle ABC = \angle ACB = 40^\circ$. If P is a point in the interior of the triangle such that $\angle PBC = 20^\circ$ and $\angle PCB = 30^\circ$, prove that $BP = BA$.

M303. *Proposed by Neven Jurič, Zagreb, Croatia.*

A curious relation among squares states that the sum of $n+1$ consecutive squares, beginning with the square of $n(2n+1)$, is equal to the sum of the squares of the next n consecutive integers. (For example, when $n = 1$ we have $3^2 + 4^2 = 5^2$, and when $n = 2$ we have $10^2 + 11^2 + 12^2 = 13^2 + 14^2$.) Show that this property holds for any $n \geq 1$.

M304. *Proposed by Mihály Bencze, Brasov, Romania.*

Let a, b, and c be real numbers such that both $a+b+c$ and $ab+bc+ca$ are rational numbers. Show that $a^4+b^4+c^4$ is a rational number if and only if the product abc is a rational number.

M305. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Find all real solutions to the following system of equations:

$$\begin{aligned} \sqrt{x}+\sqrt{y}+\sqrt{z} &= 3\,, \\ x\sqrt{x}+y\sqrt{y}+z\sqrt{z} &= 3\,, \\ x^2\sqrt{x}+y^2\sqrt{y}+z^2\sqrt{z} &= 3\,. \end{aligned}$$

M306. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Find all solutions to the following addition problem, in which each letter represents a distinct digit:

			T	E	N
			T	E	N
		N	I	N	E
	E	I	G	H	T
+	T	H	R	E	E
	F	O	R	T	Y

M301. *Proposé par D.E. Prithwijit, University College Cork, République d'Irlande.*

Soit $t_n = n^2+20$, pour $n \geq 1$, le terme général d'une suite. Montrer que pour tout $n \geq 1$, le plus grand commun diviseur de t_n et t_{n+1} doit être un diviseur de 81.

M302. *Proposé par Babis Stergiou, Chalkida, Grèce.*

Soit ABC un triangle avec $\angle ABC = \angle ACB = 40^\circ$. Si P est un point à l'intérieur du triangle de sorte que $\angle PBC = 20^\circ$ et $\angle PCB = 30^\circ$, montrer que $BP = BA$.

M303. *Proposé par Neven Jurič, Zagreb, Croatie.*

Une curieuse relation entre les carrés de nombres naturels montre que la somme de $n+1$ carrés consécutifs, commençant par le carré de $n(2n+1)$, est égale à la somme des carrés des n entiers consécutifs suivants. (Par exemple, si $n=1$ on a $3^2+4^2=5^2$, et si $n=2$ on a $10^2+11^2+12^2=13^2+14^2$.) Montrer que cette propriété est vraie pour tout $n\geq 1$.

M304. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit a, b et c trois nombres réels tels que les sommes $a+b+c$ et $ab+bc+ca$ sont des nombres rationnels. Montrer qu'alors $a^4+b^4+c^4$ est un nombre rationnel si et seulement si le product abc est un nombre rationnel.

M305. *Proposé par José Luis Díaz-Barrero, Université Polytechnique de Catalogne, Barcelone, Espagne.*

Trouver toutes les solutions réelles du système d'équations suivant :

$$\begin{aligned} \sqrt{x}+\sqrt{y}+\sqrt{z} &= 3\,, \\ x\sqrt{x}+y\sqrt{y}+z\sqrt{z} &= 3\,, \\ x^2\sqrt{x}+y^2\sqrt{y}+z^2\sqrt{z} &= 3\,. \end{aligned}$$

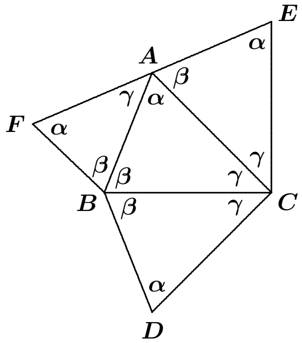
M306. *Proposé par Bruce Shawyer, Université Memorial de Terre-Neuve, St. John's, NL.*

