SKOLIAD No. 103

Robert Bilinski

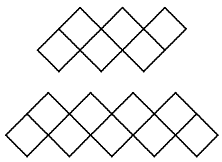
Please send your solutions to the problems in this edition by ***1 February, 2008***. A copy of **MATHEMATICAL MAYHEM Vol. 5** will be presented to one pre-university reader who sends in solutions before the deadline. The decision of the editor is final.

We received solutions to Skoliad 96 and 97 from Govinda Murali, student, St.Joseph's Public School, Cherthala, India, which arrived too late to be considered for publishing.

In this issue, we present the Team Questions from the 6th annual CNU Regional Mathematics Contest. We thank Ron Persky, C.N.U., Newport News, VA.

6th Annual CNU Regional High School Mathematics Contest (2005)

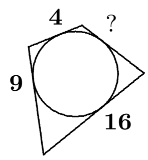
1. To the right are two zigzag shapes made from identical little squares 1 cm on a side. The first shape has 6 squares and a perimeter of 14 cm. The second has 9 squares and a perimeter of 20 cm. What is the perimeter of the zigzag shape with 35 squares?



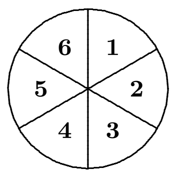
2. Three cards each have one of the digits from 1 through 9 written on them. When the three cards are arranged in some order, they make a three digit number. The largest number that can be made plus the second largest number that can be made is 1233. What is the largest number that can be made?

3. You begin counting on your left hand starting with the thumb, then the index finger, the middle finger, the ring finger, then the little finger, and back to the thumb, and so on. What is the 2005th finger you count?

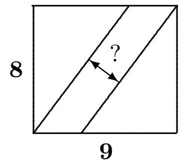
4. A quadrilateral circumscribes a circle. Three of its sides have length 4, 9, and 16 cm, as shown. What is the length in cm of the fourth side?



5. A pizza is cut into six pie-shaped pieces. Trung can choose any piece to eat first, but after that, each piece he chooses must have been next to a piece that has already been eaten (to make it easy to get out of the pan). In how many different orders could he eat the six pieces?



6. The picture shows an 8×9 rectangle cut into three pieces by two parallel slanted lines. The three pieces all have the same area. How far apart are the slanted lines?



7. Find a positive integer N so that there are exactly 25 integers x satisfying $2 \leq \frac{N}{x} \leq 5$.

8. Amy, Bart, and Carol ate some carrot sticks. Amy ate half the number that Bart ate, plus one third the number that Carol ate, plus one. Bart ate half the number that Carol ate, plus one-third the number that Amy ate, plus two. Carol ate half the number that Amy ate, plus one-third the number that Bart ate, plus three. How many carrot sticks did they eat altogether?

9. A motorized column is advancing over flat country at the rate of 15 kilometres per hour. The column is 1 kilometre long. A dispatch rider is sent from the rear to the front on a motorcycle travelling at a constant speed. He returns immediately at the same speed and his total time is 3 minutes. How fast is he going?

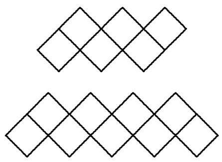
10. Find the remainder when the polynomial $x+x^3+x^9+x^{27}+x^{81}+x^{243}$ is divided by x^2-1.

11. Determine the perimeter of a right triangle with hypotenuse H and area A.

12. When a positive integer n is divided by 3, the remainder is 1. When $n+1$ is divided by 2, the remainder is 1. What is the remainder when $n-1$ is divided by 6?

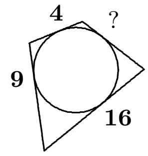
6$^{\text{ième}}$ concours CNU Régional de Mathématiques Secondaires (2005)

1. On retrouve à droit deux formes en zigzag fait de tuiles carrées identiques de 1cm de côté. La première forme a 6 carrés et un périmètre de 14 cm. La seconde a 9 carrés et un périmètre de 20 cm. Quel est le périmètre du zigzag ayant 35 carrés ?

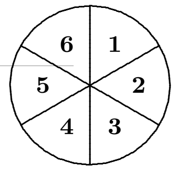


2. Trois cartes ont chacune un des chiffres de 1 à 9 écrit dessus. Quand les trois cartes sont arrangées dans un ordre, elles forment un nombre à trois chiffres. Le plus grand de ces nombres additionné du deuxième plus grand donne 1233. Quel est le plus grand nombre qui peut être fait ?

