THE OLYMPIAD CORNER

No. 263

R.E. Woodrow

We begin this number of the *Corner* with selected problems from the Thai Mathematical Olympiad 2003. Thanks go to Christopher Small, Canadian Team Leader to the IMO in Athens, for collecting them for us.

THAI MATHEMATICAL OLYMPIAD 2003
Selected Problems

1. Triangle ABC has $\angle A = 70^\circ$ and $CA + AI = BC$, where I is the incentre of triangle ABC. Find $\angle B$.

2. Let $f : \mathbb{Q} \to \mathbb{Q}$, where $\mathbb{Q}$ is the set of all rational numbers, such that

$$f(x+y) \;=\; f(x) + f(y) + 2547$$

for all x, $y \in \mathbb{Q}$ and $f(2004) = 2547$. Find $f(2547)$.

3. Let a, b, and c be positive real numbers such that $a+b+c \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$. Prove that $a^3 + b^3 + c^3 \geq a + b + c$.

4. Let ABC be an equilateral triangle. Let A', B', and C' be points on the segments BC, CA, and AB, respectively. Suppose that $|AC'| = 2|CB'|$, $|BA'| = 2|AC'|$, $|CB'| = 2|BA|$, and $[ABC] = 126$. Find the area of the triangle enclosed by the lines AA', BB', and CC'.

5. Find all pairs (x, y) which satisfy the system of equations

$$\begin{aligned} x^{x+y} &= y^{xy}\,, \\ x^2 y &= 1\,. \end{aligned}$$

6. Let $ABCD$ be a convex quadrilateral. Prove that

$$[ABCD] \;\leq\; \tfrac{1}{4}\left(AB^2 + BC^2 + CD^2 + DA^2\right).$$

7. Define f on the set of rational numbers in the interval $[0, 1]$ as follows: $f(0) = 0$, $f(1) = 1$, and

$$f(x) \;=\; \begin{cases} \dfrac{f(2x)}{4} & \text{if } 0 < x < \dfrac{1}{2}\,, \\[2mm] \dfrac{3}{4} + \dfrac{f(2x-1)}{4} & \text{if } \frac{1}{2} \leq x < 1\,. \end{cases}$$

If we write x in base-2 representation as $x = (0.b_1 b_2 b_3 \ldots)_2$, find $f(x)$ in base-2 representation.

8. Find all primes p such that $p^2 + 2543$ has less than 16 distinct positive divisors.

9. Given a right triangle ABC with $\angle B = 90°$, let P be a point on the angle bisector of $\angle A$ inside ABC and let M be a point on the side $\overline{AB}$ (with $A \neq M \neq B$). Lines AP, CP, and MP intersect $\overline{BC}$, $\overline{AB}$, and $\overline{AC}$ at D, E, and N, respectively. Suppose that $\angle MPB = \angle PCN$ and $\angle NPC = \angle MBP$. Find $[APC]/[ACDE]$.

Next we look at two tests of the 25th Albanian Mathematical Olympiad for High Schools. Thanks again go to Christopher Small, Canadian Team Leader to the IMO in Athens, for collecting them.

25th ALBANIAN MATHEMATICAL OLYMPIAD FOR HIGH SCHOOLS

Test 1

1. There are 20 pupils in a village school. Any two of them have the same grandfather. Show that there exists a grandfather who has at least 14 grandchildren.

2. Let M, N, and P be the respective mid-points of sides BC, CA, and AB of triangle ABC, and let G be the intersection point of its medians. Prove that if $BN = \frac{\sqrt{3}}{2}AB$ and $BMGP$ is a cyclic polygon, then triangle ABC is equilateral.

3. Let x_k and y_k (for $k = 1, 2, \dots, n$) be positive real numbers that satisfy $kx_ky_k \geq 1$.

(a) Prove that $\sum\limits_{k=1}^{n} \dfrac{x_k - y_k}{x_k^2 + y_k^2} \leq \frac{1}{4}n\sqrt{n+1}$.

(b) When does equality hold in part (a)?

4. Find prime numbers p and q such that $p^2 - p + 1 = q^3$.

5. Find all pairs of positive integers (x, n) such that $x^{n+1} + 2^{n+1} + 1$ is divisible by $x^n + 2^n + 1$.

Test 2

1. Some people take part in a meeting. Every participant is acquainted with at most three people in the group, and if two participants are not acquainted, then they have a common acquaintance in the group.

(a) What is the maximal number of participants in this meeting?

(b) If there are three participants who are mutually acquainted with each other, what is the maximal number of participants in this meeting?

2. Prove the inequality

$$\frac{1}{\sqrt{a+\frac{1}{b}+0.64}}+\frac{1}{\sqrt{b+\frac{1}{c}+0.64}}+\frac{1}{\sqrt{c+\frac{1}{a}+0.64}} \geq 1.2\,,$$

where $a > 0$, $b > 0$, $c > 0$, and $abc = 1$.

3. Solve the following equation in integers:

$$y^2 \;=\; 1+x+x^2+x^3+x^4\,.$$

4. Prove that for any integer $n \geq 2$, the number $2^n - 1$ is not divisible by n.

5. In an acute-angled triangle ABC, let H be the orthocentre, and let d_a, d_b, and d_c be the distances from H to the sides BC, CA, and AB, respectively. Prove that $d_a + d_b + d_c \leq 3r$, where r is the radius of the incircle of triangle ABC.

To continue your return to problem-solving pleasures, we give the 11th Form of the Final Round of the 44th Ukrainian Mathematical Olympiad. Thanks again go to Christopher Small for collecting them for our use.

44th UKRAINIAN MATHEMATICAL OLYMPIAD
11th Form, Final Round

1. (V.M. Leifura) Solve the equation

$$\arcsin\lfloor\sin x\rfloor \;=\; \arccos\lfloor\cos x\rfloor\,,$$

where $\lfloor a\rfloor$ is the greatest integer not exceeding a.

