Contributor Profiles: Michel Bataille

Michel Bataille was born in 1952 in a small village near Dieppe in Normandy. His family had a very modest income (his father was a workman in a factory), but, with the help of a state scholarship, he was able to attend secondary studies in Dieppe until 1969 and then graduate at the University of Rouen. He started teaching secondary school in 1976. In his spare time he prepared for the "agrégation" (the highest competitive examination for secondary school teachers in France), which he passed in 1983.

Apart from two years in Marrakech, Morocco (from 1977 to 1979), he has always taught in the suburbs of Rouen, France. He initially taught students between the ages of 12 and 16; but then, for ten years, he prepared students for the "baccalauréat", the French school-exit certificate. Since 1994, he has been teaching at the undergraduate level. Bataille's students sit competitive entrance examinations to various engineering schools. They are generally from a modest background, and they specialize in technology during their secondary studies. His task is to bring them to the required level in mathematics. The challenge of improving their knowledge is quite enjoyable, since "most of them are well-motivated", says Bataille.

He has been married since 1972. His wife is also a math teacher (however, he claims that her interests lie more in the realm of botany and gardening nowadays). They have two children.

His hobbies include cinema, reading (in French or English), crosswords, listening to music (indie rock and electronica), and walking. During the '90s, his leisure time was spent improving his English, linguistics, and billiards. Most of these activities have now been replaced by problem solving.

His first published solution appeared in ***CRUX with MAYHEM*** in March 1999. Since then, he has developed a real taste for problem solving and has contributed more and more solutions. Besides ***CRUX with MAYHEM***, Bataille has become a regular solver and poser in several problem sections (in the MAA publications, the Mathematical Gazette, and the Bulletin of the Association of French Math Teachers).

From an Editor's point of view, having a regular contributor like Michel Bataille is a real joy. His solutions are always correct and well-organized. Not only that, but he submits proposals on a regular basis, which are always well thought out and explained, with just the appropriate amount of detail. It has been a real pleasure for this Editor-in-Chief to have developed an ongoing friendship with such a warm human being and such a good mathematician.

EDITORIAL

Jim Totten

I would like to launch an appeal to our readers at this time, especially to our long-time readers. The Canadian Mathematical Society (CMS), which publishes ***CRUX with MAYHEM***, has asked our editorial board to provide a list of favourite problems from past issues of ***Crux Mathematicorum*** and ***CRUX with MAYHEM***. Since many on our editorial board have not been associated with CRUX for as long as some of our readers, we decided that a "Readers' Choice" selection of favourite problems from over the years would likely be more productive, as well as being more meaningful. We are looking especially for favourites from the earlier volumes.

I first mentioned this project in my Year End Finale in the December 2006 issue, and I want to reiterate it here. If you have some favourite problems from the pages of ***Crux Mathematicorum*** or ***CRUX with MAYHEM***, please forward those problem numbers to us. I would also appreciate if you could provide the reference to the volume, issue, and page numbers, in case a typo creeps into the list of pure problem numbers. Thank you. We are looking for somewhere between 100 and 200 problems, and we are hoping to receive your list of favourites by February 28, 2007. Once we have your favourite problems, the CMS is planning to make them available on CD (or perhaps a flash memory stick).

Our sincere thanks go out to Shawn Godin, who has been the Mayhem Editor for the past six years. Shawn has molded that section of our journal almost single-handedly into an excellent feature for high school students and teachers. However, he has been juggling many duties for quite some time now and felt it was time to move on.

Having said that, the Board of ***CRUX with MAYHEM*** extends a warm welcome to Jeff Hooper as the new Mayhem Editor. Jeff has acted as the Assistant Mayhem Editor for the past year, so the transition should be almost seamless.

You will have noticed that we introduced a new feature to ***CRUX with MAYHEM*** this past year, namely the "Contributor Profiles" section. It did not appear in every issue, but only when there was space available. I have received limited feedback on this new feature, but all the feedback has been positive. I personally have appreciated getting to know a little more about many of our regular contributors and what they look like.

If you can think of ways for us to produce a better magazine, be sure to send me those suggestions. We are always interested in ways to improve.

SKOLIAD No. 99

Robert Bilinski

Please send your solutions to the problems in this edition by ***1 August, 2007***. A copy of **MATHEMATICAL MAYHEM Vol. 1** will be presented to one pre-university reader who sends in solutions before the deadline. The decision of the editor is final.

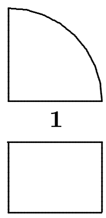
Our problems this month come from the Collège Montmorency Contest, 2004–2005. We thank André Labelle, of Collège Montmorency, who looks after this contest designed for secondary students from the Laval region.

Montmorency Contest 2004–05

Sec V, November 2004

1. The golden ratio $N = \frac{1+\sqrt{5}}{2} \approx 1.618033989\ldots$ has the remarkable property that its multiplicative inverse $1/N$ is equal to its decimal part $0.618033989\ldots$. Find another number with this property.

2. Consider a quarter circle of radius 1.

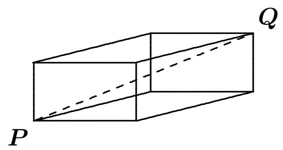


(a) Find a rectangle having the same area and the same perimeter as the quarter circle.

(b) For a complete circle of radius 1, is it possible to find such a rectangle, having an area and a perimeter equal to that of the circle? Justify your answer.

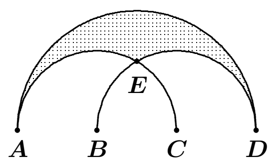
3. A barrel is filled with water. We empty half of its contents and then add a litre of water. After doing this operation seven consecutive times, we are left with three litres of water in the barrel. How many litres were in the barrel at the beginning?

4. The areas of three faces of a rectangular parallelepiped are 18 cm^2, 40 cm^2 and 80 cm^2. Find: (a) its volume; (b) the length of its diagonal $\overline{PQ}$.

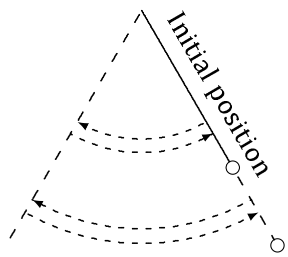


5. Evaluate $\sqrt{6+\sqrt{6+\sqrt{6+\cdots}}}$.

6. Let A, B, C, and D be collinear points such that $\overline{AB} = \overline{BC} = \overline{CD} = 1$. Consider three semi-circles of respective diameters $\overline{AC}$, $\overline{BD}$ and $\overline{AD}$. Let E be the intersection of the semi-circles with centres B and C. Determine the area of the curvilinear triangle AED (shaded in the drawing).



7. The oscillation period of a pendulum is proportional to the square root of its length (for example, to triple the oscillation period, we multiply the length by nine). Two pendulums of different lengths are released from the initial position shown. The shorter one measures 25 cm, and its oscillation period is 1 second. The two pendulums are aligned again for the first time after 7 seconds in their initial position. Find the length of the longer pendulum. (Air resistance is neglected.)



8. In a refinery, a cylindrical storage tank has a spiral staircase one meter wide attached to its exterior. The staircase goes from the bottom to the top while making exactly 2 complete revolutions. If the tank has a height of 10 m and a diameter of 8 m, find the length of the exterior edge of the staircase.

Concours Montmorency 2004–05

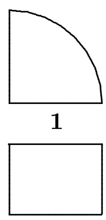
Sec V, novembre 2004

1. Le nombre d'or $N = \frac{1+\sqrt{5}}{2} \approx 1,618033989\ldots$ a la remarquable propriété que son inverse multiplicatif $1/N$ est égal à sa partie décimale $0,618033989\ldots$. Trouver un autre nombre ayant cette propriété.

2. Considérons un quart de cercle de rayon 1.

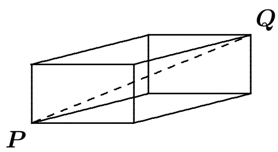
(a) Trouver le rectangle qui a à la fois la même aire et le même périmètre que ce quart de cercle.

(b) Est-il possible que pour un cercle complet de rayon 1, on puisse trouver un tel rectangle ayant à la fois la même aire et le même périmètre que ce cercle ? Justifier votre réponse.



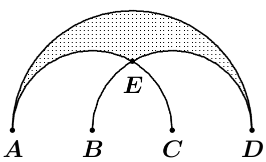
3. Un baril est rempli d'eau. On en vide la moitié et on ajoute un litre d'eau. Après avoir effectué cette opération sept fois, il en reste trois litres. Combien y avait-il de litres d'eau dans le baril au départ ?

4. Les aires de trois des faces d'un parallélépipède rectangle sont de 18 cm^2, 40 cm^2 et 80 cm^2. Trouver : (a) le volume ; (b) la longueur de la grande diagonale $\overline{PQ}$.

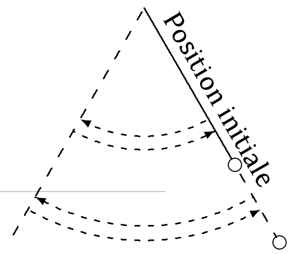


5. Évaluer $\sqrt{6+\sqrt{6+\sqrt{6+\cdots}}}$.

6. Soit A, B, C et D des points alignés tels que $\overline{AB} = \overline{BC} = \overline{CD} = 1$. On considère trois demi-cercles de diamètres respectifs $\overline{AC}$, $\overline{BD}$ et $\overline{AD}$. Soit E l'intersection des demi-cercles de centres B et C. Déterminer l'aire du triangle curviligne AED (partie hachurée sur le dessin).



7. On sait que la période d'oscillation d'un pendule est proportionnelle à la racine carrée de la longueur de celui-ci (par exemple, pour tripler la période d'oscillation d'un pendule, il faut multiplier par neuf sa longueur). En lâchant deux pendules de longueurs différentes à la position initiale indiquée et en négligeant la résistance de l'air, si le plus court mesure 25 cm, que sa période d'oscillation est de une seconde et que lorsqu'on les retrouvent à nouveau parallèles pour la première fois au bout de sept secondes, ils occupent leur position initiale, évaluer la longueur du plus long.



8. Dans une raffinerie, un réservoir d'essence cylindrique est muni d'un escalier d'un mètre de large qui longe de bas en haut la paroi extérieure dans une spirale et en fait exactement deux fois le tour. Sachant que le réservoir a 10 mètres de haut et 8 mètres de diamètre, trouver la longueur de la rampe extérieure de l'escalier.

Next we give the solutions to the 2005 BC Colleges Senior High School Mathematics Contest Final Round Part B [2006 : 193–195].

1. Les chiffres 1, 2, 3, 4 et 5 sont tous utilisés une fois pour écrire un nombre à cinq chiffres $abcde$ tel que le nombre à trois chiffres abc est divisible par 4, bcd est divisible par 5, cde est divisible par 3. Quel est le chiffre a ?

Solution par Jean-David Houle, étudiant, Cégep de Drummondville, Drummondville, QC.

Puisque 5 divise bcd, nous devons avoir d divisible par 5 et dans les choix possibles de chiffres, donc $d = 5$. Puisque 4 divise abc, alors ce nombre se termine par 2 ou 4, selon que b est pair ou impair. Puisque 3 divise cde, alors 3 divise la somme $c + d + e$, et donc $c + e \equiv 1 \pmod 3$ (car $d = 5$). Donc deux possibilités s'offrent à nous :

Si $c = 2$, alors $e \equiv 2 \pmod 3$ avec $e \in \{1, 3, 4\}$, ce qui est impossible.

Si $c = 4$, alors $e \equiv 0 \pmod 3$ avec $e \in \{1, 2, 3\}$, donc $e = 3$.

Donc $c = 4$, ce qui implique que b est pair avec $b \in \{1, 2\}$. Donc b vaut 2, ce qui laisse $a = 1$. Donc, $abcde = 12453$, et $a = 1$.

Solutioné aussi par Glenier L. Bello-Burguet, étudiant 4to de ESO, Instituto Hermanos D'Elhuyar, Logroño, Espagne ; Natalia Desy, étudiant, SMP Xaverius 1, Palembang, L'Indonésie ; et Alex Remorov, étudiant, William Lyon Mackenzie Collegiate Institute, Toronto, ON.

2. An urn contains three white, six red, and four black balls.

(a) If one ball is selected at random, what is the probability that the ball selected is red?

(b) If two balls are selected at random, what is the probability that they are both black?

(c) If two balls are selected at random, what is the probability that they are both black, given that they are the same colour?

Solution by Glenier L. Bello-Burguet, student 4to de ESO, Instituto Hermanos D'Elhuyar, Logroño, Spain.

(a) $P(\text{red}) = \dfrac{\#\text{ of red}}{\text{total }\#\text{ of outcomes}} = \dfrac{6}{3+6+4} = \dfrac{6}{13}$.

(b) $P(2\text{ black}) = \binom{4}{2}\Big/\binom{13}{2} = \dfrac{6}{78} = \dfrac{1}{13}$.

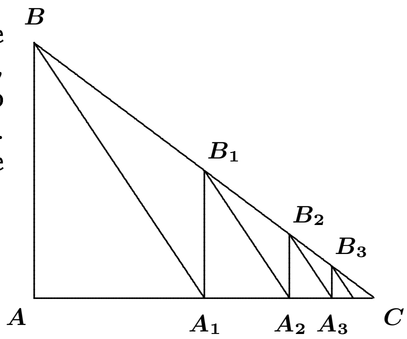
(c) We have

$$P(2\text{ black}|2\text{ same}) = \frac{\#\text{ of black pairs}}{\#\text{ of black pairs} + \#\text{ of red pairs} + \#\text{ of white pairs}}$$
$$= \frac{\binom{4}{2}}{\binom{4}{2}+\binom{6}{2}+\binom{3}{2}} = \frac{1}{4}\,.$$

Also solved by Jean-David Houle, student, Cégep de Drummondville, Drummondville, QC; Natalia Desy, student, SMP Xaverius 1, Palembang, Indonesia; and Alex Remorov, student, William Lyon Mackenzie Collegiate Institute, Toronto, ON.

