SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

3138. [2006 : 173, 176] Proposed by Paul Bracken. University of Texas, Edinburg, TX, USA.

Let $a_1$ be a non-zero real number, and define the sequence $\{a_n\}_{n=1}^{\infty}$ by $a_{n+1} = n^2/a_n$ for $n \geq 1$. Prove that

$$\sum_{n=1}^{N} \frac{1}{a_n} = \left( \frac{1}{\pi a_1} + \frac{\pi a_1}{4} \right) \ln(N) + O(1).$$

Solution by Joel Schlosberg. Bayside, NY, USA, modified by the editor.

It is straightforward to prove by induction that for $n \geq 0$,

$$a_{2n+1} = \left( \frac{2^{2n} n!^2}{(2n)!} \right)^2 a_1.$$

According to Stirling’s Formula, $n! = \sqrt{2\pi n} \left( \frac{n}{e} \right)^n \left[ 1 + O \left( \frac{1}{n} \right) \right]$ for $n \to \infty$. Therefore, for $n \geq 1$,

$$\frac{2^{2n} n!^2}{(2n)!} = \frac{2^{2n} 2\pi n \left( \frac{n}{e} \right)^{2n}}{\sqrt{2\pi(2n)} \left( \frac{n}{e} \right)^{2n}} \left[ 1 + O \left( \frac{1}{n} \right) \right] = \sqrt{\pi n} \left[ 1 + O \left( \frac{1}{n} \right) \right].$$

Then, for $n \geq 1$,

$$\frac{1}{a_{2n+1}} = \frac{1}{\pi n a_1} \left[ 1 + O \left( \frac{1}{n} \right) \right] = \frac{1}{\pi a_1} \left[ \frac{1}{n} + O \left( \frac{1}{n^2} \right) \right]$$

and

$$\frac{1}{a_{2n+2}} = \frac{a_{2n+1}}{(2n+1)^2} = \frac{\pi n a_1}{4n^2 \left( 1 + \frac{1}{2n} \right)^2} \left[ 1 + O \left( \frac{1}{n} \right) \right]$$

$$= \frac{\pi a_1}{4n} \left[ 1 + O \left( \frac{1}{n} \right) \right] = \frac{\pi a_1}{4} \left[ \frac{1}{n} + O \left( \frac{1}{n^2} \right) \right].$$
Therefore, for $N \geq 4$,

\[
\sum_{n=1}^{N} \frac{1}{a_n} = \frac{1}{a_1} + \frac{1}{a_2} + \sum_{n=1}^{N-1} \frac{1}{a_{2n+1}} + \sum_{n=1}^{N-2} \frac{1}{a_{2n+2}}
\]

\[
= \frac{1}{\pi a_1} \sum_{n=1}^{N-1} \left[ \frac{1}{n} + O\left(\frac{1}{n^2}\right) \right] + \frac{\pi a_1}{4} \sum_{n=1}^{N-2} \left[ \frac{1}{n} + O\left(\frac{1}{n^2}\right) \right] + O(1)
\]

\[
= \left( \frac{1}{\pi a_1} + \frac{\pi a_1}{4} \right) \ln \frac{N}{2} + O(1).
\]

Noting that $\ln(N/2) = \ln N - \ln 2 = \ln N + O(1)$, we arrive at the desired result.

Also solved by MICHEL BATAILLE, Rouen, France; WALther JANous, Ursulinen-gymnasium, Innsbruck, Austria; and the proposer.


Let $\varepsilon$ be the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$. Two parallel tangents to $\varepsilon$ intersect a third tangent at $M_1(x_1, y_1)$ and $M_2(x_2, y_2)$. Show that the value of

\[
\left( \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 \right) \left( \frac{x_2^2}{a^2} + \frac{y_2^2}{b^2} - 1 \right)
\]

is independent of the chosen tangents.

A combination of similar solutions by Apostolos K. Demis, Varvakeio High School, Athens, Greece; and Peter Y. Woo, Biola University, La Mirada, CA, USA.

We shall see that for all choices of the three tangents, the product in question is always 1. We denote by $O$ the centre of the given ellipse, and by $C_1$, $C_2$, and $M$ the points of tangency of the three tangents $C_1M_1$, $C_2M_2$, and $M_1M_2$.

The affine transformation $(x, y) \rightarrow (x, \frac{a}{b}y)$ transforms the given ellipse to the circle with centre $O$ and radius $a$. It also transforms the points $M_1(x_1, y_1)$, $M_2(x_2, y_2)$, $C_1$, $C_2$, and $M$ to the points $M_1'(x_1, \frac{a}{b}y_1)$, $M_2'(x_2, \frac{a}{b}y_2)$, $C_1'$, $C_2'$, and $M'$, respectively, and it transforms the tangents $C_1M_1$, $C_2M_2$, and $M_1M_2$ of the ellipse to the tangents $C_1'M_1'$, $C_2'M_2'$, and $M_1'M_2'$, respectively, of the circle.

Consequently, $M_1'O \perp M_2'O$ (bisectors of $\angle C_1'OM'$ and $\angle M'OC_2'$, where $\angle C_1'OM' + \angle M'OC_2' = \pi$), and $OM' \perp M_1'M_2'$ (since $M_1'M_2'$ is
tangent to the circle and $OM'$ is a radius). Because $\triangle M'_1M'O \sim \triangle OM'M'_2$, it follows that $M'M'_1 \cdot M'M'_2 = M'O^4$. Then

\[
\left( M'_1O^2 - M'O^2 \right) \cdot \left( M'_2O^2 - M'O^2 \right) = a^4,
\]

\[
\left( e^{x_1 + \frac{a^2}{b^2} y_1^2} - a^2 \right) \cdot \left( e^{x_2 + \frac{a^2}{b^2} y_2^2} - a^2 \right) = a^4,
\]

\[
\left( x_1^2 + \frac{y_1^2}{b^2} - 1 \right) \cdot \left( x_2^2 + \frac{y_2^2}{b^2} - 1 \right) = 1.
\]

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOEL SCHLOSBERG, Bayside, NY, USA; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer.