2. (V.V. Lymanskiy) The acute-angled triangle ABC is given. Let O be the centre of its circumcircle. The perpendicular bisector of the side AC intersects the side AB and the line BC at the points P and Q, respectively. Prove that $\angle PQB = \angle PBO$.

3. (V.A. Yasinskiy) The edge SA of the tetrahedron $SABC$ is perpendicular to the plane ABC. Two different spheres σ_1 and σ_2 contain points A, B, and C. Both these spheres are tangent internally to a sphere σ centred at S. Let r_1 and r_2 be the radii of σ_1 and σ_2, respectively. Find the radius R of σ.

4. (V.A. Yasinskiy) Prove that there does not exist an integer $n > 1$ such that n divides $3^n - 2^n$.

5. (V.A. Yasinskiy) Given are 2004 points in the plane. They are the vertices of a convex polygon and no four of them are cyclic. A triangle having three of the points as its vertices is called *thick* if the other 2001 points lie inside its circumcircle, and it is called *thin* if the other points lie outside its circumcircle. Prove that the number of *thick* triangles is equal to the number of *thin* triangles.

6. (O.O. Malakhov) Find the sum of the real roots of the equation

$$x + \frac{x}{\sqrt{x^2-1}} = 2004 .$$

7. (V.M. Radchenko) Does there exist a function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x^2y + f(x+y^2)) = x^3 + y^3 + f(xy)$ for all $x, y \in \mathbb{R}$.

8. (V.A. Yasinskiy) Let a, b, and c be positive real numbers such that $abc \geq 1$. Prove that $a^3 + b^3 + c^3 \geq ab + bc + ca$.

9. (V.A. Yasinskiy) A convex 2004-gon has vertices $A_1, A_2, \ldots, A_{2004}$. Is it possible to colour each of its sides and its diagonals with one of 2003 different colours in such a way that the following two conditions hold?

(i) There are 1002 segments of each colour.

(ii) If an arbitrary vertex and two arbitrary colours are given, then, starting from this vertex and using the segments of these two colours exclusively, one can visit every other vertex only once.

10. (I.P. Nagel) Let ω be the inscribed circle of the triangle ABC. Let L, N, and E be the points of tangency of ω with the sides AB, BC, and CA, respectively. Lines LE and BC intersect at the point H, and lines LN and AC intersect at the point J (all the points H, J, N, E lie on the same side of the line AB). Let O and P be the mid-points of the segments EJ and NH, respectively. Find $S(HJNE)$ if $S(ABOP) = u^2$ and $S(COP) = v^2$. (Here $S(\mathcal{F})$ is the area of figure $\mathcal{F}$).

We turn to our files of readers' comments and solutions to problems given in the September 2006 number of the *Corner*. The first group are for problems of the Belarus Mathematical Olympiad 2003, given in [2006 : 277].

Comment by Pierre Bornsztein, Maisons-Laffitte, France.

These six problems are from the IMO Short-list 2001.
Solutions can be found at

`http://www.mathlinks.ro/Forum/viewtopic.php?t-15624`

or in D Djukić, V. Janković, I. Matić, N. Petrović, *The IMO Compendium*, Springer, p.675.

3. Find all functions f from the real numbers to the real numbers such that, for any real numbers x and y,

$$f(xy)\big(f(x) - f(y)\big) \;=\; (x - y)f(x)f(y)\,.$$

Solved by Michel Bataille, Rouen, France.

Clearly, the zero function is a solution.

We will show that the non-zero solutions are the functions $\Phi_{a,K}$ defined by

$$\Phi_{a,K}(x) \;=\; \begin{cases} ax & \text{if } x \in K\,, \\ 0 & \text{if } x \notin K\,, \end{cases}$$

where K is a subgroup of the multiplicative group $\mathbb{R}^* = \mathbb{R}\setminus\{0\}$ and $a \in \mathbb{R}^*$.

Consider $a \in \mathbb{R}^*$ and a subgroup K of $\mathbb{R}^*$. We show that $f = \Phi_{a,K}$ satisfies

$$f(xy)\big(f(x) - f(y)\big) \;=\; (x - y)f(x)f(y) \tag{1}$$

for all x, $y \in \mathbb{R}$.

If x, $y \in K$, then $xy \in K$ and (1) holds since it can be rewritten as $axy(ax - ay) = (x - y)ax \cdot ay$.

If x, $y \notin K$, then $\Phi(x) = \Phi(y) = 0$ and (1) is true.

If, say, $x \in K$, $y \notin K$, certainly $xy \notin K$ (otherwise $y = xy \cdot \dfrac{1}{x}$ would be in K). Thus, $\Phi(xy) = \Phi(y) = 0$ and (1) holds.

Conversely, let f be any function from $\mathbb{R}$ to $\mathbb{R}$ satisfying (1), and assume that f is not the zero function. Then $f(x_0) \neq 0$ for some $x_0 \in \mathbb{R}$. Taking $x = 1$ and $y = 0$ in (1) yields $f(0) = 0$ (hence, $x_0 \neq 0$). Then $y = 1$ gives

$$f(x)\big(f(x) - ax\big) \;=\; 0\,, \tag{2}$$

where we set $a = f(1)$. This relation (2) with $x = x_0$ shows that $a \in \mathbb{R}^*$ and, more generally, that $f(x) = ax$ if $f(x) \neq 0$.