3. In the diagram, ABC is a right-triangle with $\overline{AB} = 3$ and $\overline{AC} = 4$. Furthermore, each line segment A_iB_i is perpendicular to AC, A_1 bisects AC, and A_{i+1} bisects A_iC. Find the total length of the sequence of the diagonal segments:

$$\overline{BA_1} + \overline{B_1A_2} + \overline{B_2A_3} + \cdots$$



Official solution.

Since A_1 bisects line segment AC, the length of the segment AA_1 is $\overline{AA_1} = 2$ and the length of the segment B_1A_1 is $\overline{B_1A_1} = \frac{3}{2}$. The length of the segment $\overline{BA_1}$ is given by the Theorem of Pythagoras as

$$\overline{BA_1} = \sqrt{\overline{BA}^2 + \overline{AA_1}^2} = \sqrt{3^2+2^2} = \sqrt{13}\,.$$

Each of the triangles $B_iA_iA_{i+1}$ is similar to the triangle BAA_1, with all dimensions reduced by a factor of $\frac{1}{2}$ at each step. Thus, the total length of the sequence of diagonal segments is

$$\sqrt{13}+\frac{\sqrt{13}}{2}+\frac{\sqrt{13}}{2^2}+\cdots+\frac{\sqrt{13}}{2^n}+\cdots$$
$$=\ \sqrt{13}\left(1+\frac{1}{2}+\cdots+\frac{1}{2^i}+\cdots\right)\ =\ \sqrt{13}\left(\frac{1}{1-\frac{1}{2}}\right)\ =\ 2\sqrt{13}\,.$$

Also solved by Jean-David Houle, student, Cégep de Drummondville, Drummondville, QC; Natalia Desy, student, SMP Xaverius 1, Palembang, Indonesia; and Alex Remorov, student, William Lyon Mackenzie Collegiate Institute, Toronto, ON.

4. The equation

$$x^2-3x+q\ =\ 0$$

has two real roots, α and β. Knowing that $\alpha^3+\beta^3=81$, find the value of q.

Hint: It is best not to use the quadratic formula.

Solution by Natalia Desy, student, SMP Xaverius 1, Palembang, Indonesia.

We have $\alpha+\beta=3$ and $\alpha\beta=q$. Since $\alpha^3+\beta^3=81$, we have $(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)=81$ or $27-9q=81$, which gives $q=-6$.

Also solved by Natalia Desy, student, SMP Xaverius 1, Palembang, Indonesia (second solution); Jean-David Houle, student, Cégep de Drummondville, Drummondville, QC; and Alex Remorov, student, William Lyon Mackenzie Collegiate Institute, Toronto, ON.

5. A four-digit number which is a perfect square is created by writing Anne's age in years followed by Tom's age in years. Similarly, in 31 years, their ages in the same order will again form a four-digit perfect square. Determine the present ages of Anne and Tom.

Solved by Alex Remorov, student, William Lyon Mackenzie Collegiate Institute, Toronto, ON.

Let Anne's age be a and Tom's age be t. Let a have m digits and let t have n digits. We know that $m+n=4$. If $m<2$, then $n\geq 3$. But in 31 years, a must be a 2–digit number. Thus, the second 4–digit number would have at least 5 digits, an impossibility. By symmetry, we cannot have n less than 2, which means $m=n=2$.

Let the squares be x^2 and y^2 in that order, where $0<x<y$. Since both squares have 4 digits, we have $y^2=x^2+3131$, or $(y-x)(y+x)=31\times 101$. Since $0<x<y$, we get $y-x=31$ and $y+x=101$. This yields $(x,y)=(35,66)$. Hence, $x^2=1225$ and $y^2=4356$, which implies that Anne is 12 and Tom is 25.

Also solved by Natalia Desy, student, SMP Xaverius 1, Palembang, Indonesia; and Jean-David Houle, student, Cégep de Drummondville, Drummondville, QC.

That brings us to the end of another issue. This month's winners of a past Volume of ***Mathematical Mayhem*** are Jean-David Houle, Natalia Desy, and Alex Remorov. Congratulations, Jean-David, Natalia, and Alex. Continue sending in your contests and solutions.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

Mayhem Problems

Please send your solutions to the problems in this edition by **1 June 2007**. *Solutions received after this date will only be considered if there is time before publication of the solutions.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English.

The editor thanks Jean-Marc Terrier and Martin Goldstein of the University of Montreal for translations of the problems.

M276. *Proposed by Babis Stergiou, Chalkida, Greece.*

In rectangle $ABCD$, points E and F divide side DC into three equal parts $DE = EF = FC$ and points G and H divide side BC into three equal parts $BG = GH = HC$. The line AE cuts the lines DG and DH at points K and L, respectively. Similarly, the line AF cuts the lines DG and DH at points M and N, respectively. Show that $KN \parallel CD$.

M277. *Proposed by Edward J. Barbeau, University of Toronto, Toronto, ON.*

Let $f(n,k)$ be the number of ways of distributing k candies to n children so that each child receives at most two candies. For example, if $n = 3$, then $f(3,7) = 0$, $f(3,6) = 1$, and $f(3,4) = 6$. Determine the value of

$$f(2007,1) + f(2007,4) + f(2007,7) + \cdots + f(2007,4012)\,.$$

M278. *Proposed by J. Walter Lynch, Athens, GA, USA.*

Find sixteen 16–digit palindromes, in each of which the product of the non-zero digits and the sum of the digits are both equal to 16. How many such numbers are there?

M279. *Proposed by K.R.S. Sastry, Bangalore, India.*

Determine an infinite set of rational number solutions (α, β) to the equation $\alpha^2 + \beta^2 = \alpha^3 + \beta^3$.

M280. *Proposed by the Mayhem Staff.*

An equilateral triangle lies in the plane with two of its vertices at points $(0, 0)$ and $(0, n)$. Determine the number of points (x, y) with integer coordinates which lie in the interior of the triangle.

M281. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

A square of side length s is inscribed symmetrically inside a sector of a circle with radius of length r and central angle of 60°, such that two vertices lie on the straight sides of the sector and two vertices lie on the circular arc of the sector. Determine the exact value of s/r.

. . . .

M276. *Proposé par Babis Stergiou, Chalkida, Grèce.*

Dans un rectangle $ABCD$ on divise le côté DC en trois parties égales $DE = EF = FC$ et le côté BC en trois parties égales $BG = GH = HC$. La droite AE coupe les droites DG et DH aux points respectifs K et L. De manière analogue, la droite AF coupe les droites DG et DH aux points respectifs M et N. Montrer que $KN \parallel CD$.

M277. *Proposé par Edward J. Barbeau, Université de Toronto, Toronto, ON.*

Soit $f(n, k)$ le nombre de possibilités de distribuer k bonbons à n enfants, de sorte que chaque enfant en reçoive au plus deux. Par exemple, si $n = 3$, alors $f(3, 7) = 0$, $f(3, 6) = 1$ et $f(3, 4) = 6$. Déterminer la valeur de

$$f(2007, 1) + f(2007, 4) + f(2007, 7) + \cdots + f(2007, 4012) .$$

M278. *Proposé par J. Walter Lynch, Athens, GA, USA.*

Trouver seize palindromes, chacun comprenant seize chiffres, de telle sorte que le produit des chiffres non nuls et la somme des chiffres de chaque palindrome soient tous deux égaux à seize. Combien y a-t-il de tels nombres ?

M279. *Proposé par K.R.S. Sastry, Bangalore, Inde.*

Déterminer un ensemble infini de solutions en nombres rationnels (α, β) de l'équation $\alpha^2 + \beta^2 = \alpha^3 + \beta^3$.

M280. *Proposé par l'Équipe de Mayhem.*

Dans le plan, un triangle équilatéral a deux de ses sommets aux points $(0, 0)$ et $(0, n)$. Déterminer le nombre de points (x, y) à coordonnées entières situés à l'intérieur du triangle.

M281. *Proposé par Bruce Shawyer, Université Memorial de Terre-Neuve, St. John's, NL.*

Un carré de côté s est inscrit symétriquement dans un secteur de cercle de rayon r et d'angle au centre de 60°, de telle sorte que deux de ses sommets soient sur les parties rectilignes du secteur et les deux autres sur l'arc de cercle. Déterminer la valeur exacte de s/r.

Mayhem Solutions

M226. *Proposed by John Ciriani, Kamloops, BC.*

Antonino has a drawer full of identical black socks and identical white socks. If he were to select two socks at random from his drawer, the probability that they match would be $\frac{1}{2}$. How many of each colour of sock does Antonino have? (There is more than one answer.)

Solved by Richard I. Hess, Rancho Palos Verdes, CA, USA.

Let there be b black and w white socks. Let P_b be the probability of getting a black pair of socks, and P_w be the probability of getting a white pair, when two socks are drawn at random. Then

$$P_b = \frac{b}{b+w}\cdot\frac{b-1}{b+w-1} \quad \text{and} \quad P_w = \frac{w}{b+w}\cdot\frac{w-1}{b+w-1}\,.$$

Since $P_b + P_w = \frac{1}{2}$, we have

$$\frac{b}{b+w}\cdot\frac{b-1}{b+w-1} + \frac{w}{b+w}\cdot\frac{w-1}{b+w-1} = \frac{1}{2}\,,$$

which simplifies to

$$\begin{aligned} 2\big[b(b-1)+w(w-1)\big] &= (b+w)(b+w-1)\,, \\ 2b^2-2b+2w^2-2w &= b^2-b+w^2-w+2bw\,, \\ b^2-2bw+w^2 &= b+w\,, \\ (b-w)^2 &= b+w\,. \end{aligned}$$

By symmetry, we may assume that $b > w$. Let $d = b - w$. The above equation simplifies to $d^2 - d - 2w = 0$, or $w = \frac{1}{2}d(d-1)$, which gives us $b = w + d = \frac{1}{2}d(d+1)$.

[*Ed:* The solutions for (b, w) are pairs of consecutive triangular numbers; that is, they are pairs of consecutive terms from the sequence $\left\{\frac{n(n+1)}{2}\right\}_{n=1}^{\infty} = 1, 3, 6, 10, 15, \ldots$.]

Also solved by Daniel Tsai, student, Taipei American School, Taipei, Taiwan.

M227. *Proposed by Kenneth S. Williams, Carleton University, Ottawa, ON.*

Let N be a positive integer such that N leaves a remainder of 2 or 4 when divided by 6 and there are integers x and y such that $N = x^2 + 27y^2$. Prove that there exist integers a and b with $N = a^2 + 3b^2$ where b is not divisible by 3.

Solution by Messiah College Problem Solving Group, Messiah College, Grantham, PA, USA, modified by the editor.

Let x and y be integers such that $N = x^2 + 27y^2$. We are given that $N \equiv 2 \pmod 6$ or $N \equiv 4 \pmod 6$. Therefore, N is even and is not divisible by 3. Since N is even, x^2 and $27y^2$ must have the same parity (both even or both odd), which implies that x and y have the same parity. Since N is not divisible by 3, we see that x is not divisible by 3 (because if x were divisible by 3, then the same would be true for $x^2 + 27y^2 = N$).

Define $a = \frac{1}{2}(x+9y)$ and $b = \frac{1}{2}(x-3y)$. Since x and y have the same parity, both $x+9y$ and $x-3y$ are even; therefore, a and b are integers. Note that $a^2 + 3b^2 = N$ and that b is not divisible by 3 (since x is not divisible by 3). Thus, a and b are examples of the required integers.

There were no other solutions submitted.

M228. *Proposed by K.R.S. Sastry, Bangalore, India.*

(a) The zeroes of the polynomial $P(x) = x^2 - 5x + 2$ are precisely the dimensions of a rectangle in centimetres. Determine the perimeter and the area of the rectangle.

(b) The zeroes of the polynomial $P(x) = x^3 - 70x^2 + 1629x - 12600$ are precisely the inner dimensions of a rectangular room in metres. Find the total surface area and the volume of the interior of the room (when doors and windows are closed).

Solution by Lacey K. Moore, student, Angelo State University, San Angelo, TX, USA.

(a) Let l and w be the dimensions of the rectangle in centimetres. Since we know that l and w are the zeroes of $P(x) = x^2 - 5x + 2$, we have

$$P(x) \;=\; (x-l)(x-w) \;=\; x^2 - (l+w)x + lw\,.$$

Thus,

$$x^2 - 5 + 2 \;=\; x^2 - (l+w)x + lw\,.$$

This implies that $l + w = 5$ and $lw = 2$; therefore, the perimeter of the rectangle is $2(l+w) = 2(5) = 10$ cm, and the area is $lw = 2$ cm^2.

(b) Let l, w, and h be the dimensions of the rectangular room in metres. Since l, w, and h are the zeroes of $P(x) = x^3 - 70x^2 + 1629x - 12600$, we have

$$\begin{aligned} P(x) &= (x-l)(x-w)(x-h) \\ &= x^3 - (l+w+h)x^2 + (lh+lw+wh)x - lhw\,. \end{aligned}$$

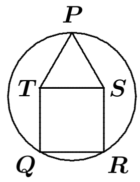
Thus,

$$\begin{aligned} &x^3 - 70x^2 + 1629x - 12600 \\ &\qquad = x^3 - (l+w+h)x^2 + (lh+lw+wh)x - lhw\,. \end{aligned}$$

This implies that $lh + lw + wh = 1629$ and $lhw = 12600$; thus, the surface area of the interior of the room is $2(lw + lh + wh) = 2(1629) = 3258\ \mathrm{m}^2$, and the volume is $lwh = 12600\ \mathrm{m}^3$.