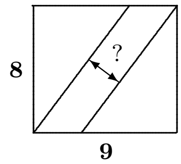
3. On commence à compter sur la main gauche avec le pouce, l'index, le majeur, l'annulaire, l'auriculaire puis on revient au pouce et ainsi de suite. Quel est le $2005^{\text{ième}}$ doigt utilisé ?



4. Un quadrilatère circonscrit un cercle. Trois de ses côtés ont pour longueur 4, 9 et 16 cm comme illustré. Quel est la longueur du quatrième côté ?



5. Une pizza est coupée en six morceaux. Trung peut choisir n'importe quel morceau pour commencer, mais après cela, chaque morceau choisi doit être à côté d'un morceau qui a déjà été choisi au préalable (pour simplifier la sortie du plat). De combien de manières peut-il manger sa pizza ?



6. L'image montre un rectangle 8×9 coupé en trois pièces par deux lignes parallèles obliques. Les trois morceaux ont tous la même aire. Quelle est la distance qui sépare les lignes obliques ?

7. Trouver un entier positif N tel qu'exactement 25 entiers x satisfont à $2 \leq \dfrac{N}{x} \leq 5$.

8. Amy, Bart et Carol ont mangé des mini-carottes. Amy a mangé la moitié du nombre à Bart, plus un tiers du nombre à Carol plus un. Bart a mangé la moitié du nombre à Carol, plus un tiers du nombre à Amy plus deux. Carol a mangé le même nombre qu'Amy plus un tiers du nombre à Bart plus trois. Combien de mini-carottes ont-ils mangé ensemble ?

9. Une colonne motorisée est en train d'avancer en terrain plat au rythme de 15 kilomètres par heure. La colonne est longue de 1 kilomètre. Un courrier est envoyé de l'arrière vers l'avant sur une motocyclette roulant à vitesse constante. Il retourne immédiatement à la même vitesse et son parcours a duré 3 minutes. Quelle était sa vitesse ?

10. Trouver le reste de la division du polynôme $x+x^3+x^9+x^{27}+x^{81}+x^{243}$ par x^2-1.

11. Déterminer le périmètre d'un triangle rectangle ayant une hypothénuse de H et une aire de A.

12. Quand un entier n est divisé par 3, le reste est 1. Quand $n+1$ est divisé par 2, le reste est 1. Quel est le reste quand $n-1$ est divisé par 6 ?

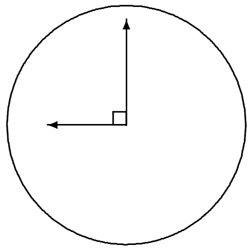
Next we give the solutions to the 2006 Maritime Mathematics Contest [2006 : 481-483].

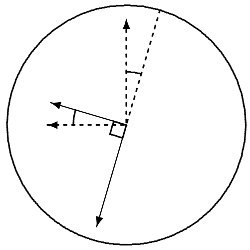
2006 Maritime Mathematics Contest

1. At 9 o'clock, the hour and minute hands on a clock form a right angle. After 9 o'clock, what is the next time at which the clock hands form a right angle?

Official solution, modified by the editor.

The diagrams at right show the hands of a clock at 9:00 (top diagram) and again the next time they form a right angle (bottom diagram). The next time appears to be a little past 9:30.





Let x be the number of minutes after 9:00 the next time the hands form a right angle. The hour hand completes one revolution in 12 hours; hence, in x minutes, it moves through $\frac{x}{12 \times 60}$ revolutions. Similarly, since the minute hand takes 60 minutes to complete one revolution, it moves through $x/60$ revolutions in x minutes. In the interval from 9:00 to the required time, the minute hand has moved through exactly half a revolution more than the hour hand. Therefore,

$$\frac{x}{60} = \frac{x}{720} + \frac{1}{2},$$

which yields $11x = 360$, or $x = 32\frac{8}{11}$. Thus, the required time is $32\frac{8}{11}$ minutes after 9:00.

2. For a positive number such as 3.14, we call 3 the *integer part* and 0.14 the *fractional part*. Find a positive number such that the fractional part, the integer part, and the number itself are three consecutive terms

(a) in an arithmetic sequence; (b) in a geometric sequence.

(The sequence a_1, a_2, a_3, a_4, ... is called *arithmetic* if there is a number d such that $a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = \ldots = d$; it is called *geometric* if there is a number $r \neq 0$ such that $a_2/a_1 = a_3/a_2 = a_4/a_3 = \ldots = r$.)

Official solution, modified by the editor.

Let x be a positive number. Let n be its integer part and y its fractional part. Thus, $x = n + y$, where n is an integer and $0 \leq y < 1$.

