SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

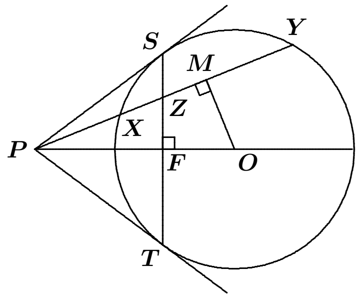
3102. [2006 : 44, 47] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Let D be the mid-point of the side BC of $\triangle ABC$. Let E and F be the projections of B onto AC and C onto AB, respectively. Let P be the point of intersection of AD and EF. Show that, if $AD = \frac{\sqrt{3}}{2} BC$, then P is the mid-point of AD.

III. *Solution by Nobutaka Shigeki, Kitakyusyu City, Fukuoka, Japan, modified by the editor.*

First we will establish a lemma.

Lemma. Let PS and PT be the tangents to a circle at S and T from an exterior point P, as shown is the figure. Let X and Y lie on the circle and be collinear with P. If Z is the point of intersection of ST and XY, then



$$\frac{1}{PX} + \frac{1}{PY} = \frac{2}{PZ}. \qquad (1)$$

Proof: Let M be the mid-point of XY, let O be the centre of the circle, and let F be the intersection of PO with ST. Then

$$\frac{1}{PX} + \frac{1}{PY} = \frac{PX + PY}{PX \cdot PY} = \frac{2PM}{PX \cdot PY}.$$

Therefore, equation (1) is equivalent to $\frac{2PM}{PX \cdot PY} = \frac{2}{PZ}$; that is

$$PM \cdot PZ = PX \cdot PY. \qquad (2)$$

Clearly, the points Z, F, O, and M are concyclic. Thus,

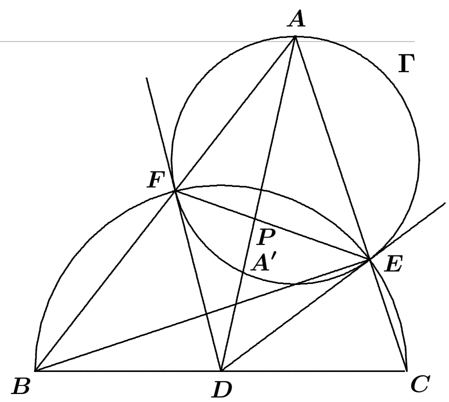
$$PM \cdot PZ = PF \cdot PO = PS^2,$$

because $\triangle PSF$ is similar to $\triangle POS$. Since $PS^2 = PX \cdot PY$, we have (2) and thus (1). ■

Now we turn our attention to the given problem. Clearly, E and F are the intersections of the circle having BC as diameter with the lines AC and AB, respectively. It follows that $\angle ABC = 180° - \angle FEC = \angle AEF$. On the other hand, since $\triangle BDF$ is isosceles, we have $\angle ABC = \angle BFD$. Thus, $\angle AEF = \angle BFD$. Similarly, $\angle AFE = \angle DEC$.

Let Γ be the circumcircle of $\triangle AEF$. By the Tangent-Chord Theorem, we see that DF is tangent to Γ at F and that DE is tangent to Γ at E. Let A' be the second point of intersection of DA with Γ. Applying the above lemma to Γ, we obtain

$$\frac{2}{DP} = \frac{1}{DA'} + \frac{1}{DA}. \qquad (3)$$



Let $r = DB = DE = DF$. Then $DA' \cdot DA = r^2$. Since we are given that $AD = \sqrt{3}r$, we deduce that $DA' = r/\sqrt{3}$. From (3), we have

$$\frac{2}{DP} = \frac{\sqrt{3}}{r} + \frac{1}{\sqrt{3}\,r}.$$

Therefore, $DP = (\sqrt{3}/2)r = \frac{1}{2}DA$, which means that P is the mid-point of AD.

[*Ed.:* By refining the argument at the end of the proof, one can show that $AD = \frac{\sqrt{3}}{2}BC$ if and only if P is the mid-point of AD.

The above proof assumes that $\triangle ABC$ is acute-angled. However, if there is an obtuse angle at B or at C, the result is still valid. The above proof extends to this case by simply modifying the argument used to show that DE and DF are tangent to Γ.]

3137. [2006 : 173, 176] *Proposed by Tina Balfour and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.*

Find all solutions in non-negative integers to the following Diophantine equations:

(a) $5^m + 3^m = 2^k$; (b) ★ $5^m + 3^n = 2^k$.

(a) *Composite of similar solutions by Brian D. Beasley, Presbyterian College, Clinton, SC, USA; and David E. Manes, SUNY at Oneonta, Oneonta, NY, USA.*

Note first that there are no solutions when $k = 0$. It is also clear that $(m, k) = (0, 1)$ and $(1, 3)$ are solutions. We now show that there are no other solutions.

Suppose $m \geq 2$. Then $k \geq 6$. Since $2^k \equiv 0 \pmod{16}$ for $k \geq 4$, we have $5^m + 3^m \equiv 0 \pmod{16}$.

However, the least non-negative residues of 5^m modulo 16 for $m \geq 1$ are 5, 9, 13, and 1, which repeat in cycles of length 4, while those of 3^m are 3, 9, 11, and 1, which also repeat in cycles of length 4. Consequently, $5^m + 3^m \equiv 8 \pmod{16}$ or $5^m + 3^m \equiv 2 \pmod{16}$, and our claim follows.

(b) *Solution by Mercedes Sánchez Benito, Universidad Complutense, Madrid, Spain, Óscar Ciaurri Ramírez, Universidad de La Rioja, Logroño, Spain, and Manuel Benito Muñoz and Emilio Fernández Moral, IES Sagasta, Logroño, Spain, modified by the editor.*

If n is even, we have $5^m + 3^n \equiv 1^m + (-1)^n \equiv 2 \pmod 4$. Since $2^k \equiv 2 \pmod 4$ if and only if $k = 1$, the unique solution for n even is $m = n = 0$ and $k = 1$.

Let n be odd. For $m = 0$, we have to find solutions to $1 + 3^n = 2^k$. However, Leo Hebreus (or Levi ben Gerson, 14^{th} century) proved that for all $n > 2$, the integer $3^n \pm 1$ has an odd divisor; hence, the unique solution of $1 + 3^n = 2^k$ for $m = 0$ and n odd is $n = 1$ and $k = 2$.