Also solved by Alper Cay, Uzman Private School, Kayseri, Turkey; and Richard I. Hess, Rancho Palos Verdes, CA, USA.

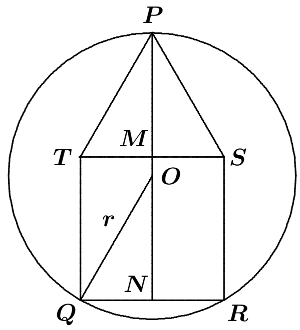
M229. *Proposed by Edward J. Barbeau, University of Toronto, Toronto, ON.*



An equilateral triangle sits atop a square as in the diagram. All sides have length 1. A circle passes through points P, Q, and R. What is the radius of the circle?

Solution by Showadai-cho, Takatsuki City, Osaka, Japan.

Let O be the centre of the circle, and let r be its radius. The perpendicular from point P to side QR bisects segments TS and QR at points M and N, respectively. We know that O lies on the segment MN, with $OM = r - PM = r - \frac{\sqrt{3}}{2}$ and $ON = \sqrt{r^2 - \left(\frac{1}{2}\right)^2}$. Since $OM + ON = 1$, we obtain the equation



$$r - \tfrac{\sqrt{3}}{2} + \sqrt{r^2 - \tfrac{1}{4}} = 1\,.$$

We now solve for r:

$$\begin{aligned} \sqrt{r^2 - \tfrac{1}{4}} &= 1 + \tfrac{\sqrt{3}}{2} - r\,, \\ r^2 - \tfrac{1}{4} &= \left(1 + \tfrac{\sqrt{3}}{2}\right)^2 - 2r\left(1 + \tfrac{\sqrt{3}}{2}\right) + r^2\,, \\ r\left(2 + \sqrt{3}\right) &= 1 + \sqrt{3} + \tfrac{3}{4} + \tfrac{1}{4}\,, \end{aligned}$$

from which we conclude that $r = 1$.

Also solved by Alper Cay, Uzman Private School, Kayseri, Turkey. There were two incorrect solutions submitted.

M230. *Proposed by the Mayhem Staff.*

Al, Betty, Cecil, Dora, and Eugene are going to divide n coins among themselves knowing that:

1. Everyone receives at least one coin.
2. Al gets fewer coins than Betty, who gets fewer than Cecil, who gets fewer than Dora, who gets fewer than Eugene.
3. Each person knows only the total n and how many coins he or she got.

What is the smallest possible value of n such that nobody can deduce the number of coins received by each of the others without more information?

Solved by Richard I. Hess, Rancho Palos Verdes, CA, USA.

Let the number of coins received by Al, Betty, Cecil, Dora, and Eugene be A, B, C, D, and E, respectively. We have $1 \leq A < B < C < D < E$, and the sum of the coins is n.

Examining values of n in increasing order and all the possible values for A, B, C, D, and E for each n gives the following table:

n	A	B	C	D	E	Who knows?
15	1	2	3	4	5	Everyone
16	1	2	3	4	6	Everyone
17	1	2	3	4	7	Dora & Eugene
17	1	2	3	5	6	Dora & Eugene
18	1	2	3	4	8	Dora & Eugene
18	1	2	3	5	7	Eugene
18	1	2	4	5	6	Cecil & Eugene
19	1	2	3	4	9	Dora & Eugene
19	1	2	3	5	8	Eugene
19	1	2	3	6	7	Dora
19	1	2	4	5	7	Cecil
20	1	2	3	4	10	Dora & Eugene
20	1	2	3	5	9	Eugene
20	1	2	3	6	8	No one
20	1	2	4	5	8	No one
20	1	2	4	6	7	Eugene

Thus, for the two cases where $n = 20$ and $E = 8$, no one can deduce the number of coins received by each of the others, and 20 is the smallest such value of n.

There were no other solutions submitted.

M231. *Proposed by Edward J. Barbeau, University of Toronto, Toronto, ON.*

Cordelia and Kent play the following game. Cordelia goes first and they take alternate turns. Each selects a number from 1 to 6 inclusive that has not already been selected; the game ends in six moves. At the end of each move, the player making the move takes the sum of all the numbers selected by either player up to that point and claims all of its positive divisors. When the game is over, the score of each player is the highest number k for which the player has claimed all the consecutive numbers 1, 2, 3, ..., k from 1 to k inclusive. The winner is the player with the highest score; if both have the same score, neither wins and the game is a draw. For example, suppose the six moves are as follows: C:2; K:4; C:1; K:3; C:5; K:6. The respective claims by C are 1, 2; 1, 7; 1, 3, 5, 15; and by K are 1, 2, 3, 6; 1, 2, 5, 10; 1, 3, 7, 21. Cordelia and Kent have the same score, 3, and the game is a draw. The example does not demonstrate very good play. Is there any way that Cordelia can be prevented from winning assuming she is playing as an expert?

Solution by the proposer.

Cordelia can assure a win. She begins the game by selecting 5, claiming 1 and 5.

Suppose that Kent selects a number other than 2. Then Cordelia on the third move can achieve a sum of 12, and altogether claim all the numbers from 1 to 6, inclusive. Kent cannot match this. Note that no sum can exceed 21, and that sums of 10 and 20 are not possible for Kent. Therefore, to claim 5, Kent must play 3 on the fourth move to achieve a sum of 15. But then Kent cannot achieve any sums 8, 12, 16, and 20, and hence cannot claim 4.

Now suppose that, on the second move, Kent selects a 2, claiming 1 and 7. Then on her second move, Cordelia selects 1. So far, Cordelia can claim 1, 2, 4, 5, and 8. Thus far, the numbers 5, 2, 1 have been selected. In order to claim 2, Kent must now select an even number to get an even sum (at the end of Kent's third move, the sum, 21, is odd).

If Kent selects 4 on his second move, he can now altogether claim 1, 2, 3, 4, 6, 7, and 12, but then Cordelia selects 3 to get a sum of 15 and claims altogether 1, 2, 3, 4, 5, 8, 15. Kent will never claim 5; whence, Cordelia wins.

On the other hand, if Kent selects 6 for his second move, he makes the sum 14 and can claim 1, 2, 7. Then Cordelia's third move is to select 4, making a sum of 18, altogether claiming 1, 2, 3, 4, 5, 6, 8, 9, 18. Kent cannot make a sum divisible by 4, and Cordelia again wins.

There were no other solutions submitted.

Problem of the Month

Ian VanderBurgh

Problem (1993 Euclid Contest) In a sequence of p zeroes and q ones, the i^{th} term, t_i, is called a *change point* if $t_i \neq t_{i-1}$, for $i = 2, 3, 4, \dots, p+q$. For example, the sequence 0, 1, 1, 0, 0, 1, 0, 1 has $p = q = 4$, and five change points t_2, t_4, t_6, t_7, t_8. For all possible sequences of p zeroes and q ones with $1 \leq p \leq q$, determine

(i) the minimum and maximum number of change points, and

(ii) the average number of change points.

The notation of this problem makes it look scary, but the problem isn't really so bad. Basically, we are being asked how many times two consecutive terms in a sequence of 0s and 1s are different. We are given that the number of 0s in the sequence is p and the number of 1s is q. If this notation makes you queasy, try working on a particular case like $p = 5$ and $q = 7$. Answering the two questions in this special case is still an interesting task.

We'll solve (i) first. Before we launch into its solution, the first teaching point from this problem arises. In order to show that M is the maximum, we need to do two things: we must justify why we cannot have more than M change points, and we must show that we can have exactly M change points. (Why this second step? Well, if there can't be more than, say, 10 change points in a particular sequence, then there can't be more than 1000 either! But M can't be both 10 and 1000.) Similar things need to be shown for the minimum, and more generally, in any optimization problem (that is, maximum or minimum problem).

Solution to (i): Let's first look for the minimum, m. Certainly $m \geq 0$, because we can't have a negative number of change points. Can there be 0 change points? No. Since any sequence contains both 0s and 1s, there must be a 0 next to a 1 somewhere. Therefore, m is at least 1.

Could the minimum be 1? Yes—the sequence 0, 0, ..., 0, 1, 1, ..., 1 has only one change point. We have shown that the number of change points must be at least 1 and can in fact be 1. So the minimum is 1.

How about the maximum, M? This is trickier. Each sequence with p zeroes and q ones has $p+q$ terms; thus, M is certainly no larger than $p+q$. But the first term cannot be a change point (check the definition). Therefore, M is no larger than $p+q-1$.

Could every term but the first be a change point? Try fiddling for a minute or two to see what you can discover. You could perhaps try a few different possible values for p and q.

Any luck? Let's look first at the case $p = q$. In this case, yes, there can be $p+q-1 = 2p-1$ change points, because the sequence could alternate between 0 and 1—for example, 0, 1, 0, 1, ..., 0, 1. Thus, if $p = q$, the maximum number of change points is $M = p+q-1 = 2p-1$.

What if $p < q$? Notice that every change point involves a 0, either in that position or in the position before. Each of the p zeroes can contribute to at most two change points (and to exactly two if it has 1s on both sides of it). Since there are p zeroes, there can be at most $2p$ change points.

Can this upper bound be achieved? (Oops!—that's fancy mathematics-speak for "Can we actually find a sequence with $2p$ change points?") Yes—for example, $1, 0, 1, 0, 1, \ldots, 0, 1, 1, \ldots, 1$ is a sequence with $p < q$ which has $2p$ change points (two for each 0). Therefore, $M = 2p$ if $p < q$.

Note that the number of 0s controls the maximum number of change points here. The number of 1s is less important, because there are more 1s than needed.

We need to look at (ii) next. Enter stage left the second teaching point. To figure out the average number of change points, we need to figure out the total number of change points over all sequences and divide by the total number of sequences. This seems to require looking at individual sequences and determining the number of change points in each sequence. We might then have to figure out how many sequences have 1 change point, how many have 2, and so on. This would actually be really painful. If you're feeling particularly ambitious, you could of course try this!

There is a sneaky way to do this. If we could determine the total number of sequences in which position 2 is a change point, the number in which position 3 is a change point, and so on, we could add these totals to get the total number of change points. (Of course, we still have to divide by the total number of sequences.)

Solution to (ii): Let's first find the total number of sequences. Since there are $p+q$ positions in total, we can choose p places to put the 0s. This means that there are $\binom{p+q}{p}$ sequences in total.

Now consider position k in the sequences, where k can be any integer from 2 to $p+q$. In order for position k to be a change point, we must have $t_k = 0$ and $t_{k-1} = 1$, or $t_k = 1$ and $t_{k-1} = 0$. How many sequences are there with $t_k = 0$ and $t_{k-1} = 1$? Such a sequence has $p-1$ zeroes and $q-1$ ones to put in the remaining $p+q-2$ positions; thus, there are $\binom{p+q-2}{p-1}$ such sequences. Similarly, there are $\binom{p+q-2}{p-1}$ sequences with $t_k = 1$ and $t_{k-1} = 0$. Therefore, there are $2\binom{p+q-2}{p-1}$ sequences with a change point in position k. Notice that this total is independent of k.

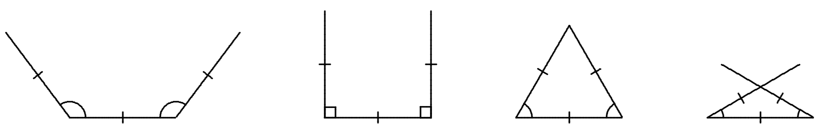
Since there are $p+q-1$ possible positions for a change point and the number of change points in a fixed position is a constant, the total number of change points is $2(p+q-1)\binom{p+q-2}{p-1}$, which means an average of

$$\frac{2(p+q-1)\binom{p+q-2}{p-1}}{\binom{p+q}{p}} = \frac{\frac{2(p+q-1)(p+q-2)!}{(p-1)!(q-1)!}}{\frac{(p+q)!}{p!q!}} = \frac{2(p+q-1)!p!q!}{(p+q)!(p-1)!(q-1)!} = \frac{2pq}{p+q}.$$

SPAs and the Harmonic Mean

Bruce Shawyer

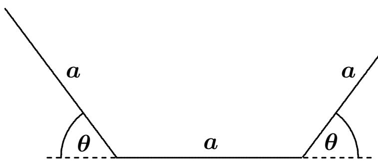
Definition: An *SPA* is a *Symmetric Polygonal Arc*, consisting of three equal straight line segments that have equal angles between adjacent segments. For example:



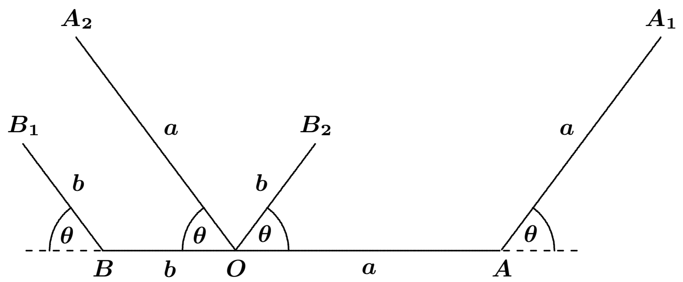
An SPA could be a portion of a regular polygon, or it could be a whole equilateral triangle, depending on its angle.