Let $a_1, a_2, \ldots, a_n$ be $n$ distinct positive real numbers, where $n \geq 2$. For $i = 1, 2, \ldots, n$, let $p_i = \prod_{j \neq i} (a_j - a_i)$. Show that $\prod_{i=1}^{n} a_i^{\frac{n}{i}} < 1$.

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria, modified by the editor.

We will apply the following theorem on convex functions of higher order (see [1, pp. 4–5]).

Theorem. Let $I$ be an open interval, and let $f : I \to \mathbb{R}$ be a function which is $n$-times differentiable. The following statements are equivalent:

(i) If $x_0, x_1, \ldots, x_n$ are any $n+1$ distinct points in $I$, then $\sum_{i=0}^{n} \frac{f(x_i)}{w'(x_i)} > 0$,

where $w(x) = \prod_{j=0}^{n} (x - x_j)$.

(ii) $f^{(n)}(x) > 0$ for all $x \in I$.

A function $f$ satisfying the above conditions is said to be strictly $n$-convex.

For our problem, we take $I = (0, \infty)$ and $f(x) = (-1)^n \ln x$. Then $f^{(n-1)}(x) = (n-2)!/x^{n-1} > 0$ for all $x > 0$. Therefore, $f$ is strictly $(n-1)$-convex.

Using the $n$ distinct numbers $a_1, a_2, \ldots, a_n$ given in the problem, we let $w(x) = \prod_{j=1}^{n} (x - a_j)$. Then for each $i = 1, 2, \ldots, n$, we have

\[
w'(a_i) = \prod_{j \neq i} (a_i - a_j) = (-1)^{n-1} p_i.
\]
According to condition (i) in the theorem, \( \sum_{i=1}^{n} \frac{f(a_i)}{w'(a_i)} > 0 \). Then
\[
\sum_{i=1}^{n} \frac{\ln a_i}{p_i} = -\sum_{i=1}^{n} (-1)^n \ln a_i = -\sum_{i=1}^{n} \frac{f(a_i)}{w'(a_i)} < 0 .
\]
Taking exponentials, we obtain \( \prod_{i=1}^{n} a_i^{\frac{1}{p_i}} < 1 \).

References

Also solved by JOEL SCHLOSBERG, Bayside, NY, USA; and the proposer.


Let \( a, b, \) and \( c \) be the sides of a scalene triangle \( ABC \). Prove that
\[
\sum_{\text{cyclic}} \frac{(a+1)bc}{(\sqrt{a} - \sqrt{b})(\sqrt{a} - \sqrt{c})} < \frac{a^4 + b^4 + c^4}{abc} .
\]

Solution by Michel Bataille, Rouen, France, modified by the editor.

Write the left side as \( S_1 + S_2 \), where
\[
S_1 = \sum_{\text{cyclic}} \frac{abc}{(\sqrt{a} - \sqrt{b})(\sqrt{a} - \sqrt{c})}, \quad S_2 = \sum_{\text{cyclic}} \frac{bc}{(\sqrt{a} - \sqrt{b})(\sqrt{a} - \sqrt{c})} .
\]

Observe that
\[
S_1 = abc \cdot \frac{(\sqrt{c} - \sqrt{b}) + (\sqrt{a} - \sqrt{c}) + (\sqrt{b} - \sqrt{a})}{(\sqrt{c} - \sqrt{b})(\sqrt{a} - \sqrt{c})(\sqrt{b} - \sqrt{a})} = 0 .
\]

We can simplify \( S_2 \) by making use of the identity
\[
x^2 y^2 (y-x) + y^2 z^2 (z-y) + z^2 x^2 (x-z) = (x-y)(y-z)(z-x)(xy+yz+zx).
\]
Dividing both sides by \( (x-y)(y-z)(z-x) \) and setting \( x = \sqrt{a}, y = \sqrt{b}, \) and \( z = \sqrt{c} \), we obtain
\[
S_2 = \sqrt{ab} + \sqrt{bc} + \sqrt{ca} .
\]
Thus, the given inequality turns out to be equivalent to
\[
abc \left( \sqrt{ab} + \sqrt{bc} + \sqrt{ca} \right) < a^4 + b^4 + c^4 . \quad (1)
\]
Now, from the Weighted AM–GM Inequality, we have
\[ a^\frac{3}{3} b^\frac{3}{3} c = (a^4)^\frac{3}{3} (b^4)^\frac{3}{3} (c^4)^\frac{1}{3} < \frac{3}{3} a^4 + \frac{3}{3} b^4 + \frac{1}{3} c^4 \]
(where the inequality is strict because \( a, b, \) and \( c \) are distinct). Similarly, we have \( a^\frac{3}{3} b^2 c^2 < \frac{3}{3} a^4 + \frac{1}{3} b^4 + \frac{3}{3} c^4 \) and \( ab^\frac{3}{3} c^2 < \frac{1}{3} a^4 + \frac{3}{3} b^4 + \frac{3}{3} c^4 \). If we add all three inequalities, we obtain (1).

Note that the result actually holds whenever \( a, b, \) and \( c \) are distinct positive real numbers.