Trouver toutes les solutions du problème d'addition suivant, dans lequel chaque lettre représente un chiffre distinct :

$$\begin{array}{cccccc} & & & \mathrm{T} & \mathrm{E} & \mathrm{N} \\ & & & \mathrm{T} & \mathrm{E} & \mathrm{N} \\ & & \mathrm{N} & \mathrm{I} & \mathrm{N} & \mathrm{E} \\ & \mathrm{E} & \mathrm{I} & \mathrm{G} & \mathrm{H} & \mathrm{T} \\ + & \mathrm{T} & \mathrm{H} & \mathrm{R} & \mathrm{E} & \mathrm{E} \\ \hline & \mathrm{F} & \mathrm{O} & \mathrm{R} & \mathrm{T} & \mathrm{Y} \end{array}$$

Mayhem Solutions

Some readers pointed out that the solution to M225 which appeared in [2006 : 496–7] *was incorrect. We apologize for this. At this point, we do not have a solution to the problem.*



M251. *Proposed by K.R.S. Sastry, Bangalore, India.*

Let α, β, γ be the angle measures at angles A, B, C, respectively, in $\triangle ABC$. On the sides of $\triangle ABC$, externally, are triangles DBC, EAC, and FBA as in the diagram.

Prove that $AD = EF$ if and only if $\alpha = \pi/2$.

Combination of solutions by Hasan Denker, Istanbul, Turkey; and Jean-David Houle, student, McGill University, Montreal, QC.

It can be seen that triangles ABC and DBC are congruent. From this we conclude that $AB = DB$ and $AC = DC$. Let AG and DG be the altitudes to BC in triangles ABC and DBC, respectively. We can then further conclude that $AG = AB\sin\beta$ and $GD = CD\sin\gamma = AC\sin\gamma$. Consequently,

$$AD \;=\; AG + GD \;=\; AB\sin\beta + AC\sin\gamma\,. \qquad (1)$$

In $\triangle ACE$ we have $AC/\sin\alpha = EA/\sin\gamma$, from which it follows that $EA = AC\sin\gamma/\sin\alpha$. Similarly, in $\triangle AFB$ we have $AF = AB\sin\beta/\sin\alpha$. Since $\alpha+\beta+\gamma = \pi$, we have $EF = EA + AF$, and we obtain

$$EF \;=\; \frac{AC\sin\gamma}{\sin\alpha} + \frac{AB\sin\beta}{\sin\alpha}\,.$$

Using equation (1), we conclude that $EF = AD/\sin\alpha$.

Now, if $\alpha = \pi/2$, then $EF = AD$. Conversely, if $EF = AD$, then $\sin\alpha = 1$, which gives $\alpha = \pi/2$ (since $0 < \alpha < \pi$). Thus, $EF = AD$ if and only if $\alpha = \pi/2$.

Also solved by HOUDA ANOUN, Bordeaux, France; COURTIS G. CHRYSSOSTOMOS and BOTIS A. JIANNHS, Larissa, Greece; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; VEDULA N. MURTY, Dover, PA, USA; and ALEX REMOROV, student, William Lyon Mackenzie Collegiate Institute, Toronto, ON. There was one incorrect solution submitted.

M252. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let x, y, z be positive real numbers. Prove that

$$\left(\frac{x}{y} + \frac{z}{\sqrt[3]{xyz}}\right)^2 + \left(\frac{y}{z} + \frac{x}{\sqrt[3]{xyz}}\right)^2 + \left(\frac{z}{x} + \frac{y}{\sqrt[3]{xyz}}\right)^2 \;\geq\; 12\,.$$

Essentially the same solution by Mohammed Aassila, Strasbourg, France; Jean-David Houle, student, McGill University, Montreal, QC; D. Kipp Johnson, Beaverton, OR, USA; Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; Phil McCartney, Northern Kentucky University, Highland Heights, KY, USA; Alex Remorov, student, William Lyon Mackenzie Collegiate Institute, Toronto, ON; and Panos E. Tsaoussoglou, Athens, Greece.