In this article, we will explore some properties of SPAs and their connection with the harmonic mean. My reason for being interested in SPAs came from a *MAYHEM* problem where two equilateral triangles were placed adjacent to one another on the same line (see problem M214 [2005 : 427, 428; 2006 : 428]).

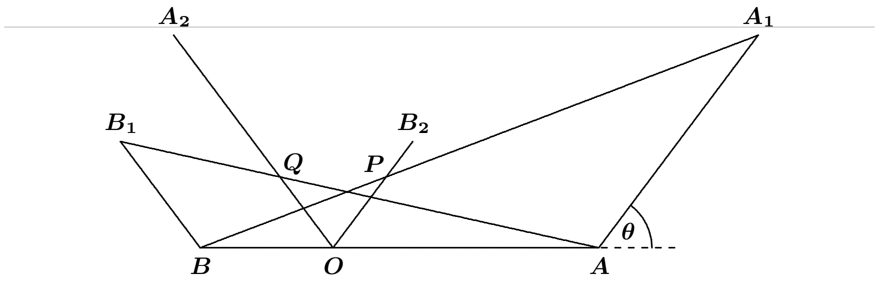
It is convenient to use an external angle as a parameter, say θ. The other convenient parameter is the length of each line segment, say a. We will refer to the middle line segment as the *base* of the SPA.



Place two SPAs with different length parameters a and b and the same angle parameter θ on the same base line with one point in common, as shown below.



Join BA_1 and AB_1. Let P be the point of intersection of the lines BA_1 and OB_2, and let Q be the point of intersection of the lines AB_1 and OA_2.



Then $OP = OQ = \dfrac{ab}{a+b}$. This is one half of the harmonic mean of a and b, and is independent of the parameter θ.

We will prove this remarkable fact in two different ways, each of which is instructive.

First Proof: Note that $\triangle A_1AB$ and $\triangle POB$ are similar. We therefore have $\dfrac{OP}{OB} = \dfrac{AA_1}{AB}$. Since $A_1A = AO = a$ and $OB = b$, we obtain $\dfrac{OP}{b} = \dfrac{a}{a+b}$, giving $OP = \dfrac{ab}{a+b}$. Similarly, using similar triangles B_1BA and QOA, we see that $OQ = \dfrac{ab}{a+b}$.

Second Proof: This proof uses coordinates. Let $O = (0,0)$, $A = (a,0)$, and $B = (-b,0)$. Note that $AA_1 = OA = a$. The coordinates of A_1 are $\big(a(1+\cos\theta), a\sin\theta\big)$, and the coordinates of A_2 are $(-a\cos\theta, a\sin\theta)$. The coordinates of B_1 are $\big(-b\cos\theta, b\sin\theta\big)$, and the coordinates of B_2 are $(b\cos\theta, b\sin\theta)$.

To determine the coordinates of P, we find the equations of the lines BA_1 and OB_2. The point of intersection of these lines is

$$P = \left(\frac{ab\cos\theta}{a+b}, \frac{ab\sin\theta}{a+b}\right).$$

Similarly, by finding the equations of the lines AB_1 and OA_2, we obtain the coordinates of Q:

$$Q = \left(-\frac{ab\cos\theta}{a+b}, \frac{ab\sin\theta}{a+b}\right).$$

From this, it is clear that $OP = OQ = \dfrac{ab}{a+b}$.

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THE OLYMPIAD CORNER

No. 259

R.E. Woodrow

As we begin another year and a new volume of ***CRUX with MAYHEM***, it is appropriate to look back over the 2006 numbers of the *Corner* and thank all those who provided us with problems, comments and solutions:

Arkady Alt	José Luis Díaz-Barrero	Toshio Seimiya
Houda Anoun	Ovidiu Furdui	Achilleas Sinefakopoulos
Miguel Amengual Covas	Geoffrey A. Kandall	Nick Skombris
Michel Bataille	Ioannis Katsikis	Babis Stergiou
Robert Bilinski	Gustavo Krimker	B.J. Venkatachala
Pierre Bornsztein	Andy Liu	Edward T.H. Wang
Christopher J. Bradley	Pavlos Maragoudakis	Kaiming Zhao
Bruce Crofoot	Vedula N. Murty	Li Zhou
Paolo Custodi		

I also want to thank Joanne Canape (née Longworth), who continues to work miracles with my scribbles, turning them into high quality LaTeX files.

We continue now with the remaining problems shortlisted for the 44^{th} IMO in Japan. My thanks go to Andy Liu, Canadian Team Leader, for collecting them for our use.

44^{th} INTERNATIONAL MATHEMATICAL OLYMPIAD

Short-listed Problems

Algebra

A4. Let n be a positive integer, and let $x_1 \le x_2 \le \cdots \le x_n$ be real numbers.

(a) Prove that $\displaystyle\left(\sum_{i,j=1}^{n} |x_i - x_j|\right)^2 \le \frac{2(n^2-1)}{3}\sum_{i,j=1}^{n}(x_i - x_j)^2$.

(b) Show that the equality holds if and only if $x_1, \ldots, x_n$ is an arithmetic sequence.

A5. Let $\mathbb{R}^+$ be the set of all positive real numbers. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ that satisfy the following conditions:

(a) $f(xyz) + f(x) + f(y) + f(z) = f(\sqrt{xy})f(\sqrt{yz})f(\sqrt{zx})$ for all $x, y, z \in \mathbb{R}^+$;

(b) $f(x) < f(y)$ for all $1 \le x < y$.

A6. Let n be a positive integer, and let $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ be two sequences of positive real numbers. Suppose $(z_2, \ldots, z_{2n})$ is a sequence of positive real numbers such that $z_{i+j}^2 \geq x_i y_j$ for all $1 \leq i,\ j \leq n$. Let $M = \max\{z_2, \ldots, z_{2n}\}$. Prove that

$$\left(\frac{M + z_2 + \cdots + z_{2n}}{2n}\right)^2 \geq \left(\frac{x_1 + \cdots + x_n}{n}\right)\left(\frac{y_1 + \cdots + y_n}{n}\right).$$

Combinatorics

C4. Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ be real numbers. Let $A = (a_{ij})_{1 \leq i,j \leq n}$ be the matrix with entries

$$a_{ij} = \begin{cases} 1 & \text{if } x_i + y_j \geq 0; \\ 0 & \text{if } x_i + y_j < 0. \end{cases}$$

Suppose that B is an $n \times n$ matrix with entries 0, 1 such that the sum of the elements in each row and each column of B is equal to the corresponding sum for the matrix A. Prove that $A = B$.

C5. Every point with integer coordinates in the plane is the centre of a disc with radius $1/1000$.

(a) Prove that there exists an equilateral triangle whose vertices lie in different discs.

(b) Prove that every equilateral triangle with vertices in different discs has side-length greater than 96.

C6. Let $f(k)$ be the number of integers n that satisfy the following conditions:

(i) $0 \leq n < 10^k$, so that n has exactly k digits (in decimal notation), with leading zeroes allowed;

(ii) the digits of n can be permuted in such a way that they yield an integer divisible by 11.

Prove that $f(2m) = 10f(2m-1)$ for every positive integer m.

Geometry

G5. Let ABC be an isosceles triangle with $AC = BC$, whose incentre is I. Let P be a point on the circumcircle of the triangle AIB lying inside the triangle ABC. The lines through P parallel to CA and CB meet AB at D and E, respectively. The line through P parallel to AB meets CA and CB at F and G, respectively. Prove that the lines DF and EG intersect on the circumcircle of the triangle ABC.

G6. Each pair of opposite sides of a convex hexagon has the property that the distance between their mid-points is equal to $\sqrt{3}/2$ times the sum of their lengths. Prove that all the angles of the hexagon are equal.

G7. Let ABC be a triangle with semiperimeter s and inradius r. The semicircles with diameters BC, CA, AB are drawn on the outside of the triangle ABC. The circle tangent to all three semicircles has radius t. Prove that

$$\frac{s}{2} < t \le \frac{s}{2} + \left(1 - \frac{\sqrt{3}}{2}\right) r .$$

Number Theory

N5. An integer n is said to be *good* if $|n|$ is not the square of an integer. Determine all integers m with the following property: m can be represented, in infinitely many ways, as a sum of three distinct good integers whose product is the square of an odd integer.

N6. Let p be a prime number. Prove that there exists a prime number q such that, for every integer n, the number $n^p - p$ is not divisible by q.

N7. The sequence a_0, a_1, a_2, ... is defined as follows: $a_0 = 2$, and $a_{k+1} = 2a_k^2 - 1$ for $k \ge 0$. Prove that if an odd prime p divides a_n, then 2^{n+3} divides $p^2 - 1$.

N8. Let p be a prime number, and let A be a set of positive integers that satisfies the following conditions:

(i) the set of prime divisors of the elements in A consists of $p-1$ elements;

(ii) for any non-empty subset of A, the product of its elements is not a perfect p^{th} power.

What is the largest possible number of elements in A?

Before turning to solutions we give a reader's comment on a solution published in the October 2006 *Corner*. The problem is from the Singapore Mathematical Olympiad [2005 : 215–216; 2006 : 383].

9. Evaluate

$$\sum_{k=1}^{2002} \frac{k \cdot k!}{2^k} - \sum_{k=1}^{2002} \frac{k!}{2^k} - \frac{2003!}{2^{2002}} .$$

Comment by David Bradley, University of Maine, Orono, ME, USA.

The published solution establishes the formula

$$\sum_{i=1}^{n} \frac{(k-1)k!}{2^k} = \frac{(n+1)!}{2^n - 1}$$

by using induction on n. But it is easier and more enlightening to observe that, since $(k-1)k! = (k+1)k! - 2(k!)$, we have

$$\sum_{i=1}^{n} \frac{(k-1)k!}{2^k} = \sum_{k=1}^{n} \frac{(k+1)!}{2^k} - \sum_{k=1}^{n} \frac{k!}{2^{k-1}},$$

which telescopes to the stated right-hand side.

We now shift to readers' solutions to problems in the November 2005 *Corner* and the Selection Test for the 7$^{\text{th}}$ National Olympiad of Bosnia and Herzegovina 2002, given at [2005 : 436].

1. Let x, y, and z be real numbers such that

$$x+y+z = 3 \qquad \text{and} \qquad xy+yz+xz = a$$

(a is a real parameter). Determine the value of the parameter a for which the difference between the maximum and minimum possible values of x equals 8.

Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Pavlos Maragoudakis, Pireas, Greece. We given Bornsztein's solution.

We have $y+z = 3-x$ and $yz = a - x(3-x)$. It is easy to verify that the system

$$\begin{aligned} y+z &= s\,, \\ yz &= p\,, \end{aligned}$$

with unknowns y and z, has a real solution if and only if $s^2 - 4p \geq 0$. Hence, the unique condition we have to satisfy is $(3-x)^2 \geq 4(a - 3x + x^2)$, or $3(x-1)^2 \leq 4(3-a)$. That is, $a \leq 3$ and

$$1 - 2\sqrt{1 - \tfrac{1}{3}a} \;\leq\; x \;\leq\; 1 + 2\sqrt{1 - \tfrac{1}{3}a}\,.$$

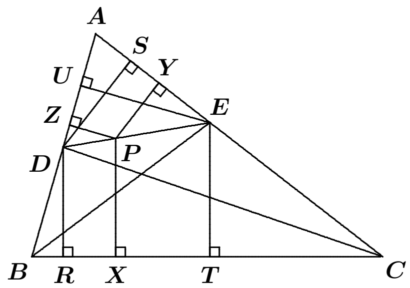
The difference between the maximum and minimum possible values of x equals 8 if and only if $4\sqrt{1-\frac{1}{3}a} = 8$, which implies that $a = -9$.

2. Triangle ABC is given in a plane. Draw the bisectors of all three of its angles. Then draw the line that connects the points where the bisectors of angles ABC and ACB meet the sides AC and AB, respectively. Through the point of intersection of the bisector of angle BAC and the previously drawn line, draw another line, parallel to the side BC. Let this line intersect the sides AB and AC in points M and N. Prove that $2MN = BM + CN$.

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; and Geoffrey A. Kandall, Hamden, CT, USA. We give the write-up by Amengual Covas, modified by the editors.

Lemma. In triangle ABC, the bisector of $\angle ACB$ meets the side AB at D, and the bisector of $\angle ABC$ meets the side AC at E. Let P be any point on the segment DE. Let X, Y, and Z be the orthogonal projections of P onto the sides BC, AC, and AB, respectively. Then $PX = PY + PZ$.

Proof: Let R and S be the orthogonal projections of D onto the sides BC and AC, respectively. Let T and U be the orthogonal projections of E onto the sides BC and AB, respectively. Note that $DR = DS$ and $ET = EU$.

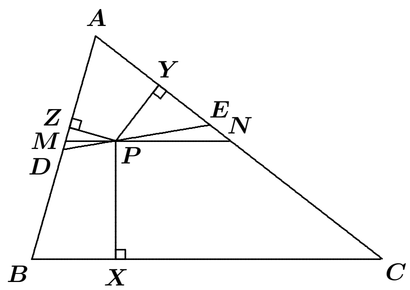


Let $r = DP/DE$. Then $0 < r < 1$ and $PE/DE = 1 - r$. Since $PY \parallel DS$, we have $PY = (1-r)DS = (1-r)DR$; similarly, since $PZ \parallel EU$, we also have $PZ = rEU = rET$. Then

$$PY + PZ \;=\; (1-r)DR + rET \;=\; PX\,,$$

since $DR \parallel PX \parallel ET$ and $DP/PE = r/(1-r)$. ■

Now we turn to the given problem. As in the lemma, we let D and E be the points where the bisectors of $\angle ACB$ and $\angle ABC$ meet the sides AB and AC, respectively. Let P be any point on the segment DE. (Later, we will require P to be on the bisector of $\angle BAC$, as required in the problem.)