Also solved by ARKADY ALI, San Jose, CA, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinen-Gymnasium, Innsbruck, Austria; JOEL SCHLOSBERG, Bayside, NY, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA, and the proposer.

Several solvers mentioned that the result is true for any three distinct positive real numbers. Most solvers began, like Bataille, by showing that the given inequality is equivalent to \( abc(\sqrt{ab} + \sqrt{bc} + \sqrt{ca}) < a^4 + b^4 + c^4 \). At that point, there were several ways to complete the solution. Janous provided several such ways himself.


If \( x_k > 0 \) for \( k = 1, 2, \ldots, n \), prove that

(a) \[ \cos \left( \frac{n}{k=1} x_k \right) - \sin \left( \frac{n}{k=1} x_k \right) \geq \frac{1}{n} \sum_{k=1}^{n} \left( \cos \frac{1}{x_k} - \sin \frac{1}{x_k} \right) ; \]

(b) \[ \frac{n}{k=1} \frac{\sin \frac{1}{x_k}}{\cos \frac{1}{x_k}} \geq \tan \left( \frac{n}{k=1} \frac{1}{x_k} \right) . \]

Editor's comment.

Unfortunately, the inequalities are incorrect as stated. Both Walther Janous, Ursulinen-Gymnasium, Innsbruck, Austria and Peter Y. Woo, Biola University, La Mirada, CA, USA gave counterexamples and then attempted to impose additional restrictions on the variables to make the inequalities correct. Janous succeeded in repairing part (a). Woo's counterexample for part (a) is \( n = 2, x_1 = 0.4, \) and \( x_2 = 0.3 \), and Janous' counterexample for part (b) is \( n = 2, x_1 = 1/\pi, \) and \( x_2 = 3/\pi \).

Solution to adjusted part (a) by Walther Janous, Ursulinen-Gymnasium, Innsbruck, Austria.

Let \( f(x) = \cos \frac{1}{x} - \sin \frac{1}{x} \). Inequality (a) can now be written as

\[ \frac{1}{n} \sum_{k=1}^{n} f(x_k) \leq f \left( \frac{1}{n} \sum_{k=1}^{n} x_k \right) . \]

This is Jensen's Inequality characterizing the concavity of the function \( f \).
Using a computer algebra system, we can show that

\[ f''(x) = \frac{(1 - 2x) \sin \frac{1}{x} - (2x + 1) \cos \frac{1}{x}}{x^4} < 0 \]

for \( x \in (0.601451551, \infty) \). (The editor verified Janous’ result using Maple.)

Thus, inequality (a) is true if \( x_k \in (0.601451551, \infty) \) for \( k = 1, 2, \ldots, n \).

**Solution to adjusted part (b) by the editor using ideas of the proposer.**

Let \( g(x) = \sin \frac{1}{x} \) and \( h(x) = \cos \frac{1}{x} \). Then

\[ g''(x) = \frac{2}{x^3} \cos \frac{1}{x} - \frac{1}{x^4} \sin \frac{1}{x} \quad \text{and} \quad h''(x) = -\left( \frac{2}{x^3} \sin \frac{1}{x} + \frac{1}{x^4} \cos \frac{1}{x} \right). \]

For \( x \in (0.928613759, \infty) \), we have \( g''(x) > 0 \) and \( h''(x) < 0 \). In fact, \( h''(x) < 0 \) if \( x \in (0.436885409, \infty) \). Since 0.928613759 > 2/\( \pi \), we have

\[ 0 < \frac{n}{\sum_{k=1}^{n} x_k} < \frac{\pi}{2} \quad \text{and} \quad 0 < \frac{1}{x_k} < \frac{\pi}{2}. \]

Now, Jensen’s inequality applied to the functions \( g(x) \) and \( h(x) \) yields

\[ \frac{1}{n} \sum_{k=1}^{n} \sin \frac{1}{x_k} \geq \sin \left( \frac{n}{\sum_{k=1}^{n} x_k} \right) > 0 \]

and

\[ 0 < \frac{1}{n} \sum_{k=1}^{n} \cos \frac{1}{x_k} \leq \cos \left( \frac{n}{\sum_{k=1}^{n} x_k} \right), \]

from which the inequality in part (b) follows immediately.

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**3143. [2006 : 239, 241] Proposed by Mihály Bence, Brăsăv, Romania.**

For \( n \geq 1 \) let \( a_n = 1 + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{n} \). Prove that

\[ \sum_{k=1}^{n} \frac{\sqrt{k}}{a_k^2} < \frac{2n + 1 + (\ln n)^2}{n + 1 + \frac{1}{2}(\ln n)^2}. \]

**Solution by Richard I. Hess, Rancho Palos Verdes, CA, USA, modified by the editor.**

Define

\[ L_n = \sum_{k=1}^{n} \frac{\sqrt{k}}{a_k^2} \quad \text{and} \quad R_n = \frac{2n + 1 + (\ln n)^2}{n + 1 + \frac{1}{2}(\ln n)^2}. \]

By direct computation, we see that \( L_1 = 1, L_2 \approx 1.2426, L_3 \approx 1.3396, L_4 \approx 1.3905 \), \( R_1 = 1.5, R_2 \approx 1.6914, R_3 \approx 1.7828, \) and \( R_4 \approx 1.8322 \).

