SKOLIAD No. 102
Robert Bilinski

Please send solutions to the problems in this edition by November 1, 2007. A copy of MATHEMATICAL MAYHEM Vol. 4 will be presented to one pre-university reader who sends in solutions before the deadline. The decision of the editor is final.


Concours 2006 de Mathématique
du secondaire de Colombie Britannique
Ronde Finale Sénior partie B, vendredi 5 mai 2006

1. Déterminer le nombre de séquences d'entiers consécutifs dont la somme est 100.

2. Un réservoir vitré cubique de côté un mètre est placé sur une table horizontale et est rempli à moitié d'eau. Ainsi, la profondeur de l'eau dans le réservoir (la distance entre la surface de l'eau et la surface de la table) est un demi-mètre. Le réservoir est tourné autour d'une des arêtes sur la table afin qu'une des faces du réservoir ait un angle de 30° avec la table. Trouver la profondeur de l'eau après la rotation.

3. Les longueurs des côtés d'un triangle sont 13, 13 et 10. Le cercle inscrit de ce triangle est un cercle ayant son centre à l'intérieur du triangle qui est tangent à chacun des côtés du triangle. (Voir le diagramme.) Trouver le rayon du cercle inscrit.

4. Cinq entiers positifs $a$, $b$, $c$, $d$ et $e$ supérieurs à un remplissent les conditions suivantes:

\[
\begin{align*}
    a(b + c + d + e) &= 128, \\
    b(a + c + d + e) &= 155, \\
    c(a + b + d + e) &= 203, \\
    d(a + b + c + e) &= 243, \\
    e(a + b + c + d) &= 275.
\end{align*}
\]

Trouver ces cinq entiers.

5. Un arbre binaire entier consiste en un noeud racine qui a deux enfants, un noeud droit et un noeud gauche, et chacun des noeuds enfants a deux enfants, jusqu'à ce que le haut de l'arbre soit atteint, où chaque noeud n'a pas d'enfants. Dans un certain arbre binaire entier, chaque noeud est numéroté,
en commençant par 1 à la racine et en numérotant de la gauche vers la droite à travers chaque niveau. Le diagramme montre les quatre premiers niveaux d'un tel arbre. La racine d'un tel arbre est placée à l'origine d'un système de coordonnées $xy$, l'axe $x$ étant à l'horizontal et l'axe $y$ étant à la verticale, comme illustré. Si l'espace entre les niveaux de l'arbre est de 2 unités dans la direction $y$ et l'espacement entre les noeuds du niveau supérieur qui contient le noyau numéroté 2006 est de 2 unités dans la direction $x$, trouver les coordonnées du noyau numéroté 2006.

![Diagramme de l'arbre](image-url)

**British Columbia Secondary School**  
**Mathematics Contest 2006**  
**Senior Final Round, Part B, Friday, May 5, 2006**

1. Determine the number of sequences of consecutive integers whose sum is 100.

2. A cubical glass tank with sides of length one metre is placed on a horizontal table and half filled with water. Thus, the depth of the water in the tank (the distance of the surface of the water from the surface of the table) is one half metre. The tank is rotated about one of the edges that is on the table so that one face of the tank makes a 30° angle with the table. Find the depth of the water in the tank after the rotation.

3. The lengths of the sides of a triangle are 13, 13, and 10. The *inscribed circle* of this triangle is the circle with centre inside the circle that is tangent to each of the three sides of the triangle. (See the diagram.) Find the radius of the inscribed circle.

4. Five positive integers $a$, $b$, $c$, $d$, and $e$ greater than one make the following conditions true:

   \[
   a(b + c + d + e) = 128, \\
   b(a + c + d + e) = 155, \\
   c(a + b + d + e) = 203, \\
   d(a + b + c + e) = 243, \\
   e(a + b + c + d) = 275.
   \]

Find the five integers.
5. A full binary tree consists of a root node which has two children, a right child node and a left child node, and each child node has two children, until the top of the tree is reached, where each node has no children. In a certain full binary tree each node is numbered, starting with 1 at the root, numbering from left to right across each level. The diagram shows the first four levels of such a tree. The root of the tree is placed at the origin of an xy-coordinate system, with the x-axis horizontal and the y-axis vertical, as shown. If the spacing between the levels of the tree is 2 units in the y-direction and the spacing between the nodes at the top level that contains the node numbered 2006 is 2 units in the x-direction, find the coordinates of the node numbered 2006.