Applying the AM–GM Inequality, we obtain the following inequalities:

$$\begin{aligned}\frac{x}{y}+\frac{z}{\sqrt[3]{xyz}} &\geq 2\sqrt{\frac{xz}{y\sqrt[3]{xyz}}}\,,\\ \frac{y}{z}+\frac{x}{\sqrt[3]{xyz}} &\geq 2\sqrt{\frac{yx}{z\sqrt[3]{xyz}}}\,,\\ \frac{z}{x}+\frac{y}{\sqrt[3]{xyz}} &\geq 2\sqrt{\frac{zy}{x\sqrt[3]{xyz}}}\,.\end{aligned}$$

We can then conclude that

$$\begin{aligned}&\left(\frac{x}{y}+\frac{z}{\sqrt[3]{xyz}}\right)^2+\left(\frac{y}{z}+\frac{x}{\sqrt[3]{xyz}}\right)^2+\left(\frac{z}{x}+\frac{y}{\sqrt[3]{xyz}}\right)^2\\ &\qquad\geq 4\left(\frac{xz}{y\sqrt[3]{xyz}}+\frac{yx}{z\sqrt[3]{xyz}}+\frac{zy}{x\sqrt[3]{xyz}}\right)\,.\end{aligned}$$

Applying the AM–GM Inequality to the right side of this last inequality, we obtain

$$4\left(\frac{xz}{y\sqrt[3]{xyz}}+\frac{yx}{z\sqrt[3]{xyz}}+\frac{zy}{x\sqrt[3]{xyz}}\right)\geq 4\cdot 3\sqrt[3]{\frac{xz}{y\sqrt[3]{xyz}}\cdot\frac{yx}{z\sqrt[3]{xyz}}\cdot\frac{zy}{x\sqrt[3]{xyz}}}=12\,.$$

Thus,

$$\left(\frac{x}{y}+\frac{z}{\sqrt[3]{xyz}}\right)^2+\left(\frac{y}{z}+\frac{x}{\sqrt[3]{xyz}}\right)^2+\left(\frac{z}{x}+\frac{y}{\sqrt[3]{xyz}}\right)^2\geq 12\,.$$

Note that equality holds if and only if $x=y=z$.

Also solved by ZAFAR AHMED, BARC, Mumbai, India; ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; ROY BARBARA, Lebanese University, Fanar, Lebanon; MIHÁLY BENCZE, Brasov, Romania; QUANG CAO MINH, Nguyen Binh Khiem High School, Vinh Long, Vietnam; SHI CHANGWEI, Xi'an City, Shaan Xi Province, China; COURTIS G. CHRYSSOSTOMOS, Larissa, Greece; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie, AB; JOE HOWARD, Portales, NM, USA; KEE-WAI LAU, Hong Kong, China; MATTI LEHTINEN, National Defence College, Helsinki, Finland; VEDULA N. MURTY, Dover, PA, USA; JOEL SCHLOSBERG, Bayside, NY, USA; and KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India.

Bencze actually outlined a solution for a more general problem: If $x_1, x_2, \ldots, x_n$ are positive real numbers, then we have

$$\sum_{\text{cyclic}}\left(\frac{x_1}{x_2}+\frac{x_3}{\sqrt[n]{x_1x_2\cdots x_n}}\right)^{\alpha}\geq 2^{\alpha}n$$

for all $\alpha\in(-\infty,0)\cup(1,\infty)$. Of course, the current problem is the case $n=3$ and $\alpha=2$.

M253. *Proposed by Fabio Zucca, Politecnico di Milano, Milano, Italy.*

Consider the set of lattice points $\{(x, y)\}$ where x and y are integers such that $0 \le x \le 7$ and $0 \le y \le 7$. Two points are selected at random from this set. All points have the same probability of being selected and the points need not be distinct. Find the probability that the area of the triangle (possibly degenerate) formed by these two points and the point $(0, 0)$ is an integer (possibly 0).

Solution by Alex Remorov, student, William Lyon Mackenzie Collegiate Institute, Toronto, ON.

Let us denote the area of a triangle XYZ by $[XYZ]$. Recall that for a triangle with vertices $A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$,

$$\begin{aligned} [ABC] &= \frac{1}{2}\left|\det\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}\right| \\ &= \frac{1}{2}\left|x_1y_2 + x_2y_3 + x_3y_1 - x_2y_1 - x_3y_2 - x_1y_3\right| . \end{aligned}$$

For the purpose of our problem, let the two points selected at random be $P(a, b)$ and $Q(c, d)$. If O is the origin, then $[PQO] = \frac{1}{2}|ad - bc|$. In order for $[PQO]$ to be an integer, $|ad - bc|$ must be even, and therefore, ad and bc must have the same parity. This will occur in the following two cases:

Case I. ad and bc are both odd.

