**Problem of the Month**

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**Problem** (1993 Euclid Contest) In a sequence of \( p \) zeroes and \( q \) ones, the \( i \)th term, \( t_i \), is called a change point if \( t_i \neq t_{i-1} \), for \( i = 2, 3, 4, \ldots, p + q \). For example, the sequence 0, 1, 0, 0, 1, 0, 1 has \( p = q = 4 \), and five change points \( t_2, t_4, t_6, t_7, t_8 \). For all possible sequences of \( p \) zeroes and \( q \) ones with \( 1 \leq p \leq q \), determine

(i) the minimum and maximum number of change points, and

(ii) the average number of change points.

The notation of this problem makes it look scary, but the problem isn't really so bad. Basically, we are being asked how many times two consecutive terms in a sequence of 0s and 1s are different. We are given that the number of 0s in the sequence is \( p \) and the number of 1s is \( q \). If this notation makes you queasy, try working on a particular case like \( p = 5 \) and \( q = 7 \). Answering the two questions in this special case is still an interesting task.

We'll solve (i) first. Before we launch into its solution, the first teaching point from this problem arises. In order to show that \( M \) is the maximum, we need to do two things: we must justify why we cannot have more than \( M \) change points, and we must show that we can have exactly \( M \) change points. (Why this second step? Well, if there can't be more than, say, 10 change points in a particular sequence, then there can't be more than 1000 either! But \( M \) can't be both 10 and 1000.) Similar things need to be shown for the minimum, and more generally, in any optimization problem (that is, maximum or minimum problem).

**Solution to (i):** Let's first look for the minimum, \( m \). Certainly \( m \geq 0 \), because we can't have a negative number of change points. Can there be 0 change points? No. Since any sequence contains both 0s and 1s, there must be a 0 next to a 1 somewhere. Therefore, \( m \) is at least 1.

Could the minimum be 1? Yes—the sequence 0, 0, ..., 0, 1, 1, ..., 1 has only one change point. We have shown that the number of change points must be at least 1 and can in fact be 1. So the minimum is 1.

How about the maximum, \( M \)? This is trickier. Each sequence with \( p \) zeroes and \( q \) ones has \( p + q \) terms; thus, \( M \) is certainly no larger than \( p + q \). But the first term cannot be a change point (check the definition). Therefore, \( M \) is no larger than \( p + q - 1 \).

Could every term but the first be a change point? Try fiddling for a minute or two to see what you can discover. You could perhaps try a few different possible values for \( p \) and \( q \).

Any luck? Let's look first at the case \( p = q \). In this case, yes, there can be \( p + q - 1 = 2p - 1 \) change points, because the sequence could alternate between 0 and 1—for example, 0, 1, 0, 1, ..., 0, 1. Thus, if \( p = q \), the maximum number of change points is \( M = p + q - 1 = 2p - 1 \).

What if \( p < q \)? Notice that every change point involves a 0, either in
that position or in the position before. Each of the \( p \) zeroes can contribute to at most two change points (and to exactly two if it has 1s on both sides of it). Since there are \( p \) zeroes, there can be at most \( 2p \) change points.

Can this upper bound be achieved? (Oops!—that's fancy mathematics-speak for “Can we actually find a sequence with \( 2p \) change points?”) Yes—for example, \( 1, 0, 1, 0, 1, \ldots, 0, 1, 1, \ldots, 1 \) is a sequence with \( p < q \) which has \( 2p \) change points (two for each 0). Therefore, \( M = 2p \) if \( p < q \).

Note that the number of 0s controls the maximum number of change points here. The number of 1s is less important, because there are more 1s than needed.

We need to look at (ii) next. Enter stage left the second teaching point. To figure out the average number of change points, we need to figure out the total number of change points over all sequences and divide by the total number of sequences. This seems to require looking at individual sequences and determining the number of change points in each sequence. We might then have to figure out how many sequences have 1 change point, how many have 2, and so on. This would actually be really painful. If you're feeling particularly ambitious, you could of course try this!

There is a sneaky way to do this. If we could determine the total number of sequences in which position 2 is a change point, the number in which position 3 is a change point, and so on, we could add these totals to get the total number of change points. (Of course, we still have to divide by the total number of sequences.)

**Solution to (ii).** Let’s first find the total number of sequences. Since there are \( p + q \) positions in total, we can choose \( p \) places to put the 0s. This means that there are \( \binom{p+q}{p} \) sequences in total.

Now consider position \( k \) in the sequences, where \( k \) can be any integer from 2 to \( p + q \). In order for position \( k \) to be a change point, we must have \( t_k = 0 \) and \( t_{k-1} = 1 \), or \( t_k = 1 \) and \( t_{k-1} = 0 \). How many sequences are there with \( t_k = 0 \) and \( t_{k-1} = 1 \)? Such a sequence has \( p-1 \) zeroes and \( q-1 \) ones to put in the remaining \( p+q-2 \) positions; thus, there are \( \binom{p+q-2}{p-1} \) such sequences. Similarly, there are \( \binom{p+q-2}{q-1} \) sequences with \( t_k = 1 \) and \( t_{k-1} = 0 \).

Therefore, there are \( 2\binom{p+q-2}{p-1} \) sequences with a change point in position \( k \). Notice that this total is independent of \( k \).

Since there are \( p + q - 1 \) possible positions for a change point and the number of change points in a fixed position is a constant, the total number of change points is \( 2(p + q - 1)\binom{p+q-2}{p-1} \), which means an average of

\[
\frac{2(p + q - 1)\binom{p+q-2}{p-1}}{\binom{p+q}{p}} = \frac{2(p + q - 1)(p + q - 2)!}{(p-1)!(q-1)!} \cdot \frac{(p+q)!}{p!q!} = \frac{2(p + q - 1)!p!q!}{(p + q)!(p-1)!(q-1)!} = \frac{2pq}{p + q}.
\]