Now we assume that $m > 0$. By considering the equation modulo 3, we obtain $(-1)^m \equiv (-1)^k \pmod 3$, which implies that m and k have the same parity. On the other hand, by examining the equation modulo 5, we get

$$2^k \equiv (-2)^n \equiv -2^n \equiv \pm 2 \pmod 5 ,$$

since n is odd. This implies that k is odd (and then so is m).

Now suppose that $m \geq 3$ and $n \geq 3$ (which means that $k \geq 7$). Setting $A = 22276800 = 2^6 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 13 \cdot 17$, we checked by a computer program that there are no solutions of $5^m + 3^n = 2^k$ modulo A for odd exponents $m \geq 3$, $n \geq 3$, and $k > 7$ (the checking is a "finite" problem, since $5^{51} \equiv 5^3 \pmod A$, $3^{243} \equiv 3^3 \pmod A$, and $2^{127} \equiv 2^7 \pmod A$). Therefore, we must have either $m = 1$ or $n = 1$.

Let $n = 1$. We must look for solutions of $5^m + 3 = 2^k$ (this problem was proposed on the XXII Spanish Mathematical Olympiad). Using the modulus $B = 65792 = 2^8 \cdot 257$, we again used a computer to search for solutions modulo B (again the checking is a "finite" problem, since $5^{256} \equiv 1 \pmod B$ and $2^{25} \equiv 2^9 \pmod B$; furthermore, 9 is the smallest power of 2 where the remainders modulo B begin to repeat). The computer program yielded the following four cases for $n = 1$ and $m > 0$:

$$(m, k) \in \{(1, 3), (3, 7)\} .$$

Since the values for k lie in the non-periodic set of remainders of powers of 2 modulo B, we see that $k = 1$ or $k = 7$. This gives us the solutions $(m, n, k) = (1, 1, 3)$ and $(m, n, k) = (3, 1, 7)$. Furthermore, any other solutions must have $m \equiv 1 \pmod B$ or $m \equiv 3 \pmod B$. Since the smallest values for m other than 1 or 3 are significantly too large to have a solution, these are the only solutions for $n = 1$.

Lastly, we will examine $m = 1$ and $n \geq 3$. This time, we use the modulus $C = 2^6 \cdot 3^4 \cdot 17$ for our computer check. Once more this becomes a finite problem since $3^{21} \equiv 3^5 \pmod C$ and $2^{223} \equiv 2^7 \pmod C$; furthermore, 5 and 7 are the smallest powers of 3 and 2, respectively, where the remainders begin to repeat. The only solution modulo C that the program generated was $(n, k) = (3, 5)$. This yields the solution $(m, n, k) = (1, 3, 5)$. Since the powers on both 2 and 3 are in the non-periodic set of remainders of their respective powers, there are no further solutions.

In conclusion, there are exactly five solutions to $5^m + 3^n = 2^k$, namely:

$$(m,n,k) \in \{(0,0,1),\ (0,1,2),\ (1,1,3),\ (3,1,7),\ (1,3,5)\}\,.$$

Part (a) *also solved by MICHEL BATAILLE, Rouen, France; ROY BARBARA, Lebanese University, Fanar, Lebanon; MERCEDES SÁNCHEZ BENITO, Universidad Complutense, Madrid, Spain, ÓSCAR CIAURRI RAMÍREZ, Universidad de La Rioja, Logroño, Spain, and MANUEL BENITO MUNOZ ̃and EMILIO FERNÁNDEZ MORAL, IES Sagasta, Logroño, Spain; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; JOEL SCHLOSBERG, Bayside, NY, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.*

Beasley conjectured that the equation in part (b) *has exactly the five solutions which are determined above.*

The reason for the late featuring of this solution is that we wanted to have the computer solution properly analyzed. We apologize for this delay. We would appreciate if our readers could find a proof for the result which is independent of computer verification.

3139. [2006 : 238, 240; 2007 : 242] *Proposed by Michel Bataille, Rouen, France.*

Let ε be the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$. Two parallel tangents to ε intersect a third tangent at $M_1(x_1,y_1)$ and $M_2(x_2,y_2)$. Show that the value of

$$\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)\left(\frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}-1\right)$$

is independent of the chosen tangents.

II. *Solution by J.A. Thas, Ghent University, Ghent, Belgium.*

The desired result is a consequence of properties of projective coordinates interpreted in the affine plane. Our conic defines a *scalar product* between the points $M_1=(x_1,y_1)$ and $M_2=(x_2,y_2)$ by

$$\langle M_1,M_2\rangle \;=\; \frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}-1\,.$$

There is likewise a scalar product defined by the dual conic (composed of the tangents to the conic) between pairs of lines: if $L_i=[u_i,v_i,w_i]$ represent the lines $u_ix+v_iy+w_i=0$ for $i=1$ and $i=2$, then

$$[L_1,L_2] \;=\; a^2u_1u_2+b^2v_1v_2-w_1w_2\,.$$

A pair of points or a pair of lines are conjugate if and only if their scalar product is zero. One easily shows that the line joining M_1 to M_2 is tangent to the conic if and only if

$$\langle M_1,M_1\rangle\langle M_2,M_2\rangle-\langle M_1,M_2\rangle^2 \;=\; 0\,. \tag{1}$$

(See, for example, H.S.M. Coxeter, *The Real Projective Plane*, 3rd edition, formula 12.76 on page 188, for the details.)

In this notation, we are required to show that $\langle M_1, M_1\rangle\langle M_2, M_2\rangle$ is independent of the chosen tangents. We will show that, for all choices of the three tangents, $\langle M_1, M_1\rangle\langle M_2, M_2\rangle = 1$. In view of (1) above, we have only to show that $\langle M_1, M_2\rangle^2 = 1$.

Consider the parallelogram formed by the given parallel tangents together with the third tangent and the tangent parallel to it. Because the three diagonals of any quadrilateral circumscribed about a conic form a self-polar triangle (this is the dual of Theorem 6.43 on page 78 of the Coxeter book cited above), the diagonals of our parallelogram, namely $y_1x - x_1y = 0$ and $y_2x - x_2y = 0$, are conjugate. This tells us that

$$\frac{x_1x_2}{a^2} + \frac{y_1y_2}{b^2} = 0,$$

as desired.

3151. [2006 : 304, 306] *Proposed by Mª Jesús Villar Rubio, Santander, Spain.*

(a) Let $r_1 < 0 < r_2 < r_3$ be the real roots of $8x^3 - 6x + \sqrt{3} = 0$. Prove that

$$r_3^2 = 4r_2^2 - 4r_2^4 \quad \text{and} \quad r_1^2 = 4r_3^2 - 4r_3^4 .$$

(b) Let $s_1 < 0 < s_2 < s_3$ be the real roots of $8x^3 - 6x + 1 = 0$. Prove that

$$r_1^2 + s_2^2 = 1, \quad s_1^2 + r_2^2 = 1, \quad \text{and} \quad r_3^2 + s_3^2 = 1 .$$

Solution by Brian D. Beasley, Presbyterian College, Clinton, SC, USA.