This is true if and only if a, b, c, and d are all odd. Since a, b, c, and d belong to the set $\{0, 1, 2, 3, 4, 5, 6, 7\}$, the probability that they are all odd is $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{16}$.

Case II. ad and bc are both even.

This is true if and only if a and d are not both odd and b and c are not both odd. The probability that a and d are not both odd is $\frac{3}{4}$, and the same is true for b and c. Therefore, the probability that ad and bc are both even is $\frac{3}{4} \cdot \frac{3}{4} = \frac{9}{16}$.

Since Case I and Case II are mutually exclusive, the probability that the area of the triangle is an integer is $\frac{1}{16} + \frac{9}{16} = \frac{5}{8}$.

Also solved by HASAN DENKER, Istanbul, Turkey; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and D. KIPP JOHNSON, Beaverton, OR, USA. One incorrect solution was also submitted.

M254. *Proposé par Edward T.H. Wang, Université Wilfrid Laurier, Waterloo, ON.*

Evaluer la somme $S_{2006} = \sum\limits_{k=1}^{2006} (-1)^k \frac{k^2 - 3}{(k+1)!}$. [On rappelle que $n! = n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1$; par exemple, $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$.]

Solution par Jean-David Houle, étudiant, Université McGill, Montréal, QC.

Pour $k \geq 1$, nous savons que

$$\begin{aligned}\frac{k^2-3}{(k+1)!} &= \frac{k(k+1)}{(k+1)!} - \frac{k+1}{(k+1)!} - \frac{2}{(k+1)!} = \frac{1}{(k-1)!} - \frac{1}{k!} - \frac{2}{(k+1)!} \\ &= \left(\frac{1}{(k-1)!} - \frac{1}{(k+1)!}\right) - \left(\frac{1}{k!} + \frac{1}{(k+1)!}\right).\end{aligned}$$

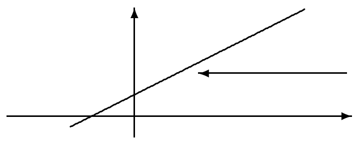
Par manipulations algébriques, on obtient :

$$\begin{aligned}S_{2006} &= \sum_{k=1}^{2006} (-1)^k \frac{k^2-3}{(k+1)!} \\ &= \left(\sum_{k=1}^{2006} \frac{(-1)^k}{(k-1)!} - \sum_{k=1}^{2006} \frac{(-1)^k}{(k+1)!}\right) - \left(\sum_{k=1}^{2006} \frac{(-1)^k}{k!} + \sum_{k=1}^{2006} \frac{(-1)^k}{(k+1)!}\right) \\ &= \left(-1+1+\sum_{k=3}^{2006} \frac{(-1)^k}{(k-1)!} - \sum_{k=1}^{2004} \frac{(-1)^k}{(k+1)!} + \frac{1}{2006!} - \frac{1}{2007!}\right) \\ &\qquad - \left(-1 + \sum_{k=2}^{2006} \frac{(-1)^k}{k!} - \sum_{k=1}^{2005} \frac{(-1)^{k+1}}{(k+1)!} + \frac{1}{2007!}\right) \\ &= \left(\frac{1}{2006!} - \frac{1}{2007!}\right) - \left(-1 + \frac{1}{2007!}\right) \\ &= \frac{1}{2006!} - \frac{2}{2007!} + 1 = \frac{2007-2}{2007!} + 1 = \frac{2005}{2007!} + 1.\end{aligned}$$

Autres solutions soumises par RICHARD I. HESS, Rancho Palos Verdes, CA, USA ; et VEDULA N. MURTY, Dover, PA, USA. Deux solutions incorrectes ont aussi été soumises.

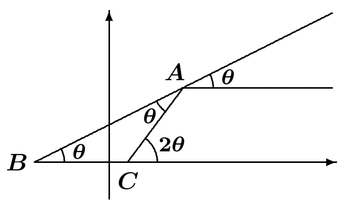
M255. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

The line with slope $\lambda > 0$ acts like a mirror to a ray of light coming along a line parallel to the x–axis. Determine the slope of the reflected ray.