Now, let

$$K \;=\; \{x \in \mathbb{R}^* \mid f(x) = ax\} \;=\; \{x \in \mathbb{R} \mid f(x) \neq 0\}\,.$$

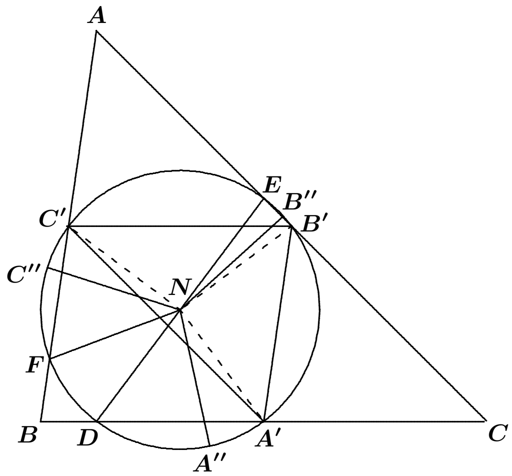
Clearly $1 \in K$. If x_1, $x_2 \in K$ with $x_1 \neq x_2$, then $f(x_1) = ax_1$ and $f(x_2) = ax_2$. Then (1) with $x = x_1$ and $y = x_2$ gives $f(x_1x_2) = ax_1x_2$; hence, $x_1x_2 \in K$. Similarly, (1) with $x = x_1$ and $y = 1/x_1$ shows that $1/x_1 \in K$ and, lastly, (1) with $x = x_1^2$ and $y = 1/x_1$ gives $x_1^2 \in K$. We have proved that K is a subgroup of $\mathbb{R}^*$ and that $f = \Phi_{a,K}$. This completes the proof.

The next block of material is for the Problems to Select Indian IMO Team 2003 given in [2006 : 278–279].

1. Let A', B', C' be the mid-points of the sides BC, CA, AB, respectively, of an acute non-isoceles triangle ABC, and let D, E, F be the feet of the altitudes through the vertices A, B, C on these sides, respectively. Consider the arc DA' of the nine-point circle of triangle ABC lying outside the triangle. Let the point of trisection of this arc closer to A' be A''. Define analogously the points B'' (on arc EB') and C'' (on arc FC'). Show that triangle $A''B''C''$ is equilateral.

Solved by Michel Bataille, Rouen, France; and D.J. Smeenk, Zaltbommel, the Netherlands. First we give Bataille's solution.

Let N be the centre of the nine-point circle $\mathcal{N}$ of triangle ABC. The perpendicular bisectors of $B'C'$ and $A'D$ coincide (both pass through N and are perpendicular to BC); hence, A' and D are symmetrical in the diameter of $\mathcal{N}$ perpendicular to BC. It follows that the mid-point of the (smaller) arc $B'C'$ and the mid-point of the arc DA' lying outside the triangle are diametrically opposite.



Without loss of generality, we may suppose that $\mathcal{N}$ is the unit circle in the complex plane, with $\triangle A'B'C'$ positively oriented (see figure). We may even suppose that the angles $\alpha = \angle B'A'C'$, $\beta = \angle C'B'A'$, and $\gamma = \angle A'C'B'$ satisfy $\frac{\pi}{2} > \beta > \alpha > \gamma$ and that the complex affix of B' is 1. It is readily seen that the affixes of C' and A' are $e^{2i\alpha}$ and $e^{-2i\gamma} = e^{2i(\pi-\gamma)}$, respectively. The affix of the mid-point of the smaller arc $B'C'$ is $e^{i\alpha}$; hence, the affix of the mid-point of the arc DA' is $e^{i(\pi+\alpha)}$. Since $\pi + \alpha < 2\pi - 2\gamma$ (note that $\alpha + 2\gamma < \alpha + \beta + \gamma - \pi$), we find D, A'', and A' in that order on the circle positively oriented. It follows that the affix of A'' is

$$e^{i(\pi+\alpha+(2\pi-2\gamma-\pi-\alpha)/3)} = e^{4\pi i/3} \cdot e^{2i(\alpha-\gamma)/3} .$$

In a similar way, we find that the affix of B'' is $e^{2i(\alpha-\gamma)/3}$ and the affix of C'' is $e^{2\pi i/3} \cdot e^{2i(\alpha-\gamma)/3}$. Thus, the affixes of B'', C'', and A'' are of the form $e^{i\theta}$, $e^{i\theta} \cdot e^{2\pi i/3}$, and $e^{i\theta} \cdot e^{4\pi i/3}$; whence, $\triangle A''B''C''$ is equilateral.

Next we give Smeenk's approach.

Set $a = BC$, $b = CA$, $c = AB$, $\alpha = \angle CAB$, $\beta = \angle ABC$, and $\gamma = \angle BCA$. Without loss of generality, we may assume that $\beta > \alpha > \gamma$. We first note that $BA' = \frac{1}{2}a$ and $BD = c\cos\beta$. Then

$$DA' \;=\; BA' - BD \;=\; \tfrac{1}{2}a - c\cos\beta \;=\; R\sin(\beta-\gamma)\,,$$

where R is the circumradius of $\triangle ABC$. Similarly, $EB' = R\sin(\alpha-\gamma)$.

Let N be the centre of the nine-point circle of $\triangle ABC$. Then we have $\angle DNA' = 2(\beta-\gamma)$ and $\angle ENB' = 2(\alpha-\gamma)$. In $\triangle NA'B'$ we have $\angle A'NB' = 2\gamma$. Therefore,

$$\angle A''NB'' \;=\; 2\gamma + \tfrac{2}{3}(\beta-\gamma) + \tfrac{2}{3}(\alpha-\gamma) \;=\; \tfrac{2}{3}(\alpha+\beta+\gamma) \;=\; 120^\circ\,.$$

In the same way, $\angle B''NC'' = \angle C''NA'' = 120^\circ$. Thus, $\triangle A''B''C''$ is equilateral.

2. Find all triples (a,b,c) of positive integers such that

(i) $a \le b \le c$;

(ii) $\gcd(a,b,c) = 1$; and

(iii) $a^3+b^3+c^3$ is divisible by each of the numbers a^2b, b^2c, c^2a.

Solution par Pierre Bornsztein, Maisons-Laffitte, France.

Soit (a,b,c) un tel triplet.