Hence, \( L_n < R_n \) for \( n = 1, 2, 3, 4 \).
Now assume that $n \geq 5$. Since the function $f(x) = x^{1/x}$ is decreasing on $(e, \infty)$, we have, for each $k \geq 5$, $1 < k^{1/k} \leq 5^{1/5} \approx 1.3797$. Note also that $\alpha_k \geq k$ for all $k \geq 1$. Hence,

$$L_n = L_4 + \sum_{k=5}^n \frac{\sqrt{k}}{\alpha_k} < L_4 + (1.3797) \sum_{k=5}^n \frac{1}{k^{1/2}} < L_4 + (1.4) \int_4^n \frac{dx}{x^2}$$

$$= L_4 + (1.4) \left( \frac{1}{4} - \frac{1}{n} \right) < (1.4) + (1.4) \left( \frac{1}{4} \right) = 1.75 .$$

Since $R_n = 2 - \frac{1}{n+1 + \frac{1}{2}(\ln n)^2} > 2 - \frac{1}{n+1} \geq 2 - \frac{1}{6} > 1.8$, it follows that $L_n < R_n$.

Also solved by WALTHER JANOUS, Ursulengymnasium, Innsbruck, Austria.

Janous’ solution, which is based on computer verifications, actually establishes the stronger inequality $\sum_{k=1}^n \frac{1}{x_k} \leq R_n$. Both Janous and Hess believe that the minimum of $R_n - L_n$ is attained when $n = 8$; thus, $L_n \leq R_n - 0.4409$ where 0.4409 $\approx R_8 - L_8$.

But they gave no proof. The solution by the proposer applied simple telescoping together with the inequality $\alpha_n < n + 1 + \frac{1}{2}(\ln n)^2$, which he claimed can be shown by induction but did not supply any proof.

### 3144. [2006 : 239, 241]

Proposed by Mihály Bencze, Brasov, Romania.

Let $A, B \in M_n(\mathbb{C})$, and let $\omega = e^{2\pi i/n}$. Prove that

$$\sum_{k=1}^n \det (A + \omega^{k-1}B) + \sum_{k=1}^n \det (B + \omega^{k-1}A) = 2n \det A + \det B .$$

[Ed. The problem has been corrected to state that $\omega = e^{2\pi i/n}$, as the proposer intended, rather than $\omega = e^{2\pi/n}$. This correction was made by the solvers.]

Solution by Michel Bataille, Rouen, France.

For $M \in M_n(\mathbb{C})$, we denote by $M^{(1)}, M^{(2)}, \ldots, M^{(n)}$ the columns of $M$. Let $x$ be an indeterminate. Since an $n \times n$ determinant is an $n$–linear function of its columns, we can expand $\det (A + xB)$ as follows:

$$\det (A + xB) = \det \left( A^{(1)} + xB^{(1)}, A^{(2)} + xB^{(2)}, \ldots, A^{(n)} + xB^{(n)} \right)$$

$$= \det \left( A^{(1)}, A^{(2)}, \ldots, A^{(n)} \right) + \sum_{j=1}^{n-1} \alpha_j x^j$$

$$+ x^n \det \left( B^{(1)}, B^{(2)}, \ldots, B^{(n)} \right)$$

$$= \det A + \sum_{j=1}^{n-1} \alpha_j x^j + x^n \det B ,$$
where \( \alpha_1, \alpha_2, \ldots, \alpha_{n-1} \) are complex numbers (independent of \( x \)). Taking \( x = 1, \omega, \omega^2, \ldots, \omega^{n-1} \) in succession and adding the results, we obtain

\[
\sum_{k=0}^{n-1} \det(A + \omega^k B) = n \det A + \sum_{j=1}^{n-1} \sum_{k=0}^{n-1} \alpha_j \omega^{kj} + \sum_{k=0}^{n-1} \omega^{kn} \det B
\]

\[
= n \det A + \sum_{j=1}^{n-1} \sum_{k=0}^{n-1} \alpha_j \omega^{kj} + \sum_{k=0}^{n-1} \omega^{kn} \det B.
\]

Now, since \( \omega^n = 1 \) and \( \sum_{k=0}^{n-1} \omega^k = \frac{1 - \omega^{n+1}}{1 - \omega} = 0 \) for \( j = 1, 2, \ldots, n - 1 \), we have

\[
\sum_{k=0}^{n-1} \det(A + \omega^k B) = n(\det A + \det B).
\]

Then \( \sum_{k=0}^{n-1} \det(B + \omega^k A) = n(\det B + \det A) \), and the result follows.

Also solved by PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.


Let \( f(x) = x - c^2 \tanh x \), where \( c > 1 \) is an arbitrary constant. It is not hard to show that \( f(x) \) is decreasing on the interval \( [-x_0, x_0] \), where \( x_0 = \ln(c + \sqrt{c^2 - 1}) \) is the positive root of the equation \( \cosh x = c \). For each \( x \in (-x_0, x_0) \), the horizontal line passing through \( (x, f(x)) \) intersects the graph of \( f \) at two other points with abscissas \( x_1(x) \) and \( x_2(x) \). Define a function \( g : (-x_0, x_0) \to \mathbb{R} \) as follows:

\[
g(x) = x + c^2 \tanh(x_1(x)) + c^2 \tanh(x_2(x)).
\]

Prove or disprove that \( g(x) > 0 \) for all \( x \in (0, x_0) \).

Editor's note: No solutions were received for this problem; hence, it remains open. The proposer believes that the conjecture is true, since there is ample empirical evidence.