Solution by D. Kipp Johnson, Beaverton, OR, USA.

Let the ray of light hit the mirror at point A, let the x-intercept of the mirror be B, and let the reflected ray of light hit the x-axis at C. If the acute angle formed by the mirror and a horizontal line is θ, then $\lambda = \tan\theta$. Since the angle of incidence and the angle of reflection are equal, we have



$\angle BAC = \angle ABC = \theta$. Then the exterior angle of $\triangle ABC$ at vertex C has measure 2θ, and the slope of the reflected ray is thus

$$\tan 2\theta \;=\; \frac{2\tan\theta}{1-\tan^2\theta} \;=\; \frac{2\lambda}{1-\lambda^2}\,.$$

(If the slope of the mirror is 1, then the reflected ray has undefined slope since it is vertical.)

Also solved by COURTIS G. CHRYSSOSTOMOS, Larissa, Greece; HASAN DENKER, Istanbul, Turkey; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JEAN-DAVID HOULE, student, McGill University, Montreal, QC; and ALEX REMOROV, student, William Lyon Mackenzie Collegiate Institute, Toronto, ON.

M256. *Proposed by the Mayhem Staff.*

Find a quadratic polynomial $f(x)$ such that, if n is a positive integer consisting of the digit 5 repeated k times, then $f(n)$ consists of the digit 5 repeated $2k$ times. (For example, $f(555) = 555555$.)

Solution by Morgan Lynch and Lacey K. Moore, Angelo State University, San Angelo, Texas, USA.

Let $f(x) = ax^2 + bx + c$. Using the given information we obtain the following system of linear equations:

$$\begin{aligned} f(5) &= 25a + 5b + c &=& \; 55\,, \\ f(55) &= 3025a + 55b + c &=& \; 5555\,, \\ f(555) &= 308025a + 555b + c &=& \; 555555\,. \end{aligned}$$

Solving this system, we determine that, $a = \frac{9}{5}$, $b = 2$, and $c = 0$. Thus,

$$f(x) \;=\; \tfrac{9}{5}x^2 + 2x \;=\; x\big(\tfrac{9}{5}x + 2\big)\,.$$

Note that:

$$\begin{aligned} \underbrace{55\cdots55}_{k\text{ times}} &= 5\big(1 + 10^1 + 10^2 + \cdots + 10^{k-1}\big) \;=\; 5\left(\frac{10^k - 1}{9}\right) \\ &= \tfrac{5}{9}\big(10^k - 1\big)\,. \end{aligned}$$

We can now verify our equation:

$$\begin{aligned} f\left(\tfrac{5}{9}(10^k - 1)\right) &= \left(\tfrac{5}{9}(10^k-1)\right)\left[\tfrac{9}{5}\left(\tfrac{5}{9}(10^k-1)\right) + 2\right] \\ &= \tfrac{5}{9}(10^k-1)(10^k+1) \;=\; \tfrac{5}{9}(10^{2k}-1) \;=\; \underbrace{55\cdots55}_{2k\text{ times}}\,. \end{aligned}$$

Also solved by HOUDA ANOUN, Bordeaux, France; COURTIS G. CHRYSSOSTOMOS, Larissa, Greece; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; AMY HOLLINGER and CORRIE MEYER, Southeast Missouri State University in Cape Girardeau, Missouri, USA; JEAN-DAVID HOULE, student, McGill University, Montreal, QC; D. KIPP JOHNSON, Beaverton, OR, USA; GUSTAVO KRIMKER, Universidad CAECE, Buenos Aires, Argentina; and ALEX REMOROV, student, William Lyon Mackenzie Collegiate Institute, Toronto, ON. One incorrect solution was also submitted.

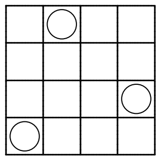
Problem of the Month

Ian VanderBurgh

This month's problem involves probability.

Problem (2007 Euclid Contest)

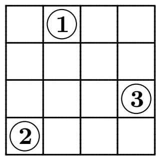
In the 4×4 grid shown, three coins are randomly placed in different squares. Determine the probability that no two coins lie in the same row or column.