On remarque d'abord que si p premier divise a et b, il divise a^2b et donc $a^3+b^3+c^3$ ainsi que a^3+b^3. Par suite, p divise c^3 et donc c, ce qui contredit (ii). Ainsi, $\gcd(a,b) = 1$. De même, $\gcd(b,c) = \gcd(c,a) = 1$.

On en déduit que a^2, b^2 et c^2 sont deux à deux premiers entre eux. Puisqu'ils divisent chacun $a^3+b^3+c^3$, cela assure que $a^2b^2c^2$ divise $a^3+b^3+c^3$. On a alors $3c^3 \ge a^3+b^3+c^3 \ge a^2b^2c^2$, d'où

$$3c \;\ge\; a^2b^2\,. \qquad (1)$$

D'autre part, c^2 divise a^3+b^3, donc

$$c^2 \;\le\; a^3+b^3 \;\le\; 2b^3\,. \qquad (2)$$

En combinant (1) et (2), il vient $108c^3 \ge 4a^6b^6 \ge a^6c^4$ et donc $a \le c \le 108/a^6$, et enfin $a^7 \le 108$. Cela entraine $a = 1$.

Si $b = 1$, alors c^2 divise $a^3+b^3 = 2$, d'où $c = 1$. Réciproquement, le triplet $(1,1,1)$ est clairement une solution.

Si $b > 1$, alors $c > b > 1$ (sans quoi, on aurait $c = b$, en contradiction avec $\gcd(b,c) = 1$). Et donc $c^3 > b^3 + 1$. Il vient alors :

$$2c^3 \;>\; 1+b^3+c^3 \;=\; a^3+b^3+c^3 \;\ge\; b^2c^2\,,$$

d'où $c > \frac{1}{2}b^2$. En combinant avec (2), il vient $2b^3 > \frac{1}{4}b^4$, ou encore $b < 8$. On vérifie alors à la main que la seule solution est $b = 2$ et $c = 3$.

Finalement, les solutions sont $(1, 1, 1)$ et $(1, 2, 3)$.

3. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that, for all x, y in $\mathbb{R}$, we have

$$f(x+y) + f(x)f(y) \;=\; f(x) + f(y) + f(xy)\,. \tag{1}$$

Solved by Michel Bataille, Rouen, France; and Li Zhou, Polk Community College, Winter Haven, FL, USA. We give the write-up by Zhou, modified by the editor.

Clearly, $f \equiv 0$, $f \equiv 2$, and $f(x) = x$ are solutions. We show that they are the only solutions.

Setting $x = 0 = y$ in (1), we get $\bigl(f(0)\bigr)^2 = 2f(0)$. If $f(0) = 2$, then, by letting $y = 0$ in (1), we see that $f(x) = 2$ for all $x \in \mathbb{R}$; that is, $f \equiv 2$.

Suppose therefore that $f(0) = 0$. Let $a = f(1)$. Setting $x = 1$ and $y = -1$, we get $af(-1) = a + 2f(-1)$; that is, $f(-1) = a/(a-2)$. Now successively substituting $(x-1, 1)$, $(-x+1, -1)$, and $(-x, 1)$ for (x, y) in (1), we get

$$f(x) + (a-2)f(x-1) \;=\; a\,, \tag{2}$$

$$f(-x) + \frac{2}{a-2}f(-x+1) - f(x-1) \;=\; \frac{a}{a-2}\,, \tag{3}$$

$$f(-x+1) + (a-2)f(-x) \;=\; a\,. \tag{4}$$

Eliminating $f(x-1)$ and $f(-x+1)$ in (2), (3), and (4) gives

$$f(x) - (a-2)f(-x) \;=\; 0\,. \tag{5}$$

Replacing x by $-x$ in (5), we obtain

$$f(-x) - (a-2)f(x) \;=\; 0\,. \tag{6}$$

If $a \notin \{1,\, 3\}$, then by eliminating $f(-x)$ in (5) and (6), we see that $f(x) = 0$ for all $x \in \mathbb{R}$; that is, $f \equiv 0$.

If $a = 3$, then (2) gives $f(x) = 3 - f(x-1)$ for all $x \in \mathbb{R}$. Hence, $f(2) = 3 - f(1) = 0$ and $f\bigl(\frac{5}{2}\bigr) = 3 - f\bigl(\frac{3}{2}\bigr) = f\bigl(\frac{1}{2}\bigr)$. On the other hand, by substituting $\bigl(2, \frac{1}{2}\bigr)$ for (x, y) in (1), we get $f\bigl(\frac{5}{2}\bigr) = f\bigl(\frac{1}{2}\bigr) + f(1) = f\bigl(\frac{1}{2}\bigr) + 3$, a contradiction.

Finally, consider $a = 1$. Then (2) gives $f(x) = f(x-1) + 1$ for all $x \in \mathbb{R}$. By induction, $f(x+n) = f(x) + n$, for all $x \in \mathbb{R}$ and $n \in \mathbb{Z}$. In particular, $f(n) = f(0+n) = f(0) + n = n$ for all $n \in \mathbb{Z}$. Substituting n for y in (1), we obtain $nf(x) = f(nx)$ for all $x \in \mathbb{R}$ and $n \in \mathbb{Z}$. Hence, if $r = m/n \in \mathbb{Q}$ and $x \in \mathbb{R}$, then

$$mf(x) \;=\; f(mx) \;=\; f(n \cdot rx) \;=\; nf(rx)\,;$$

that is, $f(rx) = rf(x)$. In particular, $f(r) = f(r \cdot 1) = rf(1) = r$. Setting $y = r$ in (1) gives $f(x + r) = f(x) + r$. Also, setting $y = x$ in (1), we get $\big(f(x)\big)^2 = f(x^2)$. Thus, $f(x) \geq 0$ if $x \geq 0$. Since $f(-x) = -f(x)$, we have $f(x) \leq 0$ if $x \leq 0$.