1. Grade 9 only. In this problem we'll be placing various arrangements of 10¢ and 20¢ coins on the nine squares of a 3 × 3 grid. Exactly one coin will be placed in each of the nine squares. The grid has four 2 × 2 subsquares each containing a corner, the centre, and the two squares adjacent to these. One example is shown in the diagram.

(a) Find an arrangement where the totals of the four 2 × 2 subsquares are 40¢, 60¢, 40¢, and 70¢ in any order. (Draw a diagram showing your arrangement.)

(b) Find an arrangement where the totals of the four 2 × 2 subsquares are 50¢, 60¢, 70¢, and 80¢ in any order. (Draw a diagram showing your arrangement.)

For each part of the problem below, illustrate your answer with a suitable arrangement and an explanation of why no other suitable arrangement contains a larger (part (c)) or a smaller (part (d)) amount of money.

(c) What is the maximum amount of money which can be placed on the grid so that each of the 2 × 2 subsquares contains exactly 50¢?
(d) What is the minimum amount of money which can be placed on the grid so that the average of the amount of money in each of the $2 \times 2$ subsquares is exactly 60¢?

Solution by the editor.

(a) Since one $2 \times 2$ subsquare contains only 40¢, all of its entries must be 10¢ coins. In particular, the centre square must contain a 10¢ coin. Since another of the $2 \times 2$ subsquares contains 70¢ and it must also contain the 10¢ coin in the centre square, its remaining entries must be 20¢. It is then easy to complete the square as in Figure 1 below. Other squares are possible by rotation.

(b) Since one $2 \times 2$ subsquare contains 80¢, all of its entries must be 20¢ coins. In particular, the centre square must contain a 20¢ coin. Since another of the $2 \times 2$ subsquares contains only 50¢ and it must also contain the 20¢ coin in the centre square, its remaining entries must be 10¢. It is then easy to complete the square as in Figure 2 below. Other squares are possible by rotation.

(c) The amount of money will be maximized when the number of 20¢ coins on the grid is maximized. If we place a 20¢ coin in the centre square, then all the remaining coins have to be 10¢ coins. If we place a 20¢ coin along one side of the grid, in the middle square of that side, then at most two more 20¢ coins may be placed on the grid (in the corners on the opposite side). But we can use four 20¢ coins if they are placed in the four corners of the grid as in Figure 3 below. Thus, the maximum number of 20¢ coins that can be used is four, giving a total of $1.30.

(d) The distribution in Figure 4 below satisfies the conditions and uses $1.30, which includes only three 20¢ coins. If we are to lower that total at all, we must use at most two 20¢ coins. But then none of the $2 \times 2$ subsquares could contain more than two 20¢ coins, and therefore none could contain more than 60¢. Since the average across all four $2 \times 2$ subsquares must be 60¢, each of the four would then have to contain exactly 60¢, which means that each would have to contain two 20¢ coins. The same two 20¢ coins would then have to be in all four $2 \times 2$ subsquares, which is impossible. Thus, the minimum is $1.30.

<table>
<thead>
<tr>
<th>10 10 20</th>
<th>10 10 10</th>
<th>20 10 20</th>
<th>10 20 10</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 10 20</td>
<td>10 20 20</td>
<td>10 10 20</td>
<td>10 20 10</td>
</tr>
<tr>
<td>20 20 20</td>
<td>20 20 20</td>
<td>20 10 20</td>
<td>10 20 10</td>
</tr>
</tbody>
</table>

Figure 1  Figure 2  Figure 3  Figure 4
2. Humankind was recently contacted by three alien races: the Kween, the Ozdaks, and the Merkuns. Little is known about these races except that Kweens always speak the truth while Ozdaks always lie. In any group of aliens a Merkun will never speak first. When it does speak, it tells the truth if the previous statement was a lie, and lies if the previous statement was truthful. Although the aliens can readily tell one another apart, of course to humans all aliens look the same.

A high-level delegation of three aliens has been sent to Earth to negotiate our fate. Among them is at least one Kween. On arrival they make the following statements (in order):

First Alien: The second alien is a Merkun.
Second Alien: The third alien is not a Merkun.
Third Alien: The first alien is a Merkun.

Which alien or aliens can you be certain are Kweens?