Let \( p > 1 \), and let \( a, b, c, d \in [1/\sqrt{p}, \sqrt{p}] \). Prove that

(a) \[
\frac{p}{1+p} + \frac{2}{1+\sqrt{p}} \leq \frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+a} \leq \frac{1}{1+p} + \frac{2\sqrt{p}}{1+\sqrt{p}};
\]

(b) \[
\frac{p}{1+p} + \frac{3}{1+\sqrt{p}} \leq \frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+d} + \frac{d}{d+a} \leq \frac{1}{1+p} + \frac{3\sqrt{p}}{1+\sqrt{p}}.
\]
Solution by Arkady Alt. San Jose, CA, USA. modified by the editor.

(a) The transposition \((a, b, c) \mapsto (b, a, c)\) in the inequality

\[
\frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+a} \leq \frac{1}{1+p} + \frac{2\sqrt{p}}{1+\sqrt{p}}
\]

(1)
gives the equivalent inequality

\[
\frac{b}{b+a} + \frac{a}{a+c} + \frac{c}{c+b} \leq \frac{1}{1+p} + \frac{2\sqrt{p}}{1+\sqrt{p}}.
\]

Since

\[
\frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+a} = 3 - \left( \frac{1}{b+a} + \frac{a}{a+c} + \frac{c}{c+b} \right),
\]

we see that (1) is satisfied if and only if

\[
\frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+a} \geq 3 - \left( \frac{1}{b+a} + \frac{a}{a+c} + \frac{c}{c+b} \right) = \frac{p}{1+p} + \frac{2\sqrt{p}}{1+\sqrt{p}}.
\]

Thus, to prove (a), it is sufficient to prove (1).

Let \(x = b/a, y = c/b,\) and \(z = a/c.\) Then (1) becomes

\[
\frac{1}{1+x} + \frac{1}{1+y} + \frac{1}{1+z} \leq \frac{1}{1+p} + \frac{2\sqrt{p}}{1+\sqrt{p}}.
\]

(2)

Note that \(xyz = 1\) and \(x, y, z \in [1/p, p].\) To prove (1), it is sufficient to prove (2) for all such \(x, y,\) and \(z.\)

By the symmetry in (2), we may assume that \(z = \max\{x, y, z\}.\) Then, since \(xyz = 1\) and \(z \leq p,\) we must have \(1 \leq z \leq p\) and \(1/p \leq xy \leq 1.\) Let \(t = \sqrt{xy}.\) Then \(t^2z = 1\) and \(1/\sqrt{p} \leq t \leq 1.\) Since \(x + y \geq 2\sqrt{xy} = 2t,\) we have

\[
\frac{1}{1+x} + \frac{1}{1+y} = \frac{2 + x + y}{1 + x + y + xy} = 1 + \frac{1-t^2}{1+x+y+t^2} \\
\leq 1 + \frac{1-t^2}{1+2t+t^2} = 1 + \frac{1-t}{1+t} = \frac{2}{1+t}.
\]

Hence,

\[
\frac{1}{1+x} + \frac{1}{1+y} + \frac{1}{1+z} \leq \frac{2}{1+t} + \frac{1}{1+z} = \frac{2}{1+t} + \frac{t^2}{1+t^2}.
\]

Let \(h(t) = \frac{2}{1+t} + \frac{t^2}{1+t^2}.\) Since \(h'(t) = \frac{-2(1-t)(1-t^3)}{(1+t)^2(1+t^2)^2},\) it follows that \(h\) is decreasing on \((0,1].\) Consequently, for \(1/\sqrt{p} \leq t \leq 1,\)

\[
h(t) \leq h(1/\sqrt{p}) = \frac{1}{1+p} + \frac{2\sqrt{p}}{1+\sqrt{p}}.
\]

This proves inequality (2) and completes the proof of (a).
(b) This is treated similarly. The transposition \((a, b, c, d) \mapsto (b, a, d, c)\) in the inequality
\[
\frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+d} + \frac{d}{d+a} \leq \frac{1}{1+p} + \frac{3\sqrt[p]{p}}{1+\sqrt[p]{p}}
\]
(4)
yields the equivalent inequality
\[
\frac{b}{b+a} + \frac{a}{a+d} + \frac{d}{d+c} + \frac{c}{c+b} \leq \frac{1}{1+p} + \frac{3\sqrt[p]{p}}{1+\sqrt[p]{p}}.
\]
Since
\[
\frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+d} + \frac{d}{d+a} = 4 - \left(\frac{b}{b+a} + \frac{a}{a+d} + \frac{d}{d+c} + \frac{c}{c+b}\right),
\]
we see that (4) is satisfied if and only if
\[
\frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+d} + \frac{d}{d+a} \geq 4 - \left(\frac{p}{p+1} + \frac{3\sqrt[p]{p}}{1+\sqrt[p]{p}}\right)
= \frac{p}{p+1} + 3
\]
Thus, to prove (b), it is sufficient to prove (4).

Let \(x = b/a, y = c/b, u = c/d, \text{ and } v = d/a\). Then (4) becomes
\[
\frac{1}{1+x} + \frac{1}{1+y} + \frac{1}{1+u} + \frac{1}{1+v} \leq \frac{1}{1+p} + \frac{3\sqrt[p]{p}}{1+\sqrt[p]{p}}.
\]
Note that \(xyuv = 1\) and \(x, y, u, v \in [1/p, p]\). To prove (4), it is sufficient to prove (5) for all such \(x, y, u, \text{ and } v\).

Let \(t = \sqrt{xy}\) and \(s = \sqrt{uv}\). By the symmetry in (5), we may assume that \(t \leq s\). Then, since \(ts = 1\), we see that \(t \leq 1 \leq s\). Furthermore, since \(s^2/u = v \leq p\), we have \(s^2/p \leq u\), and thus, \(s^2/p \leq u \leq p\).