Now, let $x \in \mathbb{R}$ be fixed. If $r \in \mathbb{Q}$ and $r < x$, then

$$f(x) \;=\; f(x - r + r) \;=\; f(x - r) + r \;\geq\; r\,,$$

since $f(x - r) \geq 0$. It follows that $f(x) \geq x$. Similarly, $f(x) \leq r$ for all $r \in \mathbb{Q}$ such that $r > x$, which implies that $f(x) \leq x$. Thus, $f(x) = x$.

7. Let $P(x)$ be a polynomial with integer coefficients such that $P(n) > n$ for all positive integers n. Suppose that for each positive integer m, there is a term in the sequence $P(1)$, $P\big(P(1)\big)$, $P\big(P\big(P(1)\big)\big)$, $\ldots$ which is divisible by m. Show that $P(x) = x + 1$.

Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Li Zhou, Polk Community College, Winter Haven, FL, USA. We give Zhou's write-up.

We will use the notation $P^{(i)}(1)$ for the i^{th} term in the sequence $P(1)$, $P\big(P(1)\big)$, $\ldots$. The condition $P(n) > n$ implies that $\deg(P) \geq 1$ and the leading coefficient of P is positive.

If $P(x) = x + b$, then $1 + b = P(1) > 1$, which implies that $b \geq 1$. It is easy to see that $P(x) = x + 1$ satisfies all the conditions. If $b \geq 2$, then $P(1) \equiv 1 \pmod{b}$ and, by induction, $P^{(i)}(1) \equiv 1 \pmod{b}$ for all $i \geq 1$.

If $P(x) = 2x + b$, then $2 + b = P(1) > 1$, which implies that $b \geq 0$. If $b = 0$, then, by induction, $P^{(i)}(1) = 2^i$ for all $i \geq 1$.

We consider together all the remaining cases: (i) $P(x) = 2x + b$ with $b \geq 1$; (ii) $P(x) = ax + b$ with $a \geq 3$; and (iii) $\deg(P) \geq 2$. In these cases, there exists $N \in \mathbb{N}$ such that $P(n) > 2n$ for all $n \geq N$. Since $1 < P(1) < P(P(1)) < P(P(P(1))) < \cdots$, there exists $k \in \mathbb{N}$ such that $P^{(k)}(1) \geq N$. Let $r = P^{(k)}(1)$ and $m = P^{(k+1)}(1) - P^{(k)}(1)$. Then $r \geq N$ and $m = P(r) - r > r$. For $1 \leq i \leq k$, we have $1 < P^{(i)}(1) \leq r < m$; thus, m does not divide any $P^{(i)}(1)$ for $1 \leq i \leq k$. Moreover, note that $P^{(k+1)}(1) = m + r \equiv r \pmod{m}$. Assume as an induction hypothesis that $P^{(i)}(1) \equiv r \pmod{m}$ for some $i \geq k + 1$. Then

$$\begin{aligned} P^{(i+1)}(1) &= P(P^{(i)}(1)) \;\equiv\; P(r) \;=\; P(P^{(k)}(1)) \\ &= P^{(k+1)}(1) \;\equiv\; r \pmod{m}\,. \end{aligned}$$

Hence, $P^{(i)}(1) \equiv r \pmod{m}$ for all $i \geq k + 1$.

8. Let ABC be a triangle, and let r, r_1 ,r_2, r_3 denote its inradius and the exradii opposite the vertices A, B, C, respectively. Suppose $a > r_1$, $b > r_2$, $c > r_3$. Prove that

(a) triangle ABC is acute, (b) $a + b + c > r + r_1 + r_2 + r_3$.

Solution by Vedula N. Murty, Dover, PA, USA, modified by the editor.

(a) Let s denote the semipermieter of $\triangle ABC$. From the known formula $\tan(A/2) = r_1/s$ and the given inequality $a > r_1$, we obtain $\tan(A/2) < a/s < 1$. Similarly, $\tan(B/2) < 1$ and $\tan(C/2) < 1$. Then $A < \frac{\pi}{2}$, $B < \frac{\pi}{2}$, and $C < \frac{\pi}{2}$. Hence, $\triangle ABC$ is acute.

(b) Since the triangle ABC is acute, we have the known inequality $s > r + 2R$, where R is the circumradius of $\triangle ABC$. We also have $r_1 + r_2 + r_3 = r + 4R$. Hence,

$$r + r_1 + r_2 + r_3 \;=\; 2r + 4R \;<\; 2s \;=\; a + b + c\,.$$

9. Let n be a positive integer and $\{A, B, C\}$ a partition of $\{1, 2, \ldots, 3n\}$ such that $|A| = |B| = |C| = n$. Prove that there exist $x \in A$, $y \in B$, $z \in C$ such that one of x, y, z is the sum of the other two.

Solution par Pierre Bornsztein, Maisons-Laffitte, France.

On dira que le triplet (a, b, c) est *bon* lorsque $a \in A$, $b \in B$, $c \in C$ et l'un des nombres est la somme des deux autres. Sans perte de généralité, on peut supposer que $1 \in A$, et que le plus petit nombre, disons k, qui n'est pas dans A est dans B. Supposons qu'il n'existe pas de bon triplet.

On commence par montrer que pour tout $x \in C$, on a $x - 1 \in A$. En effet, s'il existe $x \in C$ tel que $x - 1 \notin A$ alors, puisque $(1, x - 1, x)$ ne doit pas être bon, c'est donc que $x - 1 \notin B$, et ainsi que $x - 1 \in C$. En particulier, $x - 1 > k$. Mais comme aucun des deux triplets $(x - k, k, x)$ et $(k - 1, x - k, x - 1)$ n'est bon, c'est que $x - k \notin A$ et $x - k \notin B$. Et donc $x - k \in C$. De même, en considérant les triplets $(x - k - 1, k, x - 1)$ et $(1, x - k - 1, x - k)$, on déduit que $x - k - 1 \in C$.