(a) Set $x = \sin\theta$. The given equation $8x^3 - 6x + \sqrt{3} = 0$ may be rewritten as

$$\frac{\sqrt{3}}{2} = -4x^3 + 3x = -4\sin^3\theta + 3\sin\theta = \sin(3\theta) .$$

Then $3\theta = \frac{\pi}{3} + 2\pi k$ or $3\theta = \frac{2\pi}{3} + 2\pi k$, for any integer k. Restricting θ to the interval $[0, 2\pi]$, we find that

$$\theta \in \left\{\frac{\pi}{9}, \frac{2\pi}{9}, \frac{7\pi}{9}, \frac{8\pi}{9}, \frac{13\pi}{9}, \frac{14\pi}{9}\right\}.$$

Thus,

$$r_1 = \sin\frac{13\pi}{9} = -\sin\frac{4\pi}{9}, \quad r_2 = \sin\frac{\pi}{9}, \quad \text{and} \quad r_3 = \sin\frac{2\pi}{9} .$$

Using the identity $\sin^2 2\theta = 4\sin^2\theta - 4\sin^4\theta$, we have

$$r_3^2 = \sin^2\frac{2\pi}{9} = 4\sin^2\frac{\pi}{9} - 4\sin^4\frac{\pi}{9} = 4r_2^2 - 4r_2^4$$

and

$$r_1^2 = \sin^2\frac{4\pi}{9} = 4\sin^2\frac{2\pi}{9} - 4\sin^4\frac{2\pi}{9} = 4r_3^2 - 4r_3^4 .$$

(b) Set $x = \cos\theta$. The given equation $8x^3 - 6x + 1 = 0$ may be rewritten as

$$-\frac{1}{2} = 4x^3 - 3x = 4\cos^3\theta - 3\cos\theta = \cos(3\theta)\,.$$

Then $3\theta = \frac{2\pi}{3} + 2\pi k$ or $3\theta = \frac{4\pi}{3} + 2\pi k$, for any integer k. Restricting θ to the interval $[0, 2\pi]$, we find that

$$\theta \in \left\{\frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9}, \frac{10\pi}{9}, \frac{14\pi}{9}, \frac{16\pi}{9}\right\}.$$

Thus,

$$s_1 = \cos\frac{8\pi}{9} = -\cos\frac{\pi}{9}\,, \quad s_2 = \cos\frac{4\pi}{9}\,, \quad \text{and} \quad s_3 = \cos\frac{2\pi}{9}\,.$$

The desired result follows from the Pythagorean identity $\cos^2\theta + \sin^2\theta = 1$ applied to the above values of r_i and s_i.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; QUANG CAO MINH, Nguyen Binh Khiem High School, Vinh Long, Vietnam; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; VEDULA N. MURTY, Dover, PA, USA; JOEL SCHLOSBERG, Bayside, NY, USA; ÓLÍNA SIGURGEIRSDÓTTIR, student, Auburn University, Montgomery, AL, USA; BIN ZHAO, student, YunYuan HuaZhong University of Technology and Science, Wuhan, Hubei, China; and the proposer. Most solutions were based on the identity for $\cos 3\theta$.

3152. [2006 : 304, 307] *Proposed by Michel Bataille, Rouen, France.*

Let $x_1, x_2, \ldots, x_n$ $(n \geq 2)$ be real numbers such that $\sum\limits_{i=1}^{n} x_i = 0$ and $\sum\limits_{i=1}^{n} x_i^2 = 1$. Find the minimum and maximum of $\sum\limits_{i=1}^{n} |x_i|$.

Essentially the same solution by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; and Kee-Wai Lau, Hong Kong, China.

We have

$$\left(\sum_{i=1}^{n}|x_i|\right)^2 = \sum_{i=1}^{n}x_i^2 + \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}|x_i||x_j| \geq 1 + \left|\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}x_ix_j\right|$$
$$= 1 + \left|\left(\sum_{i=1}^{n}x_i\right)^2 - \sum_{i=1}^{n}x_i^2\right| = 2\,.$$

Equality holds for $x_1 = 1/\sqrt{2} = -x_2$ and $x_3 = x_4 = \cdots = x_n = 0$. It follows that the minimum is $\sqrt{2}$.

For the maximum, suppose that m of the x_i's are non-negative and $n-m$ are non-positive, for $1 \le m \le n$. Without loss of generality, we assume that $x_1, \dots, x_m$ are non-negative and $x_{m+1}, \dots, x_n$ are non-positive. By Schwarz's Inequality,

$$\sum_{i=1}^{m} x_i^2 \ \ge\ \frac{1}{m}\left(\sum_{i=1}^{m} x_i\right)^2$$

and

$$\sum_{i=m+1}^{n} x_i^2 \ \ge\ \frac{1}{n-m}\left(\sum_{i=m+1}^{n} x_i\right)^2 \ =\ \frac{1}{n-m}\left(\sum_{i=1}^{m} x_i\right)^2 .$$

It follows that

$$1 \ = \sum_{i=1}^{n} x_i^2 \ \ge\ \left(\frac{1}{m}+\frac{1}{n-m}\right)\left(\sum_{i=1}^{m} x_i\right)^2 \ =\ \frac{n}{m(n-m)}\left(\sum_{i=1}^{m} x_i\right)^2 .$$

Hence, $\sum\limits_{i=1}^{n} |x_i| = 2\sum\limits_{i=1}^{m} x_i \le \dfrac{2}{\sqrt{n}}\sqrt{m(n-m)}$.

For any real number r, the quadratic function $x(r-x)$ has a maximum $r^2/4$ at $x = r/2$. Thus, for $n = 2k$, where k is a positive integer, we have $m(n-m) \le k^2$ and

$$\sum_{i=1}^{n} |x_i| \ \le\ \frac{2k}{\sqrt{n}} \ =\ \sqrt{n}\,.$$

Equality holds for $x_1 = x_2 = \cdots = x_k = -x_{k+1} = \cdots = -x_n = 1/\sqrt{n}$.