As someone once suggested to me, many probability problems are just combinatorial (counting) problems where you divide by the size of the sample space whenever you want to get a probability. This problem, in particular, boils down to counting the possibilities correctly.

Before actually solving the problem, let's consider how to count the ways of placing the coins on the grid. For the moment, we will require only that no two coins be placed on the same square, without worrying about whether they are in different rows or columns. The number of ways of placing the coins depends on whether the coins are considered to be *distinguishable*. That is, can we tell them apart or are they identical?

First suppose the coins are distinguishable. We will use numbers to refer to them. Coin 1 is the one that is placed first on the grid, followed by coin 2, then coin 3. Coin 1 may be placed anywhere, which means there are 16 possible squares for it. For each of these placements of coin 1, there are 15 open squares remaining in which coin 2 may be placed, giving $16 \cdot 15$ ways of placing the first two coins. For each of these ways, there are 14 squares in which coin 3 may be placed, giving $16 \cdot 15 \cdot 14$ ways of placing all three coins. The figure at right shows one way.

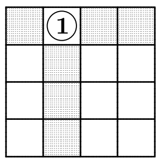


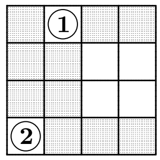
Now suppose the coins are indistinguishable. In this case, they have no numbers. What we want to count now is the number of *configurations* of the coins once they have all been placed on the grid, without regard for the order in which they are placed. Since there are 3 coins and 16 squares, the number of possible configurations is $\binom{16}{3} = \frac{16(15)(14)}{3!} = 560$. This is just our answer for the case where the coins are distinguishable divided by $3!$, the number of ways of rearranging the coins among themselves.

Now let's solve the problem.

Solution 1: Assume the coins are distinguishable. In how many ways can they be placed on the grid so that no two coins are in the same row or column?

There are 16 possible squares for coin 1. Once it has been placed, there are 3 rows and 3 columns remaining that do not contain coin 1, giving $3 \cdot 3 = 9$ squares where coin 2 may be put (the 9 white squares in the figure at right). Thus, there are $16 \cdot 9$ ways of placing the first two coins in different rows and columns.





Once the first two coins have been placed, there are 2 rows and 2 columns remaining that do not contain a coin. Thus, there are $2 \cdot 2 = 4$ squares in which coin 3 may be placed (these are the 4 white squares in the figure at right.) Altogether, there are $16 \cdot 9 \cdot 4$ ways of putting the three coins in different rows and columns.

Since the total number of ways of placing the three coins on the grid is $16 \cdot 15 \cdot 14$ (as we saw earlier), the probability that the three coins are placed in different rows and columns is $\frac{16 \cdot 9 \cdot 4}{16 \cdot 15 \cdot 14} = \frac{6}{35}$.

We can actually compute probabilities at each stage of the calculation instead of waiting until the end. The probability of placing the first two coins in different rows and columns is $\frac{16 \cdot 9}{16 \cdot 15} = \frac{3}{5}$. Given that the first two coins are in different rows and columns, the probability of placing the third coin in a different row and column from each of the first two is $\frac{4}{14} = \frac{2}{7}$ (since there are 4 acceptable squares out of 14 open squares). The probability of placing all three coins in different rows and columns is then $\frac{3}{5} \cdot \frac{2}{7} = \frac{6}{35}$.

Solution 2: This time, we regard the coins as indistinguishable. We will find the number of configurations for the coins in which the coins are in different rows and columns. This could be done by using our counting method for the case where the coins are distinguishable and then dividing by $3!$, but here is a way to count the configurations directly instead.

First, pick the 3 rows in which the coins will be put. There are $\binom{4}{3} = 4$ ways to do this. In the topmost row of these 3 rows, there are 4 possible squares for a coin. In the middle row of these rows, there are 3 possible squares for a coin (since it can't be in the same column as the coin in the topmost row). In the bottom row of these rows, there are 2 possible squares for a coin (since it can't be in the same column as either of the other two coins). Thus, there are $4 \cdot 4 \cdot 3 \cdot 2 = 96$ configurations in which no two coins are in the same row or column.