On peut alors recommencer ce raisonnement, et prouver par récurrence que, pour tout $i \geq 0$, les nombres $x - ik$ et $x - ik - 1$, tant qu'ils sont strictement positifs, appartiennent tous les deux à C. Mais pour un i bien choisi, un de ces nombres est nécessairement inférieur ou égal à k, ce qui implique qu'il appartienne à A ou à B. C'est une contradiction.

On pose $C = \{c_1, \ldots, c_n\}$. D'après la propriété précédente, et puisque $|A| = |C| = n$, c'est donc que $A = \{c_1 - 1, \ldots, c_n - 1\}$. Mais pour tout i, on a $c_i > k > 1$ donc $c_i - 1 > 1$, ce qui contredit que $1 \in A$.

Remarque. On peut prouver que si A, B et C forment une partition de $\{1, 2, \ldots, n\}$ avec $|A|$, $|B|$, $|C| > \frac{1}{4}n$, alors il existe un bon triplet.

Référence.

[1] G.J. Székely, *Contests in higher mathematics*, Springer, problem C-22.

10. Let n be a positive integer greater than 1, and let p be a prime such that n divides $p - 1$ and p divides $n^3 - 1$. Prove that $4p - 3$ is a square.

Comment by Pierre Bornsztein, Maisons-Laffitte, France.

This problem is similar to problem #4 of the 3rd Czeck-Polish-Slovak mathematical competition. A solution appears in [2006 : 375–376].

Next we turn to readers' solutions to problems of the German Mathematical Olympiad 2003 given at [2006 : 279–280].

1. Determine all pairs (x, y) of real numbers x, y which satisfy

$$\begin{aligned} x^3 + y^3 &= 7\,, \\ xy(x+y) &= -2\,. \end{aligned}$$

Solved by Pierre Bornsztein, Maisons-Laffitte, France; Geoffrey A. Kandall, Hamden, CT, USA; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Bornsztein's write-up.

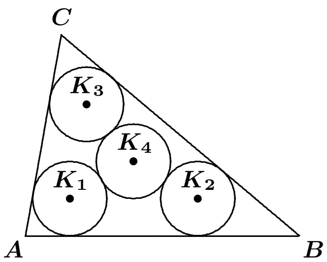
Assume that (x, y) is such a pair. Then

$$(x+y)^3 \;=\; x^3 + y^3 + 3xy(x+y) \;=\; 1\,,$$

which leads to $x + y = 1$. Thus, $xy = -2$. It follows that x and y are roots of $X^2 - X - 2 = 0$. Therefore, $(x, y) = (-1, 2)$ or $(2, -1)$, which are solutions of the problem.

2. In the interior of a triangle ABC, circles K_1, K_2, K_3, and K_4 of the same radii are defined such that K_1, K_2, and K_3 touch two sides of the triangle and K_4 touches K_1, K_2, and K_3, as shown in the figure.

Prove that the centre of K_4 is located on the line through the incentre and the circumcentre.



Solved by Michel Bataille, Rouen, France; and D.J. Smeenk, Zaltbommel, the Netherlands. We give Bataille's solution.

Let O_i be the centre of the circle K_i (for $i = 1, 2, 3, 4$), and let ρ be the common radius of the circles. Let γ (centre I, radius r) and Γ (centre O, radius R) denote the incircle and the circumcircle of $\triangle ABC$, respectively.

Circle K_1 is the image of γ under the homothety with centre A and scale factor $k = \rho/r$; hence, $\overrightarrow{AO_1} = k\overrightarrow{AI}$, so that $\overrightarrow{IO_1} = (1-k)\overrightarrow{IA}$. Similarly, $\overrightarrow{IO_2} = (1-k)\overrightarrow{IB}$ and $\overrightarrow{IO_3} = (1-k)\overrightarrow{IC}$. Therefore, $\triangle O_1O_2O_3$ is the image of $\triangle ABC$ under the homothety with centre I and factor $1-k$. It follows that the circumcentre of $\triangle O_1O_2O_3$, namely O_4, is the image of O through this homothety. As a result, O_4, I, and O are collinear, as required.

Note. The circumradius 2ρ of $\triangle O_1O_2O_3$ satisfies $2\rho = \left(1 - \frac{\rho}{r}\right)R$. Thus, $\rho = \dfrac{rR}{2r+R}$, a useful result when drawing the figure.

3. The caterpillar *Nummersatt* is sitting in the middle square of an $N \times N$ board, where N is an odd integer with $N \geq 3$. The other squares of the board each contain a positive integer, and all of these integers are different. Nummersatt wants to find a way off the board. The caterpillar can move only between adjacent squares (squares having a common side), or off the board from one of the outermost squares, having once reached such a square. On reaching a new square, Nummersatt has to eat the number on that square. The number n weighs $\frac{1}{n}$ kg, and Nummersatt cannot eat more than 2 kg.

Decide whether numbers can be distributed on the board so that there is no way off the board for Nummersatt

(a) for $N = 2003$, (b) for all odd integers $N \geq 3$.

Solved by Pierre Bornsztein, Maisons-Laffitte, France.

Nous allons prouver que, dans le cas général, Nummersatt peut toujours sortir du tableau sans manger plus de $0,9$ kg. Pour cela, on va utiliser la méthode probabiliste.

Tout d'abord, on remarque que, quitte à augmenter le poid avalé, on peut supposer que les entiers utilisés sont 1, 2, $\ldots$, N^2.

On identifie chaque case avec son centre de sorte que, si $N = 2k+1$, chaque case est un point du réseau des points entiers à coordonnées dans $I = \{-k, -k+1, \ldots, k-1, k\}$. Le carré central est $(0,0)$.