Let $[UVW]$ denote the area of a triangle UVW. Since $MN \parallel BC$, we see that $[MBP] = [MXP]$ and $[PCN] = [PXN]$. Hence,

$$[MBP] + [PCN] \;=\; [MXP] + [PXN] \;=\; [MXN]\,.$$

Since the area of $\triangle MBP$ may be expressed by $\frac{1}{2}BM \cdot PZ$, the area of $\triangle PCN$ by $\frac{1}{2}CN \cdot PY$, and that of $\triangle MXN$ by $\frac{1}{2}MN \cdot PX$, we may write the above equation as

$$BM \cdot PZ + CN \cdot PY \;=\; MN \cdot PX\,. \qquad (1)$$

Now assume that P lies on the bisector of $\angle BAC$. Then $PZ = PY$ and, by the lemma, $PX = 2PY$. Substituting into (1) and dividing by PY, we obtain the desired result.

3. If n is a natural number, prove that the number $(n+1)(n+2)\cdots(n+10)$ is not a perfect square.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

Assume, for the purpose of contradiction, that n is a natural number such that $(n+1)(n+2)\ldots(n+10)$ is a perfect square.

A common divisor of any two of the numbers $n+1$, $n+2$, $\ldots$, $n+10$ is at most 9. Also, among these 10 numbers, at most two can be multiples of 7 and at most two can be multiples of 5. It follows that at least 6 of these numbers have one of the forms x^2, $2x^2$, $3x^2$, and $6x^2$. By the pigeon-hole principle, there exist $i, j \in \{1, 2, \ldots, 10\}$ with $i < j$ such that $n + i = kx^2$ and $n + j = ky^2$ for some positive integers x and y and $k \in \{1, 2, 3, 6\}$.

Now $k(y^2 - x^2) = j - i$, and thus, $1 \le k(y^2 - x^2) \le 9$. Also, we have $kx^2 = n + i \ge 2$. The possibilities for (x, y, k) satisfying these conditions are as follows: $(1, 2, 2)$, $(1, 2, 3)$, $(2, 3, 1)$, $(3, 4, 1)$, and $(4, 5, 1)$. In every case $kx^2 \le 16$, and therefore the sequence $n + 1$, $n + 2$, $\ldots$, $n + 10$ is contained in the set $\{2, 3, \ldots, 25\}$. But then the sequence must contain either 11 or 17. Since no more than one factor of 11 or 17 can be present in the product $(n+1)(n+2)\cdots(n+10)$, we conclude that this product is not a perfect square, which contradicts our initial assumption.

4. Let a, b, and c be real numbers such that $a^2 + b^2 + c^2 = 1$. Prove the inequality

$$\frac{a^2}{1+2bc} + \frac{b^2}{1+2ca} + \frac{c^2}{1+2ab} \ge \frac{3}{5}.$$

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Pierre Bornsztein, Maisons-Laffitte, France; Ovidiu Furdui, student, Western Michigan University, Kalamazoo, MI, USA; Pavlos Maragoudakis, Pireas, Greece; and Vedula N. Murty, Dover, PA, USA. We give Furdui's solution.

In view of the inequality $2xy \le x^2 + y^2$ and the observation that $1 + 2bc = a^2 + (b + c)^2 > 0$, etc., we see that

$$\begin{aligned} \frac{a^2}{1+2bc} + \frac{b^2}{1+2ca} + \frac{c^2}{1+2ab} &\ge \frac{a^2}{1+b^2+c^2} + \frac{b^2}{1+c^2+a^2} + \frac{c^2}{1+a^2+b^2} \\ &= \frac{a^2}{2-a^2} + \frac{b^2}{2-b^2} + \frac{c^2}{2-c^2}. \end{aligned}$$

Let $f : (0, 1) \to \mathbb{R}$ be defined as $f(x) = x/(2 - x)$. Then

$$f'(x) = \frac{2}{(2-x)^2} \qquad \text{and} \qquad f''(x) = \frac{4}{(2-x)^3}.$$

Thus, f is convex, since $f''(x) > 0$. Using Jensen's Inequality, we get

$$f\left(\frac{x+y+z}{3}\right) \le \frac{f(x)+f(y)+f(z)}{3}.$$

Taking $x = a^2$, $y = b^2$, and $z = c^2$ gives $f(a^2) + f(b^2) + f(c^2) \ge 3f(\frac{1}{3})$; that is,

$$\frac{a^2}{2-a^2} + \frac{b^2}{2-b^2} + \frac{c^2}{2-c^2} \ge 3 \cdot \frac{\frac{1}{3}}{2-\frac{1}{3}} = \frac{3}{5}.$$

Note: equality is only possible if $a^2 = b^2 = c^2 = \frac{1}{3}$.

5. Let p and q be different prime numbers. Solve the following system of equations in the set of integers:

$$\begin{aligned} \frac{z+p}{x} + \frac{z-p}{y} &= q, \\ \frac{z+p}{y} - \frac{z-p}{x} &= q. \end{aligned}$$

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

Note that $xy \neq 0$. Clearing the denominators, we get

$$\begin{aligned} (z+p)y + (z-p)x &= qxy\,, \\ (z+p)x - (z-p)y &= qxy\,. \end{aligned}$$

By first subtracting the equations, then adding them, we get

$$zy = px\,, \tag{1}$$

$$xz + py = qxy\,. \tag{2}$$

Multiplying (2) by z and using (1), we get $xz^2 + xp^2 = qpx^2$, which is equivalent to $z^2 + p^2 = qpx$. This equation implies that p divides z (since p is prime). Thus, p^2 divides qpx. Since $q \neq p$ and q is prime, it then follows that p divides x. Let $x = pa$ and $z = pc$, where a and c are integers and $a \neq 0$. The system is now

$$cy = pa\,, \tag{3}$$

$$pac + y = qay\,. \tag{4}$$

From (4), we deduce that a divides y, say $y = ab$. Thus,

$$cb = p\,, \tag{5}$$

$$pc + b = qab\,. \tag{6}$$

Since p is prime, it follows from (5) that $(b,c) \in \{\pm(p,1), \pm(1,p)\}$.

Case 1. $(b,c) = (p,1)$.

From (6), we have $qa = 2$, which implies that $q = 2$ and $a = 1$. Then $(x,y,z) = (p,p,p)$, which is a solution for p any odd prime.

Case 2. $(b,c) = (-p,-1)$.

As in Case 1, we obtain $q = 2$ and $a = 1$. Thus, $(x,y,z) = (p,-p,-p)$.

Case 3. $(b,c) = (1,p)$.

Equation (6) simplifies to $p^2 + 1 = qa$. Then q must be a prime divisor of $p^2 + 1$, in which case

$$(x,y,z) = \left(\frac{p(p^2+1)}{q}, \frac{p^2+1}{q}, p^2\right)\,.$$

Case 4. $(b,c) = (-1,-p)$.

As in Case 3, we find that q must divide $p^2 + 1$. Then

$$(x,y,z) = \left(\frac{p(p^2+1)}{q}, -\frac{p^2+1}{q}, -p^2\right)\,.$$

Therefore, the solutions are $(x,y,z) \in \{(p,p,p), (p,-p,-p)\}$, if $q = 2$, for any prime $p > 2$, and

$$(x,y,z) \in \left\{\left(\frac{p(p^2+1)}{q}, \frac{p^2+1}{q}, p^2\right), \left(\frac{p(p^2+1)}{q}, -\frac{p^2+1}{q}, -p^2\right)\right\}\,,$$

for any primes p and q such that q divides $p^2 + 1$ (which implies that $p \neq q$).

6. Let the vertices of the convex quadrilateral $ABCD$ and the intersecting point S of its diagonals be integer points in the plane. Let P be the area of the quadrilateral $ABCD$ and P_1 the area of triangle ABS. Prove the following inequality:

$$\sqrt{P} \;\geq\; \sqrt{P_1} + \frac{\sqrt{2}}{2}\,.$$

Solution by Pavlos Maragoudakis, Pireas, Greece, modified by the editor.

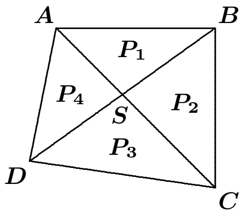
Lemma. If X_1, X_2, and X_3 are non-collinear points with integer coordinates, then $[X_1X_2X_3] \geq \frac{1}{2}$.

Proof: Let X_i have coordinates (a_i, b_i), with a_i, $b_i \in \mathbb{Z}$, for $i = 1, 2, 3$. Then

$$2[X_1X_2X_3] \;=\; \begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix} \in \mathbb{Z}_+^*\,.$$

Hence, $2[X_1X_2X_3] \geq 1$; that is, $[X_1X_2X_3] \geq \frac{1}{2}$. ∎

Let P_2, P_3, and P_4 be the areas of triangles BCS, CDS, ADS, respectively. We have $P_1/P_2 = AS/CS$, because $\triangle BAS$ has the same height from the base AS as does $\triangle BCS$ from the base CS. Similarly, $P_4/P_3 = AS/CS$. Thus, $P_1/P_2 = P_4/P_3$. Then $P_1 = P_2P_4/P_3 \leq 2P_2P_4$, since $P_3 \geq \frac{1}{2}$ by the lemma. Hence,



$$\sqrt{2P_1} \;\leq\; 2\sqrt{P_2P_4} \;\leq\; P_2 + P_4\,,$$

using the AM–GM Inequality. Then

$$\begin{aligned} P \;=\; P_1 + P_2 + P_3 + P_4 \;&\geq\; P_1 + P_3 + \sqrt{2P_1} \\ &\geq\; P_1 + \tfrac{1}{2} + \sqrt{2P_1} = \left(\sqrt{P_1} + \tfrac{\sqrt{2}}{2}\right)^2\,. \end{aligned}$$

Taking square roots gives the desired result.

Next we look at solutions from our readers to problems of the 4$^{\text{th}}$ Hong Kong Mathematical Olympiad given at [2005 : 437].

2. Find all positive integers n such that the equation $x^3+y^3+z^3 = nx^2y^2z^2$ has positive integer solutions. Be sure to give a proof.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

Let $n \geq 1$ be an integer such that there exist positive integers x, y, and z satisfying

$$x^3 + y^3 + z^3 \;=\; nx^2y^2z^2\,. \tag{1}$$

Without loss of generality, we may assume that $x \geq y \geq z$. Then we have $nx^2y^2z^2 \leq 3x^3$, which leads to

$$ny^2z^2 \;\leq\; 3x\,. \tag{2}$$

From (1), we also have $x^2(ny^2z^2 - x) = y^3 + z^3 > 0$, implying that

$$ny^2z^2 \;\geq\; x + 1\,. \tag{3}$$

Note that $(y^3 - 1)(z^3 - 1) \geq 0$, which leads to $1 + y^3z^3 \geq y^3 + z^3$. Using (1) and then (3), we get

$$1 + y^3z^3 \;\geq\; x^2(ny^2z^2 - x) \;\geq\; x^2\,. \tag{4}$$

Thus, using (2), we deduce that

$$9(1 + y^3z^3) \;\geq\; n^2y^4z^4\,. \tag{5}$$

Case 1. $y = z = 1$.

From (4), we see that $x = 1$, and the given equation yields $n = 3$.

Case 2. $yz > 1$.

Then $y^4z^4 \geq 2y^3z^3 > 1 + y^3z^3$. This together with (5) implies that $n^2 < 9$. Hence, $n = 1$ or $n = 2$.

For $n = 1$, it is easy to verify that $(x, y, z) = (3, 2, 1)$ is a solution.

For $n = 2$, using (5), we have $y^3z^3(4yz - 9) \leq 9$, which forces $yz \leq 2$. Then $yz = 2$, which implies that $y = 2$ and $z = 1$. It follows that $x \geq 2$, and the given equation reduces to $x^3 + 9 = 8x^2$. Then x^2 must divide 9, which implies that $x = 3$. But it is easy to verify that this is not a solution. Therefore, $n = 2$ is impossible.

Therefore, the desired values for n are $n = 1$ and $n = 3$.

3. For each integer $k \geq 4$, prove that if $F(x)$ is a polynomial with integer coefficients which satisfies the condition $0 \leq F(c) \leq k$ for every $c = 0, 1, \ldots, k + 1$, then $F(0) = F(1) = \cdots = F(k + 1)$.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

This problem is from the IMO 1997 shortlist. A solution can be found in P. Bornsztein, *Mégamath*, Vuibert, pb.# A1.27. or D. Djukić, V. Janković, I. Matić, N. Petrović, *The IMO Compendium*, Springer.

4. There are 212 points inside or on a circle with radius 1. Prove that there are at least 2001 pairs of these points having distances at most 1.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

We may partition the disk into 6 congruent sectors, each with a central angle of 60°, by rays from the centre O of the disk. Since the number of given points is finite, we may choose the rays so that none of the points lie on a ray, except that the centre O may be one of the points. In each of the 6 sectors, the distance between any two points is at most 1. Let $n_1, \ldots, n_6$ be the respective number of the given points in each of the 6 sectors (if O is one of the points, it is counted only once, say in n_1). Thus, $\sum_{i=1}^{6} n_i = 212$.