Now, for fixed \(s\),
\[
\max\left\{u + v \mid uv = s^2, \frac{s^2}{p} \leq u \leq p\right\}
= \max\left\{u + \frac{s^2}{u} \mid \frac{s^2}{p} \leq u \leq p\right\}
= p + \frac{s^2}{p}.
\]
Thus,
\[
\frac{1}{1+u} + \frac{1}{1+v} = \frac{2+u+v}{1+u+v+uv} = 1 - \frac{s^2-1}{1+u+v+s^2}
\leq 1 - \frac{s^2-1}{1+p + \frac{s^2}{p} + s^2}
= \frac{p}{s^2+p} + \frac{1}{1+p}.
\]
(6)
Using inequalities (6) and (3), we get
\[
\frac{1}{1+x} + \frac{1}{1+y} + \frac{1}{1+u} + \frac{1}{1+v} \leq \frac{2s}{1+s} + \frac{p}{s^2 + p} + \frac{1}{1+p}.
\]

Let \( g(s) = \frac{2s}{1+s} + \frac{p}{s^2 + p} \). Since \( g'(s) = \frac{2(s-p)(s^2 - p)}{(s+1)^2(s^2 + p)^2} \), this function has a local maximum at \( s = \sqrt{p} \), which is in the interval \( (1, p) \).

We have \( g(1) = -1 + \frac{p}{1+p} < 0 \) and \( g(p) = -\frac{2}{p+1} + \frac{1}{p+1} < 0 \); whence, \( \max_{s \in [1,p]} g(s) = g(\sqrt{p}) \), and therefore,
\[
\frac{1}{1+x} + \frac{1}{1+y} + \frac{1}{1+u} + \frac{1}{1+v} \leq \frac{2\sqrt{p}}{1 + \sqrt{p}} + \frac{p}{\sqrt{p}^2 + p} + \frac{1}{1+p} = 3\frac{\sqrt{p}}{1 + \sqrt{p}} + \frac{1}{1+p}.
\]

This proves (4) and completes the proof of (b).

Also solved by WALTHER JANOUS, Ursulengymnasium, Innsbruck, Austria (part (a)); PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.


Let \( n \geq 3 \), and let \( x_1, x_2, \ldots, x_n \) be positive real numbers such that \( x_1x_2 \cdots x_n = 1 \). For \( n = 3 \) and \( n = 4 \), prove that
\[
\frac{1}{x_1^2 + x_1x_2} + \frac{1}{x_2^2 + x_2x_3} + \cdots + \frac{1}{x_n^2 + x_nx_1} \geq \frac{n}{2}.
\]

Solution by the proposer.

Using the substitutions \( x_1 = \sqrt{\frac{a_2}{a_1}}, x_2 = \sqrt{\frac{a_3}{a_2}}, \ldots, x_n = \sqrt{\frac{a_1}{a_n}} \), the given inequality becomes
\[
\frac{a_1}{a_2 + \sqrt{a_1a_3}} + \frac{a_2}{a_3 + \sqrt{a_2a_4}} + \cdots + \frac{a_n}{a_2 + \sqrt{a_1a_3}} \geq \frac{n}{2}.
\]

Since \( \sqrt{a_1a_3} \leq \frac{a_1 + a_3}{2} \), \( \ldots \), \( \sqrt{a_na_2} \leq \frac{a_n + a_2}{2} \), it suffices to show that
\[
\frac{a_1}{a_1 + 2a_2 + a_3} + \frac{a_2}{a_2 + 2a_3 + a_4} + \cdots + \frac{a_n}{a_n + 2a_1 + a_2} \geq \frac{n}{4}.
\]

By the Cauchy-Schwarz inequality, we have
\[
(a_1 + \cdots + a_n)^2 \leq \left[ a_1(a_1 + 2a_2 + a_3) + \cdots + a_n(a_n + 2a_1 + a_2) \right] \left( \frac{a_1}{a_1 + 2a_2 + a_3} + \cdots + \frac{a_n}{a_n + 2a_1 + a_2} \right).
\]
Thus, it suffices to show that

\[ 4(a_1 + \cdots + a_n)^2 \geq n[a_1(a_1 + 2a_2 + a_3) + \cdots + a_n(a_n + 2a_1 + a_2)] \]

This inequality is an identity for \( n = 4 \). For \( n = 3 \), it is

\[ a_1^2 + a_2^2 + a_3^2 \geq a_1a_2 + a_2a_3 + a_3a_1, \]

which is true, because

\[
2(a_1^2 + a_2^2 + a_3^2 - a_1a_2 - a_2a_3 - a_3a_1)
= (a_1 - a_2)^2 + (a_2 - a_3)^2 + (a_3 - a_1)^2 \geq 0.
\]

This completes the proof. Equality holds when \( x_i = 1 \) for all \( i \).

The case \( n = 3 \) was also solved by MOHAMMED AASSILA, Strasbourg, France. There was one incorrect submission.


Let \( 0 < m < 1 \), and let \( a, b, c \in [\sqrt{m}, 1/\sqrt{m}] \). Prove that

\[
\frac{a^3 + b^3 + c^3 + 3(1 + m)abc}{ab(a + b) + bc(b + c) + ca(c + a)} \geq 1 + \frac{m}{2}.
\]

Solution by Joel Schlosberg, Bayside, NY, USA.

Let \( f(x) = -2x^2 + (m^2 + 3m)x + (m^3 - 3m^2 - 2m + 2) \). For \( x \geq m \),

\[
f'(x) = -4x + m^2 + 3m \leq -4m + m^2 + 3m = m(m - 1) < 0,
\]

which implies that \( f(x) \) is decreasing.