For $n = 2k+1$, where k is a positive integer,

$$m(n-m) \ \le\ \max\{k(2k+1-k),(k+1)(2k+1-k-1)\} \ =\ k(k+1)\,.$$

Hence,

$$\sum_{i=1}^{n} |x_i| \ \le\ \frac{2\sqrt{k(k+1)}}{\sqrt{n}} \ =\ \sqrt{n-\frac{1}{n}}\,.$$

Equality holds for

$$x_1 \ =\ x_2 \ =\ \cdots \ =\ x_k \ =\ \sqrt{\frac{k+1}{k(2k+1)}}$$

and $\qquad x_{k+1} \ =\ \cdots \ =\ x_n \ =\ -\sqrt{\dfrac{k}{(k+1)(2k+1)}}\,.$

We conclude that the maximum is $\sqrt{n+\dfrac{(-1)^n-1}{2n}}$.

Also solved by ROY BARBARA, Lebanese University, Fanar, Lebanon; PAUL BRACKEN, University of Texas, Edinburg, TX, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; JOEL SCHLOSBERG, Bayside, NY, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. There was one incorrect submission.

3153. [2006 : 304, 307] *Proposed by Michel Bataille, Rouen, France.*

For which integers n does the equation

$$\frac{3xy-1}{x+y} = n$$

have a solution in integers x and y?

Essentially the same solution by Roy Barbara, Lebanese University, Fanar, Lebanon; Brian D. Beasley, Presbyterian College, Clinton, SC, USA; Kee-Wai Lau, Hong Kong, China; Joel Schlosberg, Bayside, NY, USA; and Peter Y. Woo, Biola University, La Mirada, CA, USA.

Let k be any integer. If $n = 3k$, then the given equation becomes $3xy - 3k(x+y) = 1$, which has no solutions. If $n = 3k+1$, then $x = k$ and $y = -(3k^2+k+1)$ is a solution. If $n = 3k-1$, then $x = k$ and $y = 3k^2 - k + 1$ is a solution. Hence, the equation has a solution if and only if n is not a multiple of 3.

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, NL; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer.

3154 [2006 : 304, 307] *Proposed by Challa K.S.N.M. Sankar, Andhrapradesh, India.*

(a) If $\beta > 1$ is a real constant, determine the number of possible real solutions of the equation

$$x - \beta\log_2 x = \beta - \beta\ln\beta\,.$$

(b) If $\alpha_1 < \alpha_2$ are two positive real solutions of the equation in (a), and if x_1 and x_2 are any two real numbers satisfying $\alpha_1 \le x_1 < x_2 \le \alpha_2$, prove that, for all λ such that $0 < \lambda < 1$,

$$\lambda\log_2 x_1 + (1-\lambda)\log_2 x_2 \ \ge\ \ln\bigl(\lambda x_1 + (1-\lambda)x_2\bigr)\,.$$

Determine when equality occurs.

Solution by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina, modified and expanded by the editor.

(a) Consider the function $f(x) = x - \beta\log_2 x - \beta + \beta\ln\beta$ for $x > 0$. We have $f'(x) = 1 - \beta/(x\ln 2)$; whence, $x = \beta/(x\ln 2)$ is the only critical value. Since $f'(x) < 0$ for $0 < x < \beta/\ln 2$ and $f'(x) > 0$ for $x > \beta/\ln 2$, we see that f is decreasing on $(0, \beta/\ln 2)$ and increasing on $(\beta/\ln 2, \infty)$.

Thus, f has a relative minimum at $x = \beta/\ln 2$.

Note that $\lim_{x\to 0^+} f(x) = \infty$ and that

$$\lim_{x\to\infty} f(x) \;=\; -\beta + \beta\ln\beta + \lim_{x\to\infty} x\left(1 - \frac{\beta\log_2 x}{x}\right) = \infty\,,$$

since $\lim_{x\to\infty} \frac{\beta\log_2 x}{x} = 0$.

We now show that $f(\beta/\ln 2) < 0$.

Since $f(\beta/\ln 2) = \beta/\ln 2 - \beta\log_2(\beta/\ln 2) - \beta + \beta\ln\beta$, it suffices to show that $1/\ln 2 - \log_2(\beta/\ln 2) - 1 + \ln\beta < 0$, which is equivalent in succussion to

$$\begin{aligned}
1 - (\ln 2)\bigl(\log_2(\beta/\ln 2)\bigr) - \ln 2 + (\ln 2)(\ln\beta) &< 0\,,\\
1 - \ln 2 - \ln(\beta/\ln 2) + (\ln 2)(\ln\beta) &< 0\,,\\
1 - \ln 2 - \bigl(\ln\beta - \ln(\ln 2)\bigr) + (\ln 2)(\ln\beta) &< 0\,,\\
1 - \ln 2 + \ln(\ln 2) - (1 - \ln 2)(\ln\beta) &< 0\,,
\end{aligned}$$

which is true since $(1-\ln 2)(\ln\beta) > 0$ and $1-\ln 2+\ln(\ln 2) < 0$. Therefore, f has *exactly* two real roots, α_1 and α_2, such that $0 < \alpha_1 < \alpha_2$. That is, the given equation has exactly two real solutions.

(b) From part (a) we see that $f(x) \le 0$ for all $x \in [\alpha_1, \alpha_2]$, where $0 < \alpha_1 < \beta/\ln 2 < \alpha_2$. Thus, $\log_2 x \ge (x/\beta) - 1 + \ln\beta$.

Since $0 < \lambda < 1$, it follows that

$$\begin{aligned}
&\lambda\log_2 x_1 + (1-\lambda)\log_2 x_2\\
&\qquad \ge\; \lambda\left(\frac{x_1}{\beta} - 1 + \ln\beta\right) + (1-\lambda)\left(\frac{x_2}{\beta} - 1 + \ln\beta\right)\\
&\qquad =\; \frac{1}{\beta}\bigl(\lambda x_1 + (1-\lambda)x_2\bigr) - 1 + \ln\beta\,. \qquad (1)
\end{aligned}$$

Next we show that, for all $t > 0$,

$$t - \beta + \beta\ln\beta \;\ge\; \beta\ln t\,. \qquad (2)$$

Let $g(t) = t - \beta + \beta\ln\beta - \beta\ln t$. Then $g'(t) = 1 - \beta/t$ showing that $t = \beta$ is the only critical value. Since $g''(t) = \beta/t^2 > 0$, we see that $g(\beta) = 0$ is a relative as well as the absolute minimum of g. Hence, $g(t) \ge 0$ for all $t > 0$ and (2) follows.