In sector i, the number of pairs of points is $\frac{1}{2}n_i(n_i - 1)$, and each of these pairs has distance at most 1. Hence, the total number N of pairs of points having distances at most 1 satisfies

$$N \;\geq\; \sum_{i=1}^{6} \frac{1}{2} n_i(n_i - 1) \;=\; \frac{1}{2}\biggl(\sum_{i=1}^{6} n_i^2 - 212 \biggr) \,.$$

Then, using Jensen's Inequality for the function $f(x) = x^2$, we deduce that

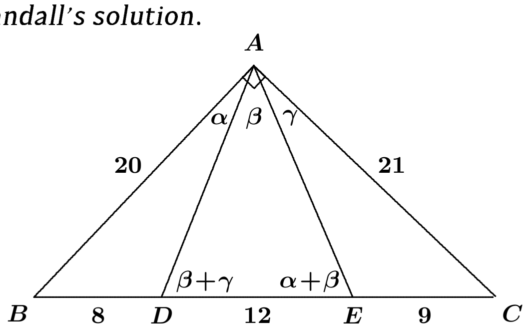
$$N \;\geq\; \frac{1}{2}\biggl(\frac{1}{6}\biggl(\sum_{i=1}^{6} n_i \biggr)^2 - 212 \biggr) \;=\; \frac{1}{2}\biggl(\frac{1}{6}(212)^2 - 212 \biggr) \;>\; 3639 \,.$$

Thus, $N \geq 3640$, which is better than required.

Next we look at readers' solutions to problems of the 15th Irish Mathematical Olympiad, First Paper, given in [2005 : 437–439].

1. In a triangle ABC, $AB = 20$, $AC = 21$, and $BC = 29$. The points D and E lie on the line segment BC, with $BD = 8$ and $EC = 9$. Calculate the angle $\angle DAE$.

Solved by Mohammed Aassila, Strasbourg, France; Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Ioannis Katsikis, Athens, Greece; Pavlos Maragoudakis, Pireas, Greece; and Geoffrey A. Kandall, Hamden, CT, USA. We give Kandall's solution.



Let $\alpha = \angle BAD$, $\beta = \angle DAE$, and $\gamma = \angle EAC$. Since $BA = BE$, we have $\angle AEB = \alpha + \beta$; similarly, since $CA = CD$, we have $\angle ADC = \beta + \gamma$. Therefore,

$$180^\circ \;=\; (\alpha + \beta) + (\beta + \gamma) + \beta \;=\; (\alpha + \beta + \gamma) + 2\beta \,.$$

But $\alpha + \beta + \gamma = 90^\circ$, since $20^2 + 21^2 = 29^2$. Consequently, $\beta = 45^\circ$.

2. (a) A group of people attends a party. Each person has at most three acquaintances in the group, and if two people do not know each other, then they have a mutual acquaintance in the group. What is the maximum number of people present?

(b) If, in addition, the group contains three mutual acquaintances (that is, three people each of whom knows the other two), what is the maximum number of people?

Solution by Pierre Bornsztein, Maisons-Laffitte, France, modified by the editor.

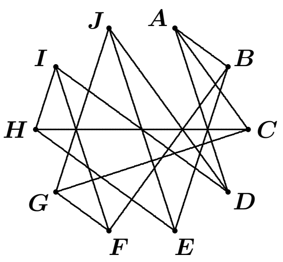
(a) The maximum is 10.

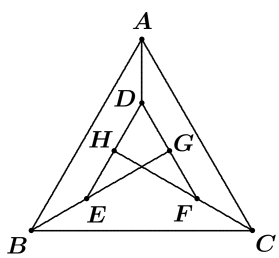
Let A be any person in the group. If A has no acquaintances, then A must be the only member of the group. If A has only one acquaintance, say B, then any other member C must be an acquaintance of B (because A and C must have a mutual acquaintance); then, since B can have at most 3 acquaintances, the group has at most 4 members.

Suppose that A has exactly two acquaintances, say B and C. Each of B and C can have at most two further acquaintances. Since any person not acquainted with A must be acquainted with either B or C, we conclude that there are at most $1 + 2 + 2 \cdot 2 = 7$ people in the group.

Now suppose that A has exactly three acquaintances, say B, C, and D. Any other person in the group must be acquainted with B, C, or D, and for each of B, C, and D there are at most two further acquaintances. Hence, there are at most $1 + 3 + 3 \cdot 2 = 10$ people in the group.

Thus, the group can have no more than 10 members. On the other hand, a party with 10 people is possible, as proved by the graph on the left below (where people are vertices and edges are acquaintances).





(b) The maximum is 8.

Let A, B, and C be three mutual acquaintances. If A has no further acquaintances, then all other persons must be acquainted with either B or C, and each of them can have at most one further acquaintance. This gives a maximum of $3 + 1 + 1 = 5$ people in the group. If A does have a further acquaintance, say D, then any other people must be acquainted with B, C, or D. For each of B and C, there is at most one such acquaintance, and at most two for D. This leads to at most $4 + 1 + 1 + 2 = 8$ people.

A party with 8 people is possible, as proved by the graph on the right above (where people are vertices and edges are acquaintances).

3. Find all triples of positive integers (p, q, n), with p and q prime, such that

$$p(p+3)+q(q+3) \;=\; n(n+3)\,.$$

Solved by Mohammed Aassila, Strasbourg, France; Pierre Bornsztein, Maisons-Laffitte, France; and Pavlos Maragoudakis, Pireas, Greece. We give Aassila's solution.

First of all, we observe that, for any positive integer m, we have $m(m+3) \equiv 1 \pmod 3$ if $3 \nmid m$, and $m(m+3) \equiv 0 \pmod 3$ if $3 \mid m$. Since we require $p(p+3)+q(q+3) \equiv n(n+3) \pmod 3$, at least one of p and q must be 3. Thus, we may assume, without loss of generality, that $p = 3$. We have $n \geq q+1$, which means that $n(n+3) - q(q+3) \geq 2q+4 > 3(3+3)$ unless $q \leq 7$. Checking the primes $q \leq 7$, we find that $q = 2$ and $q = 7$ are the only solutions.

Thus, the only solutions are $(p,q,n) = (2,3,4)$, $(p,q,n) = (3,2,4)$, $(p,q,n) = (3,7,8)$, and $(p,q,n) = (7,3,8)$.

4. Let the sequence a_1, a_2, a_3, a_4, $\ldots$ be defined by

$$a_1 \;=\; 1\,, \quad a_2 \;=\; 1\,, \quad a_3 \;=\; 1\,, \quad \text{and} \quad a_{n+1}a_{n-2} - a_n a_{n-1} \;=\; 2\,,$$

for all $n \geq 3$. Prove that a_n is a positive *integer* for all $n \geq 1$.

Solved by Mohammed Aassila, Strasbourg, France; Pierre Bornsztein, Maisons-Laffitte, France; and Pavlos Maragoudakis, Pireas, Greece. We give Aassila's write-up.

An easy induction proves that $a_n = 4a_{n-1} - a_{n-2}$ if n is even, and $a_n = 2a_{n-1} - a_{n-2}$ if n is odd. Thus, a_n is a positive integer for all $n \geq 1$.

5. Let $0 < a,\, b,\, c < 1$. Prove that

$$\frac{a}{1-a} + \frac{b}{1-b} + \frac{c}{1-c} \;\geq\; \frac{3\sqrt[3]{abc}}{1-\sqrt[3]{abc}}\,.$$

Determine the case of equality.

Solved by Mohammed Aassila, Strasbourg, France; Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Pierre Bornsztein, Maisons-Laffitte, France; José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain; Ovidiu Furdui, student, Western Michigan University, Kalamazoo, MI, USA; Vedula N. Murty, Dover, PA, USA; Pavlos Maragoudakis, Pireas, Greece; and Henry Ricardo, Medgar Evers College (CUNY), Brooklyn, NY, USA. We give the write-up of Ricardo.

The function $f(x) = x/(1-x)$ is convex on the interval $(0,1)$. In what follows, we use Jensen's Inequality and then the AM–GM Inequality:

$$\frac{a}{1-a} + \frac{b}{1-b} + \frac{c}{1-c} \;\geq\; 3 \cdot \frac{\frac{a+b+c}{3}}{1-\frac{a+b+c}{3}} \;\geq\; \frac{3\sqrt[3]{abc}}{1-\sqrt[3]{abc}}\,.$$

Equality holds if and only if $a = b = c$.

6. A $3 \times n$ grid is filled as follows. The first row consists of the numbers from 1 to n arranged from left to right in ascending order. The second row is a cyclic shift of the top row. Thus, the order goes

$$i,\ i+1,\ \ldots,\ n-1,\ n,\ 1,\ 2,\ \ldots,\ i-1$$

for some i. The third row has the numbers 1 to n in some order, subject to the rule that in each of the n columns, the sum of the three numbers is the same.

For which values of n is it possible to fill the grid according to the above rules? For an n for which this is possible, determine the number of different ways of filling the grid.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

Assume that we may fill the grid according to the rules.

Then the sum of the numbers in all the three rows is $3n(n+1)/2$. It follows that the common sum in each column is $3(n+1)/2$, which forces n to be odd. Let $n = 2p+1$. Then, among the numbers 1, 2, $\ldots$, n, there are $p+1$ odd numbers and p even numbers. Let $k+1$ be the number of the column containing n in the second row. Then the grid is as follows:

$$\begin{array}{cccccccc} 1 & 2 & \ldots & k & k+1 & k+2 & \ldots & n \\ n-k & n-k+1 & \ldots & n-1 & n & 1 & \ldots & n-k-1 \\ a_1 & a_2 & \ldots & a_k & a_{k+1} & a_{k+2} & \ldots & a_n \end{array}$$

In each of the columns 1 through $k+1$, the sum of the first two numbers has the same parity as k (for example, in column number k, the sum is $n-1+k = 2p+k$), while in each of the columns $k+2$ through n, the sum of the first two numbers has the parity opposite to k. Since the sum of the numbers in each column is constant, it follows that a_1, a_2, $\ldots$, a_{k+1} have the same parity and a_{k+2}, $\ldots$, a_n have the same parity. Thus, $k+1 = p+1$ or $k+1 = p$. Moreover, since the sum in each column is $3(n+1)/2$, if the value of k is known, then the value of a_i is determined for each i, which means that there is at most one way to fill the grid for a given k.

Case 1. $n = 2p+1$ and $k+1 = p+1$.

We may fill the grid as follows:

$$\begin{array}{cccccccc} 1 & 2 & \ldots & k & k+1 & k+2 & k+3 & \ldots & 2k+1 \\ k+1 & k+2 & \ldots & 2k & 2k+1 & 1 & 2 & \ldots & k \\ 2k+1 & 2k-1 & \ldots & 3 & 1 & 2k & 2k-2 & \ldots & 2 \end{array}$$

Case 2. $n = 2p+1$ and $k+1 = p$.

We may fill the grid as follows:

$$\begin{array}{ccccccccc} 1 & 2 & \ldots & k & k+1 & k+2 & k+3 & \ldots & 2k+1 \\ k+3 & k+4 & \ldots & 2k+2 & 2k+3 & 1 & 2 & \ldots & k+2 \\ 2k+2 & 2k & \ldots & 4 & 2 & 2k+3 & 2k+1 & \ldots & 1 \end{array}$$

Therefore, the grid may be filled if and only if n is odd. And, for such an n, there are two ways to fill the grid.

7. Suppose n is a product of four distinct primes a, b, c, d such that

(a) $a + c = d$;

(b) $a(a + b + c + d) = c(d - b)$;

(c) $1 + bc + d = bd$.

Determine n.

Solved by Mohammed Aassila, Strasbourg, France; Pierre Bornsztein, Maisons-Laffitte, France; Geoffrey A. Kandall, Hamden, CT, USA; and Pavlos Maragoudakis, Pireas, Greece. We give Kandall's generalization.

More generally, we will determine all integer solutions of the system

$$\begin{aligned} a + c &= d\,, & (1)\\ a(a + b + c + d) &= c(d - b)\,, & (2)\\ 1 + bc + d &= bd\,. & (3) \end{aligned}$$

If $d = 0$, it is easy to see that the only solutions for (a, b, c, d) are $(1, 1, -1, 0)$ and $(-1, -1, 1, 0)$.

Let us assume $d \neq 0$. From (1) and (2), we have $a(b + 2d) = c(d - b)$, which can be put in the form $b(a + c) = (c - 2a)d$. Using (1) again, we get

$$c = 2a + b\,. \qquad (4)$$

Then, in view of (1),

$$d = 3a + b\,. \qquad (5)$$

Using (4) and (5) in (3), we obtain $1 + b(2a + b) + 3a + b = b(3a + b)$, which can be rewritten as $b + 1 = a(b - 3)$. Therefore, $b \neq 3$, and

$$a = \frac{b+1}{b-3} = 1 + \frac{4}{b-3}\,.$$

Thus, $\frac{4}{b-3}$ is an integer; that is, $b - 3 \in \{\pm 1, \pm 2, \pm 4\}$. The complete table of solutions can now be easily constructed:

a	b	c	d
1	1	−1	0
−1	−1	1	0
5	4	14	19
−3	2	−4	−7
3	5	11	14
−1	1	−1	−2
2	7	11	13
0	−1	−1	−1

If we now require that a, b, c, d be primes, then $(a, b, c, d) = (2, 7, 11, 13)$, which implies that $n = 2 \cdot 7 \cdot 11 \cdot 13 = 2002$.

8. Denote by $\mathbb{Q}$ the set of rational numbers. Determine all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that

$$f\big(x + f(y)\big) \;=\; y + f(x)\,, \quad \text{for all } x,\, y \in \mathbb{Q}.$$

Solved by Mohammed Aassila, Strasbourg, France; Pierre Bornsztein, Maisons-Laffitte, France; and Pavlos Maragoudakis, Pireas, Greece. We give Aassila's write-up.