Suppose that \( a = m, b \in [m, 1], \) and \( c = 1 \). Then \( m = a \leq b \leq c = 1 \) and

\[
2[a^3 + b^3 + c^3 + 3(1 + m)abc] - (m + 2)[a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2] \]

\[
= 2[m^3 + b^3 + 1 + 3(1 + m)mb]
- (m + 2)[m^2b + mb^2 + b^2 + b + m + m^2] \]

\[
= 2b^3 - (m^2 + 3m + 2)b^2 + (-m^3 + 4m^2 + 5m - 2)b
+ (m^3 - 3m^2 - 2m + 2) \]

\[
= (1 - b)[-2b^2 + (m^2 + 3m)b + (m^3 - 3m^2 - 2m + 2)]
\]

\[
= (1 - b)f(b) \geq (1 - b)f(1) = (1 - b)m(m - 1)^2 \geq 0,
\]

which yields

\[
\frac{a^3 + b^3 + c^3 + 3(1 + m)abc}{a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2} \geq 1 + \frac{m}{2}.
\]
Suppose that $a, b, c \in [\sqrt{m}, 1/\sqrt{m}]$. Without loss of generality, we can assume that $a \leq b \leq c$. Then

$$m \leq \frac{a}{c} \leq \frac{b}{c} \leq 1.$$  

Hence, for $(m', a', b', c') = \left(\frac{a}{c}, \frac{a}{c}, \frac{b}{c}, 1\right)$, we have

$$m \leq m' = a' \leq b' \leq c' = 1,$$

and therefore,

$$\frac{a^3 + b^3 + c^3 + 3(1 + m')a' b' c'}{a^2 b' + a' b'^2 + b^2 c' + b' c'^2 + c^2 a' + c' a'^2} \geq 1 + \frac{m'}{2},$$

which can be written as

$$\frac{a^3 + b^3 + c^3 + 3(1 + m')a' b' c'}{a^2 b' + a' b'^2 + b^2 c' + b' c'^2 + c^2 a' + c' a'^2} - 1 = m' \left(\frac{1}{2} - \frac{3a' b' c'}{a^2 b' + a' b'^2 + b^2 c' + b' c'^2 + c^2 a' + c' a'^2}\right).$$

By the AM–GM Inequality,

$$\frac{a^2 b + ab^2 + b^2 c + bc^2 + c^2 a + ca^2}{6} \geq \sqrt[6]{a^6 b^6 c^6} = abc.$$

Hence,

$$\frac{1}{2} - \frac{3abc}{a^2 b + ab^2 + b^2 c + bc^2 + c^2 a + ca^2} \geq 0.$$  

Since $a : b : c = a' : b' : c'$, we have

$$\frac{a^3 + b^3 + c^3 + 3abc}{a^2 b + ab^2 + b^2 c + bc^2 + c^2 a + ca^2} - 1 = \frac{a^3 + b^3 + c^3 + 3a' b' c'}{a^2 b' + a' b'^2 + b^2 c' + b' c'^2 + c^2 a' + c' a'^2} - 1 = m' \left(\frac{1}{2} - \frac{3a' b' c'}{a^2 b' + a' b'^2 + b^2 c' + b' c'^2 + c^2 a' + c' a'^2}\right) \geq m' \left(\frac{1}{2} - \frac{3abc}{a^2 b + ab^2 + b^2 c + bc^2 + c^2 a + ca^2}\right) \geq m \left(\frac{1}{2} - \frac{3abc}{a^2 b + ab^2 + b^2 c + bc^2 + c^2 a + ca^2}\right),$$

and therefore,

$$\frac{a^3 + b^3 + c^3 + 3(1 + m)abc}{a^2 b + ab^2 + b^2 c + bc^2 + c^2 a + ca^2} \geq 1 + \frac{m}{2}.$$  

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; WAL ther JANOUS, Ursulengymnasium, Innsbruck, Austria; and the proposer. There was also one incorrect solution submitted.

Let \( P(z) \) be any non-constant complex monic polynomial. Show that there is a complex number \( w \) such that \( |w| < 1 \) and \( |P(w)| > 1 \).

1. Solution by Michel Bataille, Rouen, France.

Let \( P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0 \). Suppose, to the contrary, that \( |P(z)| < 1 \) for all complex numbers \( z \) such that \( |z| < 1 \). Consider \( Q(z) = a_0z^n + a_1z^{n-1} + \cdots + a_{n-1}z + 1 \). Note that \( Q(z) = z^nP \left( \frac{1}{z} \right) \) for \( z \neq 0 \). Thus, if \( |z| = 1 \), then \( |Q(z)| = \left| P \left( \frac{1}{z} \right) \right| < 1 \).

It follows that \( \int_0^{2\pi} |Q(e^{it})| \, dt < 2\pi \). However,

\[
\int_0^{2\pi} Q(e^{it}) \, dt = \int_0^{2\pi} \left( a_0e^{int} + a_1e^{i(n-1)t} + \cdots + a_{n-1}e^{it} + 1 \right) \, dt = 2\pi \,
\]

since \( \int_0^{2\pi} e^{ikt} \, dt = \frac{1}{ik} (e^{2k\pi i} - 1) = 0 \) for all \( k = 1, 2, \ldots, n \).

Hence, \( 2\pi = \left| \int_0^{2\pi} Q(e^{it}) \, dt \right| \leq \int_0^{2\pi} |Q(e^{it})| \, dt < 2\pi \), a contradiction.