In particular, for $t = \lambda x_1 + (1-\lambda)x_2$, we obtain

$$\lambda x_1 + (1-\lambda)x_2 - \beta + \ln\beta \;\ge\; \beta\ln\bigl(\lambda x_1 + (1-\lambda)x_2\bigr)\,. \qquad (3)$$

From (1) and (3), we then have

$$\lambda\log_2 x_1 + (1-\lambda)\log_2 x_2 \;\ge\; \ln\bigl(\lambda x_1 + (1-\lambda)x_2\bigr)\,.$$

Equality occurs if and only if $x_1 = \alpha_1$, $x_2 = \alpha_2$, and $\lambda\alpha_1 + (1-\lambda)\alpha_2 = \beta$, which yields $\lambda = \frac{\alpha_2 - \beta}{\alpha_2 - \alpha_1}$. Since $f(\beta) = \beta(\ln\beta - \log_2\beta) < 0$, we see that $\alpha_1 < \beta < \frac{\beta}{\ln 2} < \alpha_2$, which is consistent with the assumption $0 < \lambda < 1$.

Also solved by the proposer.

3155. [2006 : 304, 307] *Proposed by Virgil Nicula, Bucharest, Romania.*

In $\triangle ABC$, let D, E, F be the intersections of the altitudes from A, B, C to the sides BC, CA, AB, respectively. Let H be the orthocentre of $\triangle ABC$, let L be the intersection of AT and the line through B perpendicular to BC, and let T be the intersection of BE and DF.

Show that $BL = BC$ if and only if $\angle ACB = 45^\circ$.

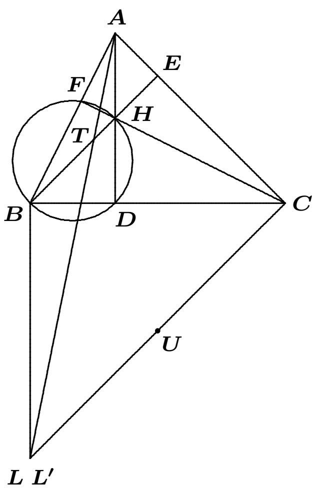
Solution by Michel Bataille, Rouen, France.

We modify the requirement of the problem to be:

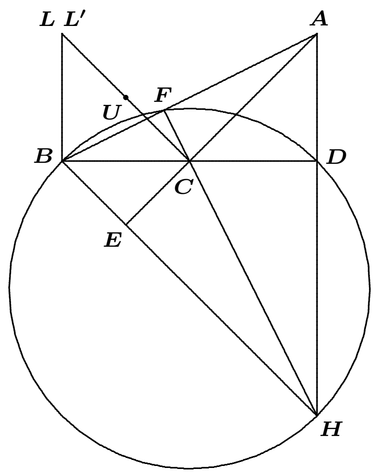
Show that $BL = BC$ if and only if $\angle ACB = 45^\circ$ or $\angle ACB = 135^\circ$, and that AT is the polar of C with respect to γ.

Note that B, F, H, and D lie on the circle γ with diameter BH and that AT is the polar with respect to γ. We call L' the point of intersection of the perpendiculars to BC at B and to AC at C.

First, suppose that $\angle ACB = 45^\circ$ or $\angle ACB = 135^\circ$. Then using the facts that $BE \parallel CL'$ and $\angle BCL' = 45^\circ$, we see that $\triangle CBL'$ and $\triangle BDH$ are isosceles right triangles, with right angles at B and D, respectively. Let U be the mid-point of CL'. Then, $\angle UBH = 90^\circ$, so that UB is tangent to γ at B and the circle $\gamma' = (BCL')$ is orthogonal to γ. Since CL' is a diameter of γ', the points C and L' are conjugate with respect to γ. Hence, L' is on the polar AT of C, and thus, $L = L'$ and $BL = BC$.



$\angle ACB = 45^\circ$



$\angle ABC = 135^\circ$

Also solved by APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and VÁCLAV KONEČNÝ, Big Rapids, MI, USA.

The following solvers only considered the case $\angle ACB = 45^\circ$: ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; GEOFFREY A. KANDALL, Hamden, CT, USA; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

3156. [2006 : 305, 307] *Proposed by Virgil Nicula, Bucharest, Romania.*

Let Γ be the circumcircle of $\triangle ABC$. Let M be an interior point on the side AB, and let N be an interior point on the side AC. Let D be an intersection point of MN with Γ. Prove that

$$\left|\frac{MB}{MA}\cdot\frac{AC}{DB}-\frac{NC}{NA}\cdot\frac{AB}{DC}\right| = \frac{BC}{DA}\,.$$

Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA.

Let $P = AD\cap BC$ and $Q = MD\cap BC$. [*Editor's comment:* Woo deals explicitly with the case where D is opposite B on arc AC of the circumcircle, and B lies between Q and C. With the use of directed distances and directed angles, we can avoid special cases except when P or Q is at infinity; these possibilities are easily handled using continuity arguments.] By Menelaus' Theorem applied to the transversal QMD of $\triangle BAP$ and to the transversal QND of $\triangle CAP$,

$$\frac{BM}{MA} = -\frac{BQ}{QP}\cdot\frac{PD}{DA} \quad\text{and}\quad \frac{CN}{NA} = -\frac{CQ}{QP}\cdot\frac{PD}{DA}\,.$$

Because $\triangle PCA \sim \triangle PDB$, we have

$$\frac{AC}{BD} = \frac{PC}{PD}\,;$$

similarly, $\triangle PBA \sim \triangle PDC$ implies that

$$\frac{AB}{CD} = \frac{PB}{PD}\,.$$

It follows that

$$\begin{aligned}
\frac{BM}{MA}\cdot\frac{AC}{BD}-\frac{CN}{NA}\cdot\frac{AB}{CD} &= -\frac{BQ}{QP}\cdot\frac{PD}{DA}\cdot\frac{AC}{BD}+\frac{CQ}{QP}\cdot\frac{PD}{DA}\cdot\frac{AB}{CD}\\
&= -\frac{BQ}{QP}\cdot\frac{PD}{DA}\cdot\frac{PC}{PD}+\frac{CQ}{QP}\cdot\frac{PD}{DA}\cdot\frac{PB}{PD}\\
&= \frac{-BQ\cdot PC+CQ\cdot PB}{QP\cdot DA}\\
&= \frac{-BQ\cdot PC+(CB+BQ)\cdot(PC+CB)}{QP\cdot DA}\\
&= \frac{CB\cdot(PC+CB+BQ)}{QP\cdot DA}\\
&= \frac{CB\cdot PQ}{QP\cdot DA} = \frac{BC}{DA}\,,
\end{aligned}$$

as desired.