(a) We want to find x so that y, n, and x are consecutive terms in an arithmetic sequence.

First suppose that $0 < x < 1$. Then $n = 0$ and $x = y$. The numbers y, n, and x are not terms in an arithmetic sequence in this case.

Now assume $x \geq 1$. Then $n \geq 1$ and $y < n \leq x$. In order for y, n, and x to be consecutive terms in an arithmetic sequence, we require

$$n - y \;=\; x - n \;=\; d\,, \tag{1}$$

for some real number d. Since $x = n + y$, we can eliminate x in (1) to get $n - y = y = d$. Then $2y = n$. Since n is a positive integer and $0 \leq y < 1$, this equation is satisfied only when $n = 1$ and $y = \frac{1}{2}$. Thus, the only possible value for x is $x = n + y = 1 + \frac{1}{2} = \frac{3}{2}$.

(b) We want to find x so that y, n, and x are consecutive terms in a geometric sequence. As in part (a), we see that this does not happen if $0 < x < 1$. So we assume $x \geq 1$. Then $n \geq 1$ and $y < n \leq x$.

In order for y, n, and x to be consecutive terms in a geometric sequence, we require

$$\frac{n}{y} \;=\; \frac{x}{n} \;=\; r\,, \tag{2}$$

for some $r \neq 0$. Since $x = n + y$, we can eliminate x in (2) to get

$$\frac{n}{y} \;=\; 1 + \frac{y}{n} \;=\; r\,. \tag{3}$$

Note that y cannot be 0, since this would make n/y undefined. So $0 < y < 1$. Therefore, $n/y > n$ and $1 + (y/n) < 1 + (1/n) \leq 2$ (since $n \geq 1$). Thus,

$$n \;<\; \frac{n}{y} \;=\; 1 + \frac{y}{n} \;\leq\; 2\,.$$

But we know that n is a positive integer. The only possibility satisfying $n < 2$ is $n = 1$.

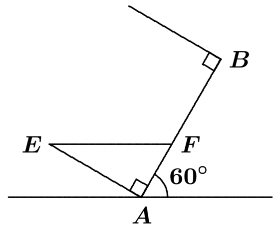
Setting $n = 1$ in (3), we get $1/y = 1 + y$, which can be rewritten as $y^2 + y - 1 = 0$. By the Quadratic Formula,

$$y \;=\; \frac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2(1)} \;=\; \frac{-1 \pm \sqrt{5}}{2}\,.$$

Since $y > 0$, we must have $y = \dfrac{-1 + \sqrt{5}}{2}$. Thus, the only possible value for x is $x = n + y = 1 + \dfrac{-1 + \sqrt{5}}{2} = \dfrac{1 + \sqrt{5}}{2}$.

3. A rectangular tank having length 60 cm, width 60 cm, and height 40 cm is filled with water to a depth of 15 cm and rests on a horizontal table. Let A, B, C, and D in cyclic order be the four bottom corners of the tank. Suppose that the edge BC is slowly raised so that the edge AD remains on the table. As water flows out, the tank is raised until the edge AB makes an angle of 60° with the table. The edge BC is then lowered until the tank once again rests on the table. At this point, what is the depth of water in the tank?

Official solution.



Consider the situation when the tank has been raised so that the edge AB makes an angle of 60° with the table. If water did indeed spill from the tank while it was being raised, the water now reaches the level of EF, where E is a top corner of the tank (the corner above A) and F is a point on AB, as shown in the diagram. (Note that, since the tank has been raised beyond 45°, $|AF|$ is less than $|AE| = 40$, and since $|AB| = 60$, point F does indeed lie on AB.)

Since EF is parallel to the table top, $\angle EFA = 60^\circ$. Hence,

$$|AE| \;=\; |AF|\tan 60^\circ \;=\; |AF|\sqrt{3}\,,$$

or $|AF| = |AE|/\sqrt{3} = 40/\sqrt{3}$.

The volume of water in the tank at this point is equal to the area of $\triangle AEF$ multiplied by 60 cm (the dimension of the tank perpendicular to $\triangle AEF$). Thus, the volume of water is $\frac{1}{2} \times 40 \times \frac{40}{\sqrt{3}} \times 60 = \frac{48000}{\sqrt{3}}$ cm^3. The original volume of water is $15 \times 60 \times 60 = 54000$ cm^3, which is greater than $\frac{48000}{\sqrt{3}}$ cm^3. Therefore, some water does indeed spill from the tank, and the volume of the remaining water is $\frac{48000}{\sqrt{3}}$ cm^3.