Setting $x = 0$ in the given condition, we find that $f\big(f(y)\big) = f(0) + y$ for all $y \in \mathbb{Q}$. If $f(x) = f(y)$ for some x, $y \in \mathbb{Q}$, then $f\big(f(x)\big) = f\big(f(y)\big)$; hence, $f(0) + x = f(0) + y$, from which we get $x = y$. Thus, f is injective.

Setting $y = 0$ in the given condition, we obtain $f\big(x + f(0)\big) = f(x)$ for all $x \in \mathbb{Q}$. Since f is injective, we get $x = x + f(0)$, and thus, $f(0) = 0$. Then $f\big(f(y)\big) = y$ for all $y \in \mathbb{Q}$.

Now, for all x, $y \in \mathbb{Q}$,

$$f(x + y) \;=\; f\Big(x + f\big(f(y)\big)\Big) \;=\; f(x) + f(y)\,.$$

Hence, by an easy induction, $f(nx) = nf(x)$ for all $n \in \mathbb{N}^*$. Now let x be any positive rational number. Setting $x = r/s$ with r, $s \in \mathbb{N}^*$, we have $sf(x) = f(sx) = f(r) = rf(1)$; hence, $f(x) = (r/s)f(1) = xf(1)$.

Now we note that, for all $x \in \mathbb{Q}$,

$$0 \;=\; f(0) \;=\; f(x - x) \;=\; f(x) + f(-x)\,,$$

and therefore, $f(-x) = -f(x) = -xf(1)$. Thus, $f(x) = xf(1)$ for all $x \in \mathbb{Q}$. Moreover, by setting $x = f(1)$, we find that $f\big(f(1)\big) = f(1)f(1)$. From above, we have $f\big(f(1)\big) = 1$. Thus, $f(1) = \pm 1$. We conclude that either $f(x) = x$ for all $x \in \mathbb{Q}$ or $f(x) = -x$ for all $x \in \mathbb{Q}$.

It is now easy to check that $f(x) = x$ and $f(x) = -x$ are solutions.

9. For each real number x, define $\lfloor x \rfloor$ to be the greatest integer less than or equal to x. Let $\alpha = 2 + \sqrt{3}$. Prove that

$$\alpha^n - \lfloor \alpha^n \rfloor \;=\; 1 - \alpha^{-n}\,, \quad \text{for } n = 0,\, 1,\, 2,\, \ldots.$$

Solved by Mohammed Aassila, Strasbourg, France; Pierre Bornsztein, Maisons-Laffitte, France; and Pavlos Maragoudakis, Pireas, Greece. We give Aassila's solution.

We have $\alpha^{-1} = 2 - \sqrt{3} < 1$; then $0 < \alpha^{-n} < 1$. By the Binomial Theorem, the odd powers of $\sqrt{3}$ in the expansion of $\big(2 + \sqrt{3}\big)^n + \big(2 - \sqrt{3}\big)^n$ cancel out. Hence, $\alpha^n + \alpha^{-n}$ is an integer. Thus,

$$\alpha^n + \alpha^{-n} \;=\; \lfloor \alpha^n \rfloor + 1\,.$$

[*Editor's note:* Maragoudakis points out that a solution, albeit with different integers involved ($\alpha = 3 + \sqrt{5}$), is published in [2005 : 384].

10. Let ABC be a triangle whose side lengths are all integers, and let D and E be the points at which the incircle of ABC touches BC and AC, respectively. If $|AD^2 - BE^2| \leq 2$, show that $AC = BC$.

Solved by Mohammed Aassila, Strasbourg, France; and Pavlos Maragoudakis, Pireas, Greece. We give Aassila's solution, modified by the editor.

We have $CE = CD = (a+b-c)/2$. By the Cosine Law,

$$AD^2 = b^2 + \frac{(a+b-c)^2}{4} - b(a+b-c)\cos C$$

$$\text{and} \quad BE^2 = a^2 + \frac{(a+b-c)^2}{4} - a(a+b-c)\cos C\,.$$

Thus,

$$BE^2 - AD^2 = a^2 - b^2 - (a+b-c)(a-b)\cos C\,.$$

By the Cosine Law, we have $\cos C = \dfrac{a^2+b^2-c^2}{2ab}$, and hence,

$$\begin{aligned} BE^2 - AD^2 &= a^2 - b^2 - (a+b-c)(a-b)\frac{a^2+b^2-c^2}{2ab} \\ &= \frac{a-b}{2ab}\big(a^2(-a+b+c) + b^2(a-b+c) + c^2(a+b-c)\big)\,. \end{aligned}$$

For the purpose of contradiction, we assume that $a \neq b$. Without loss of generality, we may assume $a > b$. If $c = 1$, then, since $a < b+c$, we have $b < a < b+1$, which is impossible for integers a and b. Therefore, $c \geq 2$.

Let $a - b = k$. If $k = 1$, then

$$BE^2 - AD^2 = \frac{(b+1)^2(c-1) + b^2(c+1) + c^2(2b+1-c)}{2(b+1)b}\,.$$

We must have $2 \leq c \leq a+b-1 = 2b$. It is not hard to show that the minimum value of $f(c) = c^2(2b+1-c)$ for $2 \leq c \leq 2b$ is $f(2) = 4(2b-1)$. Therefore,

$$\begin{aligned} BE^2 - AD^2 &\geq \frac{(b+1)^2(1) + b^2(3) + 4(2b-1)}{2(b+1)b} = \frac{4b^2+10b-3}{2b^2+2b} \\ &= 2 + \frac{3(2b-1)}{2b^2+2b} > 2\,. \end{aligned}$$

If $k \geq 2$, then

$$\begin{aligned} BE^2 - AD^2 &= \frac{k}{2(b+k)b}\big[(b+k)^2(c-k) + b^2(c+k) + c^2(2b+k-c)\big] \\ &\geq \frac{2}{2(b+k)b}\big[(b+k)^2 + b^2\big] = \frac{2b^2+2kb+k^2}{b^2+kb} > 2\,. \end{aligned}$$

Thus, in both cases, $BE^2 - AD^2 > 2$, a contradiction. Hence, $a = b$; that is, $AC = BC$.

That completes the *Corner* for this issue. Send me your nice solutions (and soon) for problems that have appeared in 2006 numbers of the *Corner*.

BOOK REVIEWS

John Grant McLoughlin

99 Points of Intersection
By Hans Walser, translated from the original German by Peter Hilton and Jean Pedersen, published by The Mathematical Association of America, 2006.
ISBN 0-88385-553-4, hardcover, 168 pages, US$48.50.
Reviewed by ***Nora Franzova****, Langara College, Vancouver, BC.*

If three straight lines pass through a common point, we call the lines concurrent. We all remember instances from the geometry of triangles where three lines are concurrent: the three altitudes have a common point, as well as the three medians, the three perpendicular bisectors of the sides, and the three angle bisectors.

In this book there are 99 spectacular examples of concurrence given as pictures-without-words.

The book opens with a broad introduction discussing the intersection of three or more straight lines and curves. The obvious intersection of diagonals of a dodecagon is followed by the intersection of circles, Fourier Flowers (functions from Fourier expansions), and Chebyshev Polynomials. This introduction seems like a gate opening to a great garden, from which the author chooses one corner to explore in the main part of the book.

In the second part of the book, 99 pictures-without-words represent 99 examples of concurrence. Some of these pictures/graphs are named (for example, "Homage to Pythagoras", "Butterflies", "Kissing Circles"), some come with a literature reference, and all come with three pictures that show the steps leading to their construction. Each page is devoted to one example only, and no proofs or explanations are given in this part. An obvious reason for that is emphasized by the author himself in the foreword to the book, where he explains his hope that the book would encourage the reader to find the points of intersection on his/her own, possibly with the help of interactive geometry software. (Software popular in Europe is replaced in the translation by packages more commonly used in the United States.)

After finding a surprising concurrence, the next step for a mathematician or any other curious mind is to figure out why the concurrence happens and look for generalizations. In the third part of the book, the author provides several proof strategies that explain many of the pictures in the previous section. The strategies range from simple geometrical steps, through vectors, to Ceva's Theorem and Jacobi's Theorem. Some of these would definitely engage high school or even elementary school students. Especially exciting and powerful is the "Pythagoras-free derivation of the Law of Cosines", which is a byproduct of an investigation of the Point of Intersection 84.

Some of the algebraic proofs are quite complicated and require the help of a Computer Algebra System. This, of course, leaves the author and many other geometry lovers hoping for an elementary geometrical proof. Readers are openly invited to help find an elementary geometrical proof for the Point of Intersection 79 (also called the "Propeller"), which the author proved with use of Maple.

With geometry more and more neglected by the school curriculum in most of North America, this book brings a renewed passion for the subject. It opens new paths to explore and shows directions for exploring. Those who like geometry and puzzles will enjoy pondering each picture.

Real Infinite Series
By Daniel D. Bonar and Michael J. Khoury, published by the Mathematical Association of America, 2006.
ISBN 0-88385-745-6, hardcover, xii+264 pages, US$49.95.
Reviewed by **John Grant McLoughlin**, *University of New Brunswick, Fredericton, NB.*

An entire book dedicated to real infinite series! At first I was surprised to see such a book. The six chapters and three appendices contain a rich collection of mathematical ideas. Chapters 1 through 3 combine to provide a detailed overview of language, convergence tests, techniques, and special series, with particular attention to the harmonic series. The remaining three chapters (plus one appendix) made a significant impression, highlights of which are shared here.

Chapter 4 consists of 107 "Gems". The authors write: "There are a number of reasons we may have labeled a result a gem. Gems may confirm a common intuitive notion with a clarifying proof, or may provide a counter-example to intuition. They may exhibit a particularly slick or unexpected proof technique. Some gems were chosen because they seemed to us at first astonishing but with a moment's study of the proof became almost common sense."

Chapter 5 consists entirely of 63 problems from Putnam Competitions from 1940 through 2002 inclusive. Detailed solutions are provided.

Yet, most impressive is Chapter 6, Final Diversions, in which the authors share, "as a parting gift . . . a sort of dessert," puzzles involving infinite series and pictorial proofs of basic facts about infinite series. Selections from "Fallacies, Flaws, and Flimflam" (*College Math. Journal* and/or Ed Barbeau's MAA book by the same name) wind up the chapter: One such example is an adaptation of a 1996 ***Crux Mathematicorum*** problem.

The appendices offer valuable references. Appendix A is unusual in that it features 101 True/False questions. Detailed explanations are again provided. This thoroughness and sense of completion pervade the entire book. Whether you teach calculus, work with Putnam participants, or simply wish to consider a familiar subject from a refreshing perspective, this book will not disappoint you. My appreciation for the topic was enhanced by this book, particularly the blend of gems and diversions.

A Parity Subtraction Game

Richard Guy

In memory of Robert Barrington Leigh

In the *Olympiad Corner* No. 222 of ***CRUX with MAYHEM***, **28**, no. 4 (May, 2002), a selection of problems from the St. Petersburg Mathematical Olympiads is given by Oleg Ivrii and Robert Barrington Leigh. The third one [2, p. 289, Problem 3 (1965)] is

> A game starts with a heap of 25 beans. Two players alternately remove 1, 2, or 3 of them. When all the beans have been taken, the winner is the player who has an even number of beans. Assuming perfect play, does the first player or the second have a sure win?

The *Olympiad Corner* editor recently received a request for a solution. The problem is from a list of supplementary problems; it may not have been used, and no solution is given in the book.

This game differs from the usual kind of take-away game in that it is not impartial. After a move has been made, even if you were not watching, you can tell which player has moved by noticing who has added to his or her collection of beans. Thus, we cannot use the Sprague-Grundy Theory; that is, we cannot calculate nim-values [1, Chap. 2]. On the other hand, it is not always a last-player-winning game; there is a mixture of normal and misère (last-player-losing) play [1, Chap. 13], which means that we cannot use the Conway Theory [1, Chap. 1] either.

We use a rather brute-force method, in effect drawing the whole game tree, though we save a good deal of space by identifying nodes in our Figure 1.

Here is a solution for the game, played with any odd number of beans.

For heaps of $8k + 3$, $8k + 5$, or $8k + 7$ beans, the first player wins; for heaps of $8k + 1$ beans, the second player wins. Hence, with 25 beans, the second player can win.

To see that this is the case, we represent positions in the game by $b(f, s)$, where b is the number of beans remaining in the heap, and f and s are the total numbers of beans already collected by the first (next) and second (previous) players, respectively.

We will use d and e for arbitrary odd and even numbers, respectively. The opening position is of shape $d(0, 0)$, and subsequent positions all satisfy $b + f + s = d$. Notice that, when a move is made, the roles of first (next) and second (previous) player are interchanged. When the next player takes t beans from $b(f, s)$, the position becomes $b-t\,(s, f+t)$. There is need for someone to devise a more perspicuous notation!

Detailed Solution

A position in such a game is, with best play, either a win for the first (next) player, or the second (previous) player. These are often labelled $\mathcal{N}$–positions and $\mathcal{P}$–positions, respectively.

Solution. In the game described above,

1. the $\mathcal{P}$–positions are $8k+1\,(e,e)$ and $8k+5\,(d,d)$;
2. the $\mathcal{N}$–positions are $8k+7\,(d,d)$, $8k+7\,(e,e)$, $8k+6\,(d,e)$, $8k+6\,(e,d)$, $8k+5\,(e,e)$, $8k+3\,(e,e)$, $8k+3\,(d,d)$, $8k+2\,(d,e)$, $8k+2\,(e,d)$ and $8k+1\,(d,d)$.
3. That leaves $8k+4\,(d,e)$ and $8k+4\,(e,d)$, which are wins for d, whoever starts, and $8k\,(d,e)$ and $8k\,(e,d)$, which are wins for e, whoever starts. That is, $8k+4\,(d,e)$ and $8k\,(e,d)$ are $\mathcal{N}$–positions, while $8k+4\,(e,d)$ and $8k\,(d,e)$ are $\mathcal{P}$–positions.