II. Solution by Bin Zhao, student, YunYuan HuaZhong University of Technology and Science, Wuhan, Hubei, China, modified by the editor.

Let \( P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0 \). Let \( \zeta = e^{i\pi/n} \). Then \( \zeta^n = 1, \zeta^{2n} = 1, \) and \( |\zeta^k| = 1 \) for \( k = 0, 1, 2, \ldots, 2n - 1 \). Now,

\[
\sum_{k=0}^{2n-1} |P(\zeta^k)| \geq \left| (P(1) + P(\zeta^2) + \cdots + P(\zeta^{2n-2})) - (P(\zeta) + P(\zeta^3) + \cdots + P(\zeta^{2n-1})) \right|
\]

\[
= \sum_{k=0}^{n-1} \left| (P(\zeta^{2k}) - P(\zeta^{2k+1})) \right| \tag{1}
\]

and

\[
\sum_{k=0}^{n-1} (P(\zeta^{2k}) - P(\zeta^{2k+1})) = \sum_{k=0}^{n-1} (\zeta^{2k})^n - (\zeta^{2k+1})^n
\]

\[
+ \sum_{j=1}^{n-1} a_j \sum_{k=0}^{n-1} ((\zeta^{2k})^j - (\zeta^{2k+1})^j). \tag{2}
\]

Note that \( (\zeta^{2k})^n - (\zeta^{2k+1})^n = (\zeta^n)^{2k} - (\zeta^n)^{2k+1} = 1 - (-1) = 2 \), and hence,

\[
\sum_{k=0}^{n-1} ((\zeta^{2k})^n - (\zeta^{2k+1})^n) = 2n. \tag{3}
\]
On the other hand, for each \( j = 1, 2, \ldots, n - 1 \), we have
\[
\sum_{k=0}^{n-1} ((\zeta^{2k})^j - (\zeta^{2k+1})^j) = (1 - \zeta^j) \sum_{k=0}^{n-1} (\zeta^{2j})^k
= (1 - \zeta^j) \frac{1 - (\zeta^{2j})^n}{1 - \zeta^{2j}} = 0 ,
\]
(4)
since \((\zeta^{2j})^n = (\zeta^{2n})^j = 1\).

Substituting (3) and (4) into (2) and using (1), we then have
\[
\sum_{k=0}^{2n-1} |P(\zeta^k)| \geq 2n ,
\]
from which we deduce that there must be some \( k \), with \( 0 \leq k \leq 2n - 1 \), for which \(|P(\zeta^k)| \geq 1\), completing the proof.

Also solved by KEE-WAI LAU, Hong Kong, China; and the proposer. There was also one incorrect solution.

The proof given by Lau used Rouche's Theorem from complex analysis. Both solutions featured above show that \( w \) can be chosen so that \(|w| = 1\) and \(|P(w)| \geq 1\). But this is not surprising, in view of the well-known Maximum Modulus Principle.


Let \( a, b, c \) be the three sides of a triangle, and let \( h_a, h_b, h_c \) be the altitudes to the sides \( a, b, c \), respectively. Prove that
\[
\frac{h_a^2}{b^2 + c^2} \cdot \frac{h_b^2}{c^2 + a^2} \cdot \frac{h_c^2}{a^2 + b^2} \leq \left( \frac{3}{8} \right)^3 .
\]

Essentially the same solution by D. Kipp Johnson, Beaverton, OR, USA; Joel Schlosberg, Bayside, NY, USA; and D.J. Smeenk, Zaltbommel, the Netherlands.

Using the formula \( h_a = b \sin C \), the Law of Sines, and the AM–GM Inequality, we obtain
\[
\frac{h_a^2}{b^2 + c^2} = \frac{b^2 \sin^2 C}{b^2 + c^2} = \frac{\sin^2 B \sin^2 C}{\sin^2 B + \sin^2 C} \leq \frac{\sin^2 B \sin^2 C}{2 \sin B \sin C} = \frac{1}{2} \sin B \sin C ,
\]
and similarly,
\[
\frac{h_b^2}{c^2 + a^2} \leq \frac{1}{2} \sin C \sin A \quad \text{and} \quad \frac{h_c^2}{a^2 + b^2} \leq \frac{1}{2} \sin A \sin B .
\]
Multiplying these and using the well-known inequality
\[
\sin A \sin B \sin C \leq \frac{3\sqrt{3}}{8}
\]
(see [1, p. 18]), we obtain
\[
\frac{h_a^2}{b^2 + c^2} \cdot \frac{h_b^2}{c^2 + a^2} \cdot \frac{h_c^2}{a^2 + b^2} \leq \frac{1}{8} \left( \sin A \sin B \sin C \right)^2
\]
\[
\leq \frac{1}{8} \left( \frac{3\sqrt{3}}{8} \right)^2 = \left( \frac{3}{8} \right)^3.
\]
Equality holds if and only if the triangle is equilateral.

References


Also solved by ARKADY ALT, San Jose, CA, USA; MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; SEFET ARSLANAGIC, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (2 solutions); MICHEL BATAILLE, Rouen, France; QUANG CAO MINH, Nguyen Binh Kiem High School, Vinh Long, Vietnam; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; WALTHER JANOUS, Ursulengymnasium, Innsbruck, Austria; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie, AB; KEE-WAI LAU, Hong Kong, China; D.M. MILOŠEVIĆ, Pranjani, Yugoslavia; VEDULA N. MURTY, Dover, PA, USA; PANOS E. TSASOUSSOGLOU, Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; BIN ZHAO, student, YunYuan Huazhong University of Technology and Science, Wuhan, Hubei, China and the proposer.

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