Let d be the final depth of water in the tank. Considering the volume of water remaining in the tank, we have $d \times 60 \times 60 = \frac{48000}{\sqrt{3}}$, which implies that $d = \frac{40}{3\sqrt{3}}$. Thus, the final depth of water in the tank is $\frac{40}{3\sqrt{3}}$ cm.

4. Suppose that the positive integers are written in a spiral as shown. Relative to the number 1, where does the number 2006 appear? (For example, 10 appears one unit up and two units to the right of 1.)

7	8	9	10
6	1	2	11
5	4	3	12
	...	14	13

Solution by Natalia Desy, student, Palembang, Indonesia, modified by the editor

The numbers on the diagonal going up to the right are 1, 9, 25, $\ldots$, which are the squares of the odd positive integers. The closest odd square to 2006 is $45^2 = 2025$. Since $45 = 2 \times 22 + 1$, we see that 2025 lies 22 units up and 22 units to the right of 1. But 2006 is located $2025 - 2006 = 19$ positions to the left of 2025. Therefore, the position of 2006 is 22 units up and 3 units to the right of 1.

5. A *square pair* is a pair (x, y) of positive integers such that $x+y$ and xy are both perfect squares. For example, $(5, 20)$ is a square pair since $5 + 20 = 25$ and $5 \times 20 = 100$ are both perfect squares. Show that no square pair exists in which one of the numbers is 3.

Official solution, modified by the editor.

Suppose that 3 and x constitute a square pair. Then $3 + x = a^2$ and $3x = b^2$ for some positive integers a and b. Since $3x$ is a perfect square, x

must be of the form $3c^2$ for some positive integer c. Substituting $x = 3c^2$ into $3 + x = a^2$, we obtain $3(1 + c^2) = a^2$; thus, $1 + c^2$ must contain 3 as a factor. We will show that this is not possible.

When we divide c by 3, the remainder must be 0, 1, or 2. If it is 0, then $c = 3k$ for some integer k; this gives $1 + c^2 = 1 + 9k^2$, which does not have 3 as a factor (since 3 is a factor of $9k^2$). If the remainder is 1, then $c = 3k + 1$ for some integer k; this gives $1 + c^2 = 9k^2 + 6k + 2$, which does not have 3 as a factor (since 3 is a factor of $9k^2 + 6k$). If the remainder is 2, then $c = 3k + 2$ for some integer k; this gives $1 + c^2 = (9k^2 + 12k + 3) + 2$, which does not have 3 as a factor (since 3 is a factor of $9k^2 + 12k + 3$).

6. Find all solutions in real numbers x and y for the system of equations:

$$\begin{aligned} 2(x + y - 2) &= y(x - y + 2)\,, \\ x^2(y - 1) + y^2(x - 1) &= xy - 1\,. \end{aligned}$$

Official solution.

Letting $a = x - 1$ and $b = y - 1$, the given equations become

$$\begin{aligned} 2(a + b) &= (b + 1)(a - b + 2)\,, \\ (a + 1)^2 b + (b + 1)^2 a &= (a + 1)(b + 1) - 1\,. \end{aligned}$$

Expanding and simplifying the second equation gives $ab(a + b + 3) = 0$; thus, $a = 0$, $b = 0$, or $a + b = -3$.

If $a = 0$, the first equation becomes $2b = (b + 1)(-b + 2)$, which is equivalent to $b^2 + b - 2 = 0$. Factoring gives $(b + 2)(b - 1) = 0$. Thus, $b = 1$ or $b = -2$.

If $b = 0$, the first equation becomes $2a = a + 2$; thus, $a = 2$.

If $a + b = -3$, we can set $a = -b - 3$ in the first equation to get $2(-3) = (b + 1)(-b - 3 - b + 2)$, which simplifies to $2b^2 + 3b - 5 = 0$. Factoring gives $(2b+5)(b-1) = 0$. Then $b = -\frac{5}{2}$ or $b = 1$. Since $a = -b-3$, we obtain $a = -\frac{1}{2}$ and $a = -4$, respectively.

To summarize, the equations have the following five solutions:

a	b	x	y
0	1	1	2
0	-2	1	-1
2	0	3	1
$-\frac{1}{2}$	$-\frac{5}{2}$	$\frac{1}{2}$	$-\frac{3}{2}$
-4	1	-3	2

An incomplete solution was received.

That brings us to the end of another issue. This month's winner of a past Volume of Mayhem is Natalia Desy. Congratulations, Natalia! Continue sending in your contests and solutions.