Also solved by MICHEL BATAILLE, Rouen, France; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; JOEL SCHLOSBERG, Bayside, NY, USA; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer.

Nicula provided a selection of familiar and not so familiar special cases of his result:

- *If* $BN \cap CM = G$ *(the centroid), then* $\left|\frac{b}{DB} - \frac{c}{DC}\right| = \frac{a}{DA}$.
- *If* $BN \cap CM = I$ *(the incentre), then* $\left|\frac{1}{DB} - \frac{1}{DC}\right| = \frac{1}{DA}$.
- *If* $BN \cap CM = H$ *(the orthocentre), then* $\left|\frac{\cos B}{DB} - \frac{\cos C}{DC}\right| = \frac{|\cos A|}{DA}$.

3157. [2006 : 305, 308] *Proposed by Mihály Bencze, Brasov, Romania.*

Let p be a fixed odd prime number. Let $\alpha(n)$ denote the largest integer k such that p^k is an integral divisor of $1^1 \cdot 2^2 \cdot 3^3 \cdots n^n$. Prove that

$$\lim_{n\to\infty} \frac{\alpha(n)}{n^2} = \frac{1}{2(p-1)}.$$

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

In order to get an idea of the behavior of $\alpha(n)$, we proceed very much like Legendre in his reasoning for the exponent of p in the prime decomposition of $n!$. The result then is

$$\alpha(n) = p\left(1+2+\cdots+\left\lfloor\frac{n}{p}\right\rfloor\right) + p^2\left(1+2+\cdots+\left\lfloor\frac{n}{p^2}\right\rfloor\right) + p^3\left(1+2+\cdots+\left\lfloor\frac{n}{p^3}\right\rfloor\right) + \cdots.$$

In a more concise form,

$$\alpha(n) = \sum_{j=1}^{N(n)} \frac{p^j}{2}\left\lfloor\frac{n}{p^j}\right\rfloor\left(\left\lfloor\frac{n}{p^j}\right\rfloor+1\right),$$

where $N(n) = \lfloor \ln n/\ln p\rfloor$. We have

$$\left(\frac{n}{p^j}-1\right)\left(\frac{n}{p^j}\right) < \left\lfloor\frac{n}{p^j}\right\rfloor\left(\left\lfloor\frac{n}{p^j}\right\rfloor+1\right) \le \left(\frac{n}{p^j}\right)\left(\frac{n}{p^j}+1\right).$$

Multiplying by p^j yields

$$n\left(\frac{n}{p^j}-1\right) < p^j\left\lfloor\frac{n}{p^j}\right\rfloor\left(\left\lfloor\frac{n}{p^j}\right\rfloor+1\right) \le n\left(\frac{n}{p^j}+1\right).$$

Thus,

$$\frac{1}{2}\left(\left(\sum_{j=1}^{N(n)}\frac{1}{p^j}\right)-\frac{N(n)}{n}\right) < \frac{\alpha(n)}{n^2} \le \frac{1}{2}\left(\left(\sum_{j=1}^{N(n)}\frac{1}{p^j}\right)+\frac{N(n)}{n}\right).$$

Letting $n \to \infty$ and noting that $\lim\limits_{n\to\infty}\frac{N(n)}{n} = 0$, we get

$$\lim_{n\to\infty}\frac{\alpha(n)}{n^2} = \frac{1}{2}\sum_{j=1}^{\infty}\frac{1}{p^j} = \frac{1}{2(p-1)}.$$

Also solved by MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOEL SCHLOSBERG, Bayside, NY, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

3158. [2006 : 305, 308] *Proposed by Mihály Bencze, Brasov, Romania.*

Let $E = \{(x,y) \in \mathbb{N}\times\mathbb{N} \mid x+y \text{ is a perfect square}\}$, and let $N(n)$ be the size of the set $\{(x,y) \in E \mid x \le n \text{ and } y \le n\}$, for $n \in \mathbb{N}$. Prove that

$$\lim_{n\to\infty}\frac{N(n)}{n\sqrt{n}} = \frac{4}{3}(\sqrt{2}-1).$$

Solution by Joel Schlosberg, Bayside, NY, USA, modified by the editor.

For each $n \in \mathbb{N}$ and $k \in \mathbb{N}$, let $\varphi(n,k)$ be the number of integer pairs (x,y) with $1 \le x,\ y \le n$ and $x+y=k$. Then $N(n) = \sum\limits_{i=1}^{\infty}\varphi(n,i^2)$.

To evaluate $\varphi(n,k)$, we first observe that if $1 \le x,\ y \le n$, then $2 \le x+y \le 2n$. It follows that $\varphi(n,k) = 0$ unless $2 \le k \le 2n$. If $2 \le k \le n+1$, then the equation $x+y=k$ is satisfied by the pairs $(1,k-1)$, $(2,k-2)$, …, $(k-1,1)$; hence, $\varphi(n,k) = k-1$. If $n+1 \le k \le 2n$, then the equation $x+y=k$ is satisfied by $(k-n,n)$, $(k-n+1,n-1)$, …, $(n,k-n)$; hence, $\varphi(n,k) = 2n+1-k$. Thus,

$$\varphi(n,k) = \begin{cases} k-1 & \text{if } 2 \le k \le n+1, \\ 2n+1-k & \text{if } n+1 \le k \le 2n, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, for each $i \in \mathbb{N}$,

$$\varphi(n,i^2) = \begin{cases} i^2-1 & \text{if } 2 \le i \le I_1, \\ 2n+1-i^2 & \text{if } I_1 \le k \le I_2, \\ 0 & \text{otherwise,} \end{cases}$$

where $I_1 = \lfloor\sqrt{n+1}\rfloor$ and $I_2 = \left\lfloor\sqrt{2n}\right\rfloor$.