Official solution, expanded by the editor.

The first alien cannot be a Merkun, since a Merkun never speaks first. Thus, the third statement must be a lie, which means that the third alien cannot be a Kween.

Suppose that the third alien is a Merkun (who is lying). Since Merkuns only lie when the previous statement is true, the second statement must be true. But then we have a contradiction. Hence, the third alien is an Ozdak.

Since there is a Kween among the three aliens, it must be one of the first two aliens. Suppose that the first is a Kween. Then the second is a Merkun and its statement is a lie. But this implies that the third alien is a Merkun, which we have already ruled out. Therefore, the first alien is not a Kween, which means that the first alien is an Ozdak and the second is a Kween.

Thus, only the second alien is a Kween (and the others are both Ozdaks).

3. (Note: In this question an “equal division” is one where the total weight of the two parts is the same.)

(a) Belinda and Charles are burglars. Among the loot from their latest caper is a set of 12 gold weights of 1g, 2g, 3g, and so on, through to 12g. Can they divide the weights equally between them? If so, explain how they can do it, and if not, why not?

(b) When Belinda and Charles take the remainder of the loot to Freddy the fence, he demands the 12g weight as his payment. Can Belinda and Charles divide the remaining 11 weights equally between them? If so, explain how they can do it, and if not, why not?

(c) Belinda and Charles also have a set of 150 silver weights of 1g, 2g, 3g, and so on, through to 150g. Can they divide these weights equally between them? If so, explain how they can do it, and if not, why not?
Official solution.

(a) Yes, it can be done. One way is to pair them from the "extremes": 1 + 12, 2 + 11, . . . , 6 + 7. Each person then takes three of the pairs. There are other possibilities.

(b) Yes, it can still be done. Belinda gets 6 + 1 and Charles gets 7. The rest can be paired from the "extremes" (2 + 11, 3 + 10, 4 + 9, 5 + 8), with each person taking two of the pairs.

(c) No, it cannot be done. There are 75 even and 75 odd weights, which implies that the total is odd and cannot be split equally.

4. A chessboard is an 8 × 8 grid of squares. One of the chess pieces, the king, moves one square at a time in any direction, including diagonally.

(a) A king (denoted by K in the diagram) stands on the lower left corner of a chessboard. It has to reach the square marked F in exactly three moves. Show that the king can do this in exactly four different ways.

(b) Assume that the king is placed back on the bottom left corner. In how many ways can it reach the upper left corner (marked G) in exactly seven moves?

Official solution.

(a) Since each move must find the king one row higher, the following four diagrams illustrate all possible routes.

(b) The king must move one row higher on each move. In each square that could be part of the king's seven-move path from bottom left to upper left corner, we place the number of ways the king can reach that square. (Each number may be obtained by adding the numbers in the squares below it and having an edge or corner in common with it.) We find that there are 127 ways the king can reach the upper left corner in seven moves.
5. (Note: For this question answers containing expressions such as $\frac{4\pi}{13}$ are acceptable. If you have a calculator you may use the button for $\pi$ if you like.)

(a) The Jones family lives in a perfectly square house, 10m by 10m, which is placed exactly in the middle of a 40m by 40m lot, entirely covered (except for the house) in grass. They keep the family pet, Dolly the sheep, tethered to the middle of one side of the house on a 15m rope. What is the area of the part of the lawn (in m$^2$) in which Dolly is able to graze? (See shaded area.)

(b) The Jones' neighbours, the Smiths, have an identical lot to the Jones but their house is located five metres to the North of the centre. Their pet sheep, Daisy, is tethered to the middle of the southern side of the house on a 20m rope. What is the area of the part of the lawn (in m$^2$) in which Daisy is able to graze?

Official solution.

(a) The part of the lawn in which Dolly can graze consists of a semicircle of radius 15 and two quarter circles of radius 10. Its area is therefore $\frac{1}{2}\pi15^2 + 2 \cdot \frac{1}{4}\pi10^2 = \frac{325}{2}\pi$.

(b) The part of the lawn in which Daisy can graze is made up from one semicircle and four other quarter circles, as shown. Its area is therefore $\frac{1}{2}\pi20^2 + 2 \cdot \frac{1}{4}\pi15^2 + 2 \cdot \frac{1}{4}\pi5^2 = 325\pi$.

That brings us to the end of another issue. Please send in solutions!