On va considérer les chemins partant de $(0,0)$ et sortant du réseau en ne passant que d'un point (x,y) à un point voisin (x',y') tel que $|x'| + |y'| > |x| + |y|$ (par exemple, si x, $y > 0$ cela n'autorise qu'un déplacement vers $(x+1,y)$ ou vers $(x,y+1)$). En particulier, un tel chemin mène nécessairement vers la sortie en un nombre fini d'étapes et sans boucle. Le long d'un tel chemin, Nummersatt se déplace en respectant les conditions de l'énoncé, celle indiquée ci-dessus et les contraintes probabilistes suivantes (il est vivement conseillé de faire un dessin) :

- De $(0,0)$, il choisit un des points $(-1,0)$, $(1,0)$, $(0,1)$, and $(0,-1)$ de façon équiprobable.
- S'il est en $(n,0)$, avec $n \geq 1$, il va en $(n,1)$ ou en $(n,-1)$ avec une probabilité de $\dfrac{1}{2(n+1)}$ dans chacun des cas, et va en $(n+1,0)$ avec une probabilité de $\dfrac{n}{n+1}$.
- S'il est en $(n-p,p)$, avec $n > p \geq 1$, il va en $(n-p+1,p)$ avec une probabilité de $\dfrac{2(n-p)+1}{2(n+1)}$ et en $(n-p,p+1)$ avec une probabilité de $\dfrac{2p+1}{2(n+1)}$.
- S'il est en $(0,n)$, avec $n \geq 1$, il va en $(1,n)$ ou en $(-1,n)$ avec une probabilité de $\dfrac{1}{2(n+1)}$ dans chacun des cas, et va en $(0,n+1)$ avec une probabilité de $\dfrac{n}{n+1}$.

Les autres cas se traitent de façon symétrique (par rapport à l'un des axes de coordonnées ou par rapport à $(0,0)$) par rapport à la situation qui vient d'être décrite pour les points (x,y) avec $x,\ y \geq 0$. Il est alors facile de vérifier que si l'on note $p(x,y)$ la probabilité que Nummersatt passe par le point $(x,y) \neq (0,0)$, alors $p(x,y) = \dfrac{1}{4(|x|+|y|)}$.

Soit X la variable aléatoire égale au poid total mangé par Nummersatt au cours de son périple. On note $E(X)$ son espérance mathématique.

Pour $(x,y) \neq (0,0)$, on note $w(x,y)$ le poids attribué au point (x,y) (si le nombre accroché à (x,y) est n, on a donc $w(x,y) = 1/n$. De plus, on convient, pour simplifier les sommations qui suivent, que $w(x,y) = 0$ lorsque (x,y) ne fait pas partie du réseau fini considéré). On note que pour un point (x,y) du réseau considéré, on a $|x|+|y| \leq 2k$.

Alors :

$$\begin{aligned}
E(X) &= \sum_{(x,y)\neq(0,0)} p(x,y)w(x,y) \\
&= \sum_{n=1}^{2k}\left[\sum_{|x|+|y|=n} p(x,y)w(x,y)\right] = \sum_{n=1}^{2k}\left[\frac{1}{4n}\sum_{|x|+|y|=n} w(x,y)\right] \\
&\leq \frac{1}{4}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)+\frac{1}{8}\left(\frac{1}{5}+\frac{1}{6}+\cdots+\frac{1}{12}\right) \\
&\qquad +\frac{1}{12}\left(\frac{1}{13}+\cdots+\frac{1}{24}\right)+\cdots \\
&= \frac{1}{4}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)+\frac{1}{8}\left(\frac{1}{5}+\frac{1}{6}+\cdots+\frac{1}{12}\right)+R,
\end{aligned}$$

où

$$R = \sum_{n=3}^{2k}\left[\frac{1}{4n}\sum_{i=1}^{4n}\frac{1}{2n^2-2n+i}\right].$$

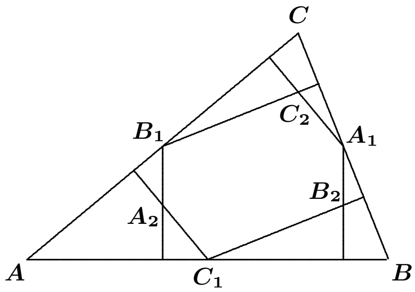
On a $\frac{1}{4}(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4})+\frac{1}{8}(\frac{1}{5}+\frac{1}{6}+\cdots+\frac{1}{12}) = \frac{143771}{221760} < 0,65$ et

$$\begin{aligned}
R &\leq \sum_{n=3}^{2k}\left[\frac{1}{4n}\sum_{i=1}^{4n}\frac{1}{2n^2-2n}\right] = \sum_{n=3}^{2k}\left(\frac{1}{4n}\right)\left(\frac{4n}{2n^2-2n}\right) \\
&= \sum_{n=3}^{2k}\frac{1}{2n(n-1)} = \frac{1}{2}\sum_{n=3}^{2k}\left(\frac{1}{n-1}-\frac{1}{n}\right) = \frac{1}{2}\left(\frac{1}{2}-\frac{1}{2k}\right) < \frac{1}{4}.
\end{aligned}$$

Ainsi $E(X) < 0,9$.

On en déduit que, parmi les chemins considérés, il en existe un pour lequel le poids total mangé par Nummersatt ne dépasse pas $0,9$ kg.

4. Let A_1, B_1, and C_1 be the mid-points of the sides of the acute-angled triangle ABC. The 6 lines through these points perpendicular to the other sides meet in the points A_2, B_2, and C_2, as shown in the figure. Prove that the area of the hexagon $A_1C_2B_1A_2C_1B_2$ equals half of the area of $\triangle ABC$.



Solved by Michel Bataille, Rouen, France; Geoffrey A. Kandall, Hamden, CT, USA; and D.J. Smeenk, Zaltbommel, the Netherlands. We give Bataille's write-up.

