SPAs and the Harmonic Mean

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Definition: An SPA is a Symmetric Polygonal Arc, consisting of three equal straight line segments that have equal angles between adjacent segments. For example:

An SPA could be a portion of a regular polygon, or it could be a whole equilateral triangle, depending on its angle.

In this article, we will explore some properties of SPAs and their connection with the harmonic mean. My reason for being interested in SPAs came from a MAYHEM problem where two equilateral triangles were placed adjacent to one another on the same line (see problem M214 [2005 : 427, 428; 2006 : 428]).

It is convenient to use an external angle as a parameter, say $\theta$. The other convenient parameter is the length of each line segment, say $a$. We will refer to the middle line segment as the base of the SPA.

Place two SPAs with different length parameters $a$ and $b$ and the same angle parameter $\theta$ on the same base line with one point in common, as shown below.
Join $BA_1$ and $AB_1$. Let $P$ be the point of intersection of the lines $BA_1$ and $OB_2$, and let $Q$ be the point of intersection of the lines $AB_1$ and $OA_2$.

Then $OP = OQ = \frac{ab}{a+b}$. This is one half of the harmonic mean of $a$ and $b$, and is independent of the parameter $\theta$.

We will prove this remarkable fact in two different ways, each of which is instructive.

First Proof: Note that $\triangle A_1AB$ and $\triangle POB$ are similar. We therefore have \( \frac{OP}{OB} = \frac{AA_1}{AB} \). Since $A_1A = AO = a$ and $OB = b$, we obtain \( \frac{OP}{b} = \frac{a}{a+b} \), giving \( OP = \frac{ab}{a+b} \). Similarly, using similar triangles $B_1BA$ and $QOA$, we see that \( OQ = \frac{ab}{a+b} \).

Second Proof: This proof uses coordinates. Let $O = (0, 0)$, $A = (a, 0)$, and $B = (-b, 0)$. Note that $AA_1 = OA = a$. The coordinates of $A_1$ are $(a(1 + \cos \theta), a \sin \theta)$, and the coordinates of $A_2$ are $(-a \cos \theta, a \sin \theta)$. The coordinates of $B_1$ are $(-b \cos \theta, b \sin \theta)$, and the coordinates of $B_2$ are $(b \cos \theta, b \sin \theta)$.

To determine the coordinates of $P$, we find the equations of the lines $BA_1$ and $OB_2$. The point of intersection of these lines is

\[
P = \left( \frac{ab \cos \theta}{a+b}, \frac{ab \sin \theta}{a+b} \right).
\]

Similarly, by finding the equations of the lines $AB_1$ and $OA_2$, we obtain the coordinates of $Q$:

\[
Q = \left( -\frac{ab \cos \theta}{a+b}, \frac{ab \sin \theta}{a+b} \right).
\]

From this, it is clear that $OP = OQ = \frac{ab}{a+b}$.

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