Now

$$\begin{aligned}
N(n) &= \sum_{i=1}^{\infty}\varphi(n,i^2) = \sum_{i=2}^{I_1}(i^2-1) + \sum_{i=I_1+1}^{I_2}(2n+1-i^2) \\
&= \sum_{i=1}^{I_1} i^2 - I_1 + (2n+1)(I_2-I_1) - \sum_{i=I_1+1}^{I_2} i^2 \\
&= 2\sum_{i=1}^{I_1} i^2 - \sum_{i=1}^{I_2} i^2 + 2n(I_2-I_1) + I_2 - 2I_1 \\
&= \frac{I_1(I_1+1)(2I_1+1)}{3} - \frac{I_2(I_2+1)(2I_2+1)}{6} \\
&\qquad + 2n(I_2-I_1) + I_2 - 2I_1 \\
&= \frac{\sqrt{n}\cdot\sqrt{n}\cdot 2\sqrt{n}}{3} - \frac{\sqrt{2n}\cdot\sqrt{2n}\cdot 2\sqrt{2n}}{6} + 2n(\sqrt{2n}-\sqrt{n}) + O(n) \\
&= n\sqrt{n}\left(\frac{2}{3} - \frac{2\sqrt{2}}{3} + 2\sqrt{2} - 2\right) + O(n) \\
&= \frac{4}{3}n\sqrt{n}\,(\sqrt{2}-1) + O(n)\,.
\end{aligned}$$

Thus,

$$\frac{N(n)}{n\sqrt{n}} = \frac{4}{3}\left(\sqrt{2}-1\right) + O\left(\frac{1}{\sqrt{n}}\right),$$

and therefore,

$$\lim_{n\to\infty}\frac{N(n)}{n\sqrt{n}} = \frac{4}{3}\left(\sqrt{2}-1\right),$$

as claimed.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

3159. [2006 : 305, 308] *Proposed by Mihály Bencze, Brasov, Romania.*

Let n be a positive integer, and let γ be Euler's constant. Prove that

$$\gamma - \frac{1}{48n^3} < 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln\left(n + \frac{1}{2} + \frac{1}{24n}\right) < \gamma - \frac{1}{48(n+1)^3}\,.$$

Solution by the proposer.

For each positive integer n, let

$$x_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln\left(n + \frac{1}{2} + \frac{1}{24n}\right) + \frac{1}{48n^3}\,.$$

Then $x_{n+1} - x_n = f(n)$, where

$$f(x) = \frac{1}{x+1} - \ln\left(x + \frac{3}{2} + \frac{1}{24(x+1)}\right) + \ln\left(x + \frac{1}{2} + \frac{1}{24x}\right) + \frac{1}{48(x+1)^3} - \frac{1}{48x^3}.$$

We have $f'(x) > 0$ for $x > 0$. [*Ed:* Using a computer algebra system, we get $f'(x) = \frac{2656x^6+10096x^5+15008x^4+10836x^3+3870x^2+652x+37}{16x^4(x+1)^4(24x^2+12x+1)(24x^2+60x+37)}$.] Furthermore, $\lim_{x\to\infty} f(x) = 0$. Therefore, $f(x) < 0$ for all $x > 0$, which implies that the sequence $\{x_n\}_{n=1}^{\infty}$ is strictly decreasing. Since $\lim_{n\to\infty} x_n = \gamma$, we must have $x_n > \gamma$ for all n. This proves the left inequality.

For each positive integer n, let

$$y_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln\left(n + \frac{1}{2} + \frac{1}{24n}\right) + \frac{1}{48(n+1)^3}.$$

Then $y_{n+1} - y_n = g(n)$, where

$$g(x) = \frac{1}{x+1} - \ln\left(x + \frac{3}{2} + \frac{1}{24(x+1)}\right) + \ln\left(x + \frac{1}{2} + \frac{1}{24x}\right) + \frac{1}{48(x+2)^3} - \frac{1}{48(x+1)^3}.$$

We have $g'(x) < 0$ for $x > 0$. [*Ed:* Using a computer algebra system, we get $g'(x) = -\frac{8864x^7+72336x^6+247520x^5+456204x^4+483110x^3+288492x^2+86997x+9472}{16x(x+1)^4(x+2)^4(24x^2+12x+1)(24x^2+60x+37)}$.] Furthermore, $\lim_{x\to\infty} g(x) = 0$. Therefore, $g(x) > 0$ for all $x > 0$, which shows that the sequence $\{y_n\}_{n=1}^{\infty}$ is strictly increasing. Since $\lim_{n\to\infty} y_n = \gamma$, we must have $y_n < \gamma$ for all n. This proves the right inequality.

Also solved by PAUL BRACKEN and N. NADEAU, University of Texas, Edinburg, TX, USA; and WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria. There was one incomplete solution.

3160. [2006 : 305, 308] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Let $\triangle ABC$ have altitude AD and orthocentre H. Let E be the mid-point of AD and M the mid-point of BC.

(a) If $AD = BC$, prove that $HM = HE$.

(b) Is the converse of (a) true?

I. *A composite of similar solutions by Roy Barbara, Lebanese University, Fanar, Lebanon; and Geoffrey A. Kandall, Hamden, CT, USA.*

Dealing with parts (a) and (b) together, we prove that $HM = HE$ if and only if $AD = BC$. We introduce coordinates with $D(0,0)$, $E(0,1)$, and $A(0,2)$ on the y-axis, while $B(b,0)$ and $C(c,0)$ define the x-axis for real numbers $c > b$. It follows that M is the point $\left(\frac{1}{2}(b+c),0\right)$ and the line CH, passing through C and perpendicular to AB, has the equation $y = \frac{1}{2}bx - \frac{1}{2}bc$. The y-intercept of CH is $H(0,-\frac{1}{2}bc)$. We thus have

$$\begin{aligned} HM^2 - HE^2 &= \left(\frac{b+c}{2}\right)^2 + \frac{b^2c^2}{4} - \left(1+\frac{bc}{2}\right)^2 \\ &= \frac{b^2+c^2}{4} - \frac{bc}{2} - 1 \\ &= \frac{1}{4}\left((b-c)^2 - 4\right) = \frac{1}{4}(BC^2 - AD^2)\,. \end{aligned}$$

Thus, $HM = HE$ if and only if $BC = AD$.

II. *A composite of similar solutions by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; and Titu Zvonaru, Comăneşti, Romania.*

Because $\triangle ABD \sim \triangle CHD$ (since $\angle BAD = 90^\circ - \angle ABD = \angle HCD$), we deduce that $\dfrac{AD}{CD} = \dfrac{BD}{HD}$. In terms of signed segments, this equality becomes

$$DA \cdot HD = DB \cdot DC\,.$$

We therefore have

$$\begin{aligned} HM^2 - HE^2 &= HD^2 + DM^2 - HE^2 \\ &= HD^2 + (DB+BM)^2 - (HD+DE)^2 \\ &= HD^2 + \left(DB + \tfrac{1}{2}BC\right)^2 - \left(HD + \tfrac{1}{2}DA\right)^2 \\ &= \tfrac{1}{4}(BC^2 - AD^2) - DA\cdot HD + DB(DB+BC) \\ &= \tfrac{1}{4}(BC^2 - AD^2) - DA\cdot HD + DB\cdot DC \\ &= \tfrac{1}{4}(BC^2 - AD^2)\,. \end{aligned}$$

Thus, $HM = HE$ if and only if $AD = BC$.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (a second solution); MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Big Rapids, MI, USA (2 solutions); JOEL SCHLOSBERG, Bayside, NY, USA; BOB SERKEY, Leonia, NJ, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

One of Konečný's solutions considered the family of all triangles ABC with fixed base BC for which $AD = BC$. The locus of orthocentres H as A moves along the line parallel to BC at the fixed distance of $AD = BC$ is a parabola whose focus is the mid-point M of BC, latus rectum is BC, and directrix is the locus of the mid-point E of AD. The equality of HM and HE is then equivalent to a basic property of conics: From any point on a parabola, the distance to the directrix equals the distance to the focus.