Denote the area of $\triangle XYZ$ by $[XYZ]$. Since $\triangle A_1B_1C_1$ is the image of $\triangle ABC$ under the homothety with centre at the centroid and factor $-\frac{1}{2}$, we have $[A_1B_1C_1] = \frac{1}{4}[ABC]$. Similarly, using the homothety h with centre A and factor $\frac{1}{2}$, we have $[AB_1C_1] = \frac{1}{4}[ABC]$.

Now, let O and H be the circumcentre and orthocentre of $\triangle ABC$, respectively. Note that these points are interior to the acute-angled $\triangle ABC$. Letting h_a be the length of the altitude from A to BC in $\triangle ABC$, we have

$$[BHC] = \tfrac{1}{2} \cdot BC \cdot (h_a - AH) = [ABC] - \tfrac{1}{2}BC \cdot AH\,.$$

Using the well-known relation $AH = 2OA_1$, we deduce that

$$[BHC] = [ABC] - BC \cdot OA_1 = [ABC] - 2[OBC]\,.$$

Observing that $h(H) = A_2$ (where h is the homothety defined above), we obtain

$$[B_1A_2C_1] = \tfrac{1}{4}[BHC] = \tfrac{1}{4}[ABC] - \tfrac{1}{2}[OBC]\,.$$

In the same way, we can show that $[C_1B_2A_1] = \frac{1}{4}[ABC] - \frac{1}{2}[OCA]$ and $[A_1C_2B_1] = \frac{1}{4}[ABC] - \frac{1}{2}[OAB]$. It follows that

$$\begin{aligned}[A_1C_2B_1A_2C_1B_2] &= [A_1B_1C_1] + [B_1A_2C_1] + [C_1B_2A_1] + [A_1C_2B_1] \\ &= \tfrac{1}{4}[ABC] + \tfrac{3}{4}[ABC] \\ &\qquad - \tfrac{1}{2}([OBC] + [OCA] + [OAB]) \\ &= [ABC] - \tfrac{1}{2}[ABC] = \tfrac{1}{2}[ABC]\,,\end{aligned}$$

as required.

5. If n is a positive integer, let $a(n)$ be the smallest positive number for which $(a(n))!$ is divisible by n. Determine all positive integers n satisfying

$$\frac{a(n)}{n} = \frac{2}{3}\,.$$

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Wang's write-up.

The only solution is $n = 9$. Note first that the given condition is simply $3a(n) = 2n$, which implies $3 \mid n$. Setting $n = 3k$, we have $a(3k) = 2k$. Clearly, $k \neq 1$ since $a(3) = 3$. For $k \geq 2$, we have $3k \mid (2k)!$, since $(2k)! = (2k) \cdots 2 \cdot 1$ contains the separate factors k and 3 even if $k = 3$. However, if $k > 3$, then $(2k-1)! = (2k-1) \cdots k \cdots 2 \cdot 1$ also contains the distinct factors k and 3, and hence, $a(3k) < 2k$. Finally, for $k = 3$, we have $a(9) = 6$, since $9 \mid 6!$ but $9 \nmid 5!$. Thus, $n = 9$ is the only solution.

6. Prove that there are infinitely many pairs (a, b) of positive integers with $a > b$ having the following properties:

(i) the greatest common divisor of a and b equals 1;

(ii) a is a divisor of $b^2 - 5$.

(iii) b is a divisor of $a^2 - 5$.

Solved by Michel Bataille, Rouen, France; and Pierre Bornsztein, Maisons-Laffitte, France. We give Bataille's write-up.

Let (a, b) be a pair of positive integers with $a > b$. We first show that (a, b) satisfies (i), (ii), and (iii) if and only if $a^2 + b^2 - 5$ is a multiple of ab.

Suppose that (i), (ii), and (iii) hold. Then $b^2 - 5 = \lambda a$ and $a^2 - 5 = \mu b$ for some integers λ and μ. Hence, $ab^2 - 5a = \lambda a^2 = 5\lambda + \lambda\mu b$. Thus, $b(ab - \lambda\mu) = 5(\lambda + a)$. If 5 divides b, then, since $a^2 = 5 + \mu b$, it follows that 5 divides a, contradicting (i). Therefore, 5 divides $ab - \lambda\mu$. Then $ab - \lambda\mu = 5k$ for some integer k, and we have $b(5k) = 5(\lambda + a)$; that is, $\lambda = bk - a$. Then $b^2 - 5 = (bk - a)a$; that is, $a^2 + b^2 - 5 = kab$.

Conversely, suppose that (a, b) satisfies $a^2 + b^2 - 5 = kab$ for some integer k. Conditions (ii) and (iii) clearly hold. If $d = \gcd(a, b)$, then $a = da'$ and $b = db'$ and so $d^2 a'^2 + d^2 b'^2 - 5 = kd^2 a'b'$. It follows that d^2 divides 5 and d must be 1, implying that condition (i) holds as well.

Now, suppose that (a, b) satisfies $a^2 + b^2 - 5 = kab$ for some integer $k > 1$. From $(ak - b)^2 + a^2 - 5 = ka(ak - b)$, we see that $(ak - b, a)$ is another pair satisfying the conditions. Starting with the pair $(4, 1)$ (for which $k = 3$), and applying repeatedly the transformation $(a, b) \to (ak - b, a)$, we obtain a sequence of distinct pairs that are solutions. Specifically, let $a_1 = 4$, $b_1 = 1$ and $a_{n+1} = 3a_n - b_n$, $b_{n+1} = a_n$ for all $n \in \mathbb{N}$, then $a_n < a_{n+1}$ for all n and (a_n, b_n) satisfies all the conditions. Besides, it is easily seen that $a_n = L_{2n+1}$ and $b_n = L_{2n-1}$ where $\{L_n\}$ is the Lucas sequence defined by $L_0 = 2$, $L_1 = 1$ and $L_{n+1} = L_n + L_{n-1}$ for $n \in \mathbb{N}$.

That completes this number of the *Corner*. Please send your nice solutions and generalizations promptly since I start using your material about eight months after the problem sets appear in the *Corner*.