Finally, the required probability is $\frac{96}{560} = \frac{6}{35}$.

This problem has an interesting history. The initial version asked the same question for 3 coins on a 5×5 grid. (Can you solve this version?) During the development of the 2007 Euclid Contest, the problem was changed to the following problem before being changed back to its original form:

> Three *different* numbers are chosen from the set
>
> $$\{11, 12, 13, 14, 21, 22, 23, 24, 31, 32, 33, 34, 41, 42, 43, 44\}.$$
>
> What is the probability that no two of these numbers have the same units digit or the same tens digit?

This problem seemed quite a lot harder than the problem with the coins, which is strange, as it is actually the same problem! Can you see why?

You might like to try solving a more general problem where k coins are placed on an $n \times n$ grid (with $k \leq n$, of course).

Note. The author wishes to acknowledge the contributions of Bruce Crofoot, Associate Editor, in the preparation of this column.

Pólya's Paragon

Greatest Common Divisors

Ian VanderBurgh

Most of us learned about greatest common divisors (gcd's) in elementary school when we first learned about prime numbers and prime factorizations. (Remember those prime factorization trees?) We used greatest common divisors again when we learned to add fractions. Since then, however, we have probably forgotten most of what we learned! Here is a refresher on gcd's along with some related calculations and manipulations.

Definition. If a and b are integers that are not both 0, the *greatest common divisor* of a and b, denoted $\gcd(a, b)$, is the largest positive integer that divides exactly into both a and b.

In other words, $\gcd(a, b)$ is the greatest of all the common divisors of a and b. (Don't you wish that all mathematical definitions made this much sense?) To emphasize, d is a *divisor* of a if d divides exactly into a (that is, if $a = qd$ for some integer q). To tidy up a loose end, we say that $\gcd(0, 0) = 0$. (Notice here that there is not, in fact, a largest positive integer that divides into both 0 and 0, since every positive integer divides into 0. This means that we either need to ignore this case entirely, or we need to say something special here, as we have done.)

Calculations. Finding the gcd of a pair of integers is not terribly difficult when the integers are small: $\gcd(2, -4) = 2$, $\gcd(3, 5) = 1$, and $\gcd(-13, 1) = 1$. It is worth noting that $\gcd(a, 0) = a$ if a is positive and $\gcd(a, 0) = -a$ if a is negative. (Those of you comfortable with absolute values can condense this to $\gcd(a, 0) = |a|$.) Also, $\gcd(b, 1) = 1$ for every integer b. Can you see why these formulas are true from the definition?

What happens if the integers are large? For example, suppose we want to calculate $\gcd(1977, 2007)$. Your first instinct might be to try to factor 1977 and 2007 to find their positive divisors, then compare lists to find the largest of all common divisors. Let's try this.

First, $1977 = 3 \times 659$. After a bit of painful trial and error, we find that 659 appears to be a prime number. (How do we know that 659 is prime? That's a subject for another Paragon!) This tells us that the positive divisors of 1977 are 1, 3, 659, and 1977.

Next, $2007 = 3 \times 669 = 3 \times 3 \times 223$. Again, after a bit of flailing around, we discover that 223 is prime; hence, the positive divisors of 2007 are 1, 3, 9, 223, 669, and 2007.

Therefore, the positive common divisors of 1977 and 2007 are 1 and 3, which implies that $\gcd(1977, 2007) = 3$.

This method works reasonably well, but it would be pretty gruesome if 1977 and 2007 were each eight digits long instead of only four, or if there were no readily apparent small prime factors. We can cut down our work a bit by looking directly at the prime factorizations instead of listing divisors, but this still requires calculating the prime factorizations, which, as it turns out, is a very demanding problem computationally.

There is a better way, which takes advantage of the following fact:

Important Fact #1: If a, b, q, and r are integers with $a = qb + r$, then $\gcd(a, b) = \gcd(b, r)$.