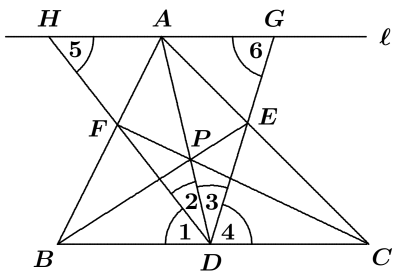
3161. [2006 : 305, 308] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Let D be a point on the side BC of $\triangle ABC$, and let P be an arbitrary point on the segment AD. Let BP meet AC at E and CP meet AB at F.

(a) If $AD \perp BC$, prove that $\angle BDF = \angle CDE$.

(b) Is the converse of (a) true?

Solution by Geoffrey A. Kandall, Hamden, CT, USA, modified by the editor.



Let ℓ be the line through vertex A parallel to the side BC. Let DE and DF meet ℓ at points G and H, respectively. Let 1, 2, 3, 4, 5, and 6 denote the angles BDF, FDA, EDA, CDE, AHD, and AGD, respectively. Clearly, $\angle 5 = \angle 1$ and $\angle 6 = \angle 4$.

First we show that $HA = AG$. Triangles AHF and BFD are similar, which implies that $\frac{AF}{FB} = \frac{HA}{BD}$. Likewise, triangles AGE and CDE are similar, yielding $\frac{CE}{EA} = \frac{DC}{AG}$. Applying Ceva's Theorem to $\triangle ABC$, we get

$$1 = \frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = \frac{HA}{BD} \cdot \frac{BD}{DC} \cdot \frac{DC}{AG} = \frac{HA}{AG}.$$

Thus, $HA = AG$.

(a) If $AD \perp BC$, then triangles DAH and DAG are congruent, implying that $\angle 1 = \angle 4$.

(b) If $\angle 1 = \angle 4$, then $\angle 5 = \angle 6$, which implies that $\triangle DGH$ is isosceles (with $DG = DH$). The median DA in $\triangle DGH$ is therefore also the altitude from D to GH. This makes $AD \perp \ell$ and thus also $AD \perp BC$.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; JOEL SCHLOSBERG, Bayside, NY, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

Konečný and Malikić pointed out that part (a) *of the problem is known; it had appeared, in almost identical form, as the fifth problem of the Canadian Mathematical Olympiad in 1994 with solution published in Crux* [1994 : 189]. *Konečný also supplied an earlier reference* (A Survey of Geometry, *Howard Eves, 1972, Allyn and Bacon, p.86*).

3162. [2006 : 306, 308] *Proposed by Eckard Specht, Otto-von-Guericke University, Magdeburg, Germany.*

Determine all integer solutions (x, y) of the equation

$$x^5 + y^7 = 2004^{1007} .$$

Solution by Kee-Wai Lau, Hong Kong, China.

We show that the equation has no integer solutions by considering residues modulo 71.

Since 71 is a prime and $(16, 71) = 1$, we have $(16^{14})^{70} \equiv 1 \pmod{71}$ by Fermat's Little Theorem. Hence,

$$\begin{aligned} 2004^{1007} &\equiv 16^{1007} = (16^{14})^{70}(16^{27}) \equiv 16^{27} = 4096^9 \\ &\equiv 49^9 \equiv 117649^3 \equiv 2^3 = 8 \; (\pmod{71} . \end{aligned}$$

With the help of a computer, we find that the quintic residues (mod 71) are

$$0,\ 1,\ 20,\ 23,\ 26,\ 30,\ 32,\ 34,\ 37,\ 39,\ 41,\ 45,\ 48,\ 51, \text{ and } 70$$

and the septic residues (mod 71) are

$$0,\ 1,\ 5,\ 14,\ 17,\ 25,\ 46,\ 54,\ 57,\ 66, \text{ and } 70.$$

It follows by tedious but straightforward calculations that the residues of $x^5 + y^7 \pmod{71}$ are precisely those k where $0 \le k \le 70$ such that $k \notin \{8, 10, 11, 60, 61, 63\}$.

Since the residue 8 is missing, we conclude that the given equation has no integer solutions.

Also solved (using essentially the same argument) by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; and the proposer. There was also an incomplete solution.

The proposer remarked that 71 is the smallest prime for which this proof works.

3163. [2006 : 394, 396] *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Calculate

$$\lim_{n\to\infty} \ln \left(\prod_{k=1}^{n} \left(\frac{k^2+n^2}{n^2} \right)^k \right)^{\frac{1}{n^2}} .$$

Composite of virtually identical solutions by those solvers identified by an asterisk beside their names in the list at the end.

Let $S_n = \ln \left(\prod_{k=1}^{n} \left(\frac{k^2+n^2}{n^2} \right)^k \right)^{\frac{1}{n^2}}$. Then

$$S_n = \frac{1}{n^2} \sum_{k=1}^{n} k \ln \left(\frac{k^2+n^2}{n^2} \right) = \frac{1}{n} \sum_{k=1}^{n} \frac{k}{n} \ln \left(1 + \frac{k^2}{n^2} \right) ,$$

a Riemann sum associated with the continuous function $f(x) = x\ln(1+x^2)$ and the regular partition $\{\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n}\}$ of the interval $[0, 1]$.

Hence,

$$\begin{aligned}\lim_{n\to\infty} S_n &= \int_0^1 x\ln(1+x^2)\,dx = \frac{1}{2}\int_1^2 \ln u\,du = \frac{1}{2}(u\ln u - u)\Big|_1^2 \\ &= \frac{1}{2}\big((2\ln 2 - 2) - (-1)\big) = \frac{1}{2}(-1 + 2\ln 2)\,.\end{aligned}$$

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