This fact is not all that intuitive (lots of number theory books contain a proof, if you're interested), but we can use this fact to do what mathematicians love to do: take a problem and turn it into a smaller (or simpler) one. Here's how:

$$\begin{array}{rcllrcl} 2007 & = & 1(1977) + 30 & \Longrightarrow & \gcd(2007, 1977) & = & \gcd(1977, 30) \\ 1977 & = & 65(30) + 27 & \Longrightarrow & \gcd(1977, 30) & = & \gcd(30, 27) \\ 30 & = & 1(27) + 3 & \Longrightarrow & \gcd(30, 27) & = & \gcd(27, 3) \\ 27 & = & 9(3) + 0 & \Longrightarrow & \gcd(27, 3) & = & \gcd(3, 0) \end{array}$$

Following through this chain, $\gcd(2007, 1977) = \gcd(3, 0)$, which equals 3. (We could have stopped earlier when we saw a gcd that was easy to calculate, but it doesn't hurt to keep going until we get a 0.) Can you tell what we did at each step? At each step, we took out as many copies of the smaller number as we could from the larger number, and determined what was left over. (In technical terms, we performed the Division Algorithm several times, calculating the remainder at each stage.) Overall, this method of calculating the gcd is called the Euclidean Algorithm. Try this algorithm on 6540 and 1236. (Did you get 12 as your answer?)

After you get comfortable with this method, you may notice two time-saving features. The first is that you don't have to write all of the equalities of gcd's down the right side—these will always be true, so we can relate the gcd of the original numbers to the gcd of the final numbers directly. The second builds on the first—the gcd will actually always be the final non-zero remainder in the Algorithm. (Can you see why?)

Manipulations. These methods seem to work really well for numbers, you may say, but can I use them in a more abstract setting, like what might appear in a contest problem?

Funny you should ask .... Here is the very first problem from the very first International Mathematical Olympiad in 1959:

Problem #1. Prove that the fraction $\dfrac{21n + 4}{14n + 3}$ is irreducible for every natural number n.

Step one here, as in any problem, is to figure out what it is really asking. This problem can be restated as "Prove that $\gcd(21n + 4, 14n + 3) = 1$ for every natural number n" (since a fraction is irreducible if its numerator and denominator have no common factors).

We try to model our method from above:

$$\begin{aligned} 21n+4 &= 1(14n+3)+(7n+1)\,, \\ 14n+3 &= 2(7n+1)+1\,, \\ 7n+1 &= (7n+1)(1)+0\,. \end{aligned}$$

Thus, $\gcd(21n+4, 14n+3) = \gcd(14n+3, 7n+1) = \gcd(7n+1, 1) = 1$, as we wanted. So, we can adapt this method!

Another fact that can be quite handy:

Important Fact #2: If $\gcd(c,b) = 1$, then $\gcd(ac,b) = \gcd(a,b)$.

This fact is actually useful in both directions—it allows us to convert $\gcd(a,b)$ to $\gcd(ac,b)$ (although it is not immediately obvious why we would ever want to do this), and it allows us to convert $\gcd(ac,b)$ to $\gcd(a,b)$. This fact is more intuitive—can you explain it to yourself?

We now try a second problem:

Problem #2. Prove that $\gcd(n^2, 2n+1) = 1$ for any natural number n.

Our initial instinct is to try to use the abstract version of the Euclidean Algorithm, but it is very difficult to make $2n+1$ go into n^2 without introducing fractions. This is where Important Fact #2 can be used: since $2n+1$ is odd, then $\gcd(2n+1, 2) = 1$. Thus,

$$\begin{aligned} \gcd(n^2, 2n+1) &= \gcd(2n^2, 2n+1) && (\text{since } \gcd(2n+1,2)=1) \\ &= \gcd(-n, 2n+1) && (\text{since } 2n^2 = n(2n+1)+(-n)) \\ &= \gcd(n, 2n+1) && (\text{since } \gcd(-1, 2n+1)=1) \\ &= \gcd(n, 1) && (\text{since } 2n+1 = 2(n)+1) \\ &= 1\,, \end{aligned}$$

as required.

I hope you have remembered a bit and learned a bit about gcd's here. By no means is what we have done comprehensive, but it should give you some ideas to think about and some strategies to use. Try applying them to one of this month's Mayhem problems!

Ian VanderBurgh
Centre for Education in Mathematics and Computing
University of Waterloo
200 University Avenue West
Waterloo, ON, Canada N2L 3G1
iwtvande@uwaterloo.ca