Proof: We use induction. To start the induction, we consider $k=0$.

- $0\,(d,e)$ is a win for e by definition, and hence, $1\,(e,e)$ will be a loss, and $1\,(d,d)$ a win, for the next player.
- $2\,(d,e)$ is a win for the next player, if he goes to $1\,(e,e)$, as is $2\,(e,d)$ if she takes both beans, going to $0\,(d,e)$.
- If there are just 3 beans remaining, the first player takes them all if he is d, but only 2 of them if she is e.
- But $4\,(d,e)$ and $4\,(e,d)$ are both wins for d. If d starts, he goes to $1\,(e,e)$, while the only possible moves for e are $3\,(d,d)$, $2\,(d,e)$, or $1\,(d,d)$, which are all next-player wins.
- Thus, from $5\,(e,e)$, the next player will go to $4\,(e,d)$. But from $5\,(d,d)$, the next player must go to $4\,(d,e)$, $3\,(d,d)$, or $2\,(d,e)$, which are next-player wins.
- $6\,(d,e)$ and $6\,(e,d)$ are $\mathcal{N}$–positions; d goes to $4\,(e,d)$ or e goes to $5\,(d,d)$.
- From $7\,(e,e)$, the next player goes to $4\,(e,d)$, where we have seen that d wins, while from $7\,(d,d)$ the next player can go to the $\mathcal{P}$–position $5\,(d,d)$.
- From $8\,(e,d)$, e wins by going to $5\,(d,d)$ while from $8\,(d,e)$, d must play to $7\,(e,e)$, $6\,(e,d)$, or $5\,(e,e)$ and the next player then wins by going to $4\,(e,d)$, $5\,(d,d)$, or $4\,(e,d)$, respectively.

In order to get the induction off the ground, we need to go a bit further, with three cases of $k=1$, since the players may remove up to three beans. From $9\,(d,d)$, the next player wins by going to $8\,(d,e)$, while $9\,(e,e)$ is a $\mathcal{P}$–position, the second player able to go to $5\,(d,d)$ if the first player takes 1 or 3, or to $4\,(e,d)$ if the first player takes 2.

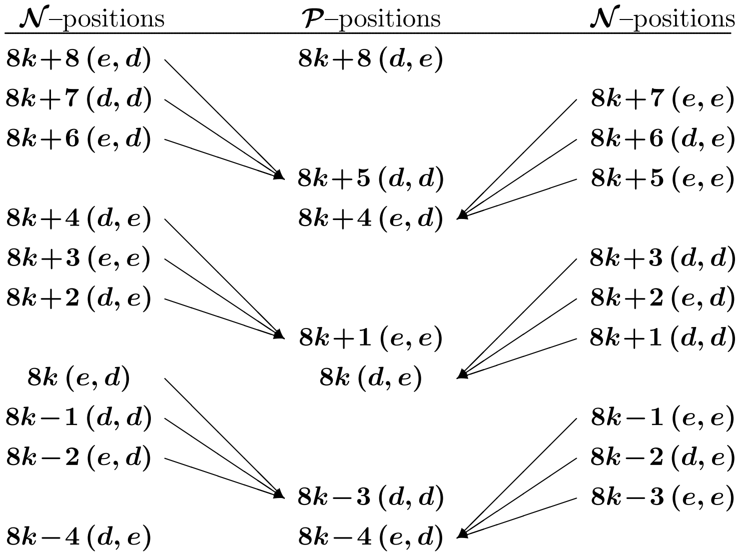


Figure 1: Condensed game tree

From $10\,(d,e)$, d wins with $9\,(e,e)$ and from $10\,(e,d)$, e wins with $8\,(d,e)$. From here on, copy the strategy from 8 beans back; for example, from $11\,(d,d)$ move to $8\,(d,e)$ and from $11\,(e,e)$ to $9\,(e,e)$.

Check the detailed solution listed above against Figure 1, which is periodic in the sense that it repeats itself every 8 rows. For example,

the strategy for	$8k+3$	$8k+4$	$8k+5$	$8k+6$	$8k+7$
is the same as for	$8k-5$	$8k-4$	$8k-3$	$8k-2$	$8k-1$

respectively. The arrows in Figure 1 are the only winning moves. There is an arrow from every $\mathcal{N}$–position to a $\mathcal{P}$–position. All other legal moves, of which there are three from each $\mathcal{P}$–position and two non-winning moves from each $\mathcal{N}$–position, lead to $\mathcal{N}$–positions; they are not shown in the figure.

Can anyone supply a general theory for such "four outcome" games?

References

[1] Elwyn Berlekamp, John Conway, and Richard Guy, *Winning Ways for Your Mathematical Plays*, 2nd edition, A.K. Peters, Natick, MA, 2001–2004.

[2] D.V. Fomin, *St. Petersburg Mathematical Olympiads, 1961–1993*, Politechnika, St. Petersburg, 1994.

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PROBLEMS

Solutions to problems in this issue should arrive no later than **1 August 2007**. *An asterisk ($\star$) after a number indicates that a problem was proposed without a solution.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English. In the solutions section, the problem will be stated in the language of the primary featured solution.

The editor thanks Jean-Marc Terrier and Martin Goldstein of the University of Montreal for translations of the problems.

We have discovered that some recently posed problems are repeats of earlier problems: problem 3182 [2006 : 463, 464] is the same as problem 3096 [2005 : 544, 547]; problem 3185 [2006 : 463, 465] is the same as problem 2935 [2004 : 174, 176], and problem 3198 [2006 : 516, 518] is the same problem as 3187 [2006 : 463, 465]. Since all three duplications have only appeared within the last two issues, we are replacing them in this issue. Any solutions for the original problems 3182 and 3185 will be ignored, since solutions to those problems have already appeared; any solutions to the original 3198 will be treated as solutions to 3187. Our thanks to Michel Bataille for bringing this to our attention.

3182. Replacement. *Proposed by Arkady Alt, San Jose, CA, USA.*

Let a, b, and c be any positive real numbers, and let p be a real number such that $0 < p < 1$.

(a) Prove that

$$\frac{a}{(b+c)^p} + \frac{b}{(c+a)^p} + \frac{c}{(a+b)^p} \ \ge\ \frac{1}{2^p}\left(a^{1-p} + b^{1-p} + c^{1-p}\right) .$$

(b) Prove that, if $p = 1/3$, then

$$\frac{a}{(a+b)^p} + \frac{b}{(b+c)^p} + \frac{c}{(c+a)^p} \ \ge\ \frac{1}{2^p}\left(a^{1-p} + b^{1-p} + c^{1-p}\right) .$$

(c)$\star$ Prove or disprove

$$\frac{a}{\sqrt{a+b}} + \frac{b}{\sqrt{b+c}} + \frac{c}{\sqrt{c+a}} \ \ge\ \frac{1}{\sqrt{2}}\left(\sqrt{a} + \sqrt{b} + \sqrt{c}\right) .$$

3185. Replacement. *Proposed by Shaun White, student, Vincent Massey Secondary School, Windsor, ON.*

Let a_n denote the units digit of $(4n)^{(3n)^{(2n)^n}}$. Find all positive integers n such that $\sum\limits_{i=1}^{n} a_i \ge 4n$.

3198. Replacement. *Proposed by Michel Bataille, Rouen, France.*

Let $p = 2n+1$ be a prime, and let s be any integer such that $1 \le s \le n$. Prove that:

(a) $4^s \sum_{k=0}^{n-s} \binom{2s+2k-1}{2s-1} \equiv 1 \pmod{p}$,

(b) $4^s \sum_{k=0}^{n-s} \binom{2s+2k}{2s-1} \equiv -1 \pmod{p}$.

3201. *Proposed by G.P. Henderson, Garden Hill, Campbellcroft, ON, in memory of Murray S. Klamkin.*

Given positive integers m and n, consider the real monic polynomials $P(x) = \sum_{i=0}^{m} a_i x^i$ and $Q(x) = \sum_{j=0}^{n} b_j x^j$ with non-negative coefficients. We are interested in whether P and Q satisfy the condition

$$P(x)Q(x) = \sum_{k=0}^{m+n} x^k .$$

(a) Prove that if m and n are both odd, there are no such polynomials.

(b) Prove that if $m = n$, there are no such polynomials.

(c) Show that for each m there is an infinite set of values of n for which there do exist such polynomials.

(d) Prove that the coefficients in every such pair of polynomials are either 0 or 1.

(Compare problem 266 in Edward J. Barbeau, Murray S. Klamkin, and William O.J. Moser, *Five Hundred Mathematical Challenges*, where $m = n = 5$.)

3202. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Let Γ be a circle with radius r, let A be any point on Γ, and let t be the tangent line to Γ at A. Let B and C be points of t on opposite sides of A such that $AB = mr$ and $AC = nr$ for some positive real numbers m and n. Let P be any point of Γ different from A. Show that $\cot\angle APB + \cot\angle APC$ is a constant for all such points P, and determine this constant value in terms of m and n.

3203. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Let AB be the diameter of a semicircle Γ. Let D be any point on the tangent to Γ at B and lying on the same side of AB as Γ, and let C be the mid-point of BD. The segments AC and AD intersect Γ for the second time at the points K and L, respectively. If M and N are the projections onto KL of A and B, respectively, show that $ML = LK = KN$.

3204. *Proposed by Mihály Bencze, Brasov, Romania.*

Let A, $J \in M_{n\times n}(\mathbb{R})$, where J is the matrix all of whose entries are 1, and let $b \in \mathbb{R}$. Set $B = bJ$, and for $k = 1, 2, \ldots, n$, denote by A_k the matrix obtained from A by replacing each element in row k with the value b. Prove that

$$\det(A+B)\det(A-B) \;=\; (\det A)^2 - \left(\sum_{k=1}^{n} \det A_k\right)^2 .$$

3205. *Proposed by Mihály Bencze and Marian Dinca, Brasov, Romania.*

Let $A_1A_2\cdots A_n$ be a convex polygon, and let P be any interior point of the polygon. For $k = 1, 2, \ldots, n$, let G_k be the centroid of the polygon $A_1A_2\cdots A_{k-1}A_{k+1}\cdots A_n$ (the polygon obtained by removing vertex A_k from $A_1A_2\cdots A_n$). If B_k is the reflection of G_k through the point P, prove that the lines A_iB_i are concurrent for $i = 1, 2, \ldots, n$.

3206. *Proposed by Mihály Bencze, Brasov, Romania.*

Let n be a positive integer and x a real number. Prove that

$$\lfloor x\rfloor^2 + \left\lfloor x+\frac{1}{n}\right\rfloor^2 + \cdots + \left\lfloor x+\frac{n-1}{n}\right\rfloor^2 \;=\; \lfloor nx\rfloor^2$$

if and only if $\lfloor nx\rfloor = \lfloor x\rfloor$, where $\lfloor\cdot\rfloor$ denotes the greatest integer function.

3207. *Proposed by Peter Y. Woo, Biola University, La Mirada, CA, USA.*

Let a convex quadrilateral $APQC$ have its sides AP, PQ, and QC tangent to a minor circular arc ABC at the points A, B, and C, respectively. Let E be the projection of B onto AC. Let a semicircle with PQ as diameter cut AC at H and K, with H between A and K.

Without using trigonometry, prove that BE bisects $\angle PEQ$ and that PH bisects $\angle APB$.

3208. *Proposed by Shaun White, student, Vincent Massey Secondary School, Windsor, ON.*

Find the largest integer k such that for all positive real numbers a, b, c, we have

$$(a^3+3)(b^3+6)(c^3+12) \;>\; k(a+b+c)^3 .$$

3209. *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

Let f be a convex function on an interval I. For $i = 1, 2, \ldots, n$, let $a_i \in I$. Define $a = \frac{1}{n}\sum\limits_{i=1}^{n} a_i$. Prove that

$$\frac{n(n-2)}{2}\,f(a) + \sum_{i=1}^{n} f(a_i) \;\ge\; \frac{n}{2(n-1)}\sum_{i\neq j} f\left(a+\frac{a_i-a_j}{n}\right) .$$

3210. *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

Prove that, for all real numbers $a_1, a_2, \ldots, a_n \in \left[\frac{1}{\sqrt{2}}, \sqrt{2}\right]$, we have

$$\sum_{i=1}^{n} \frac{3}{a_i + 2a_{i+1}} \;\geq\; \sum_{i=1}^{n} \frac{2}{a_i + a_{i+1}}\,,$$

where the subscripts are taken modulo n.

3211. *Proposed by an anonymous proposer.*

Let $ABCD$ be a quadrilateral which is inscribed in a circle Γ. Further suppose that $ABCD$ itself has an incircle. Let EF be the diameter of Γ which is perpendicular to BD, with E lying on the same side of BD as A. Let BD intersect EF at M and AC at S.

Prove that $AS : SC = EM : MF$.

[*Ed:* This problem came into the University of Regina's Math Central website, but the name of the proposer has subsequently been lost.]

3212. *Proposed by José Luis Díaz-Barrero and Francisco Palacios Quiñonero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let $a_1, a_2, \ldots, a_n$ be real numbers such that $a_k \geq 1$ for $1 \leq k \leq n$. Prove that

$$\prod_{k=1}^{n} a_k^{\left(\frac{2k}{n(n+1)}\right)^{1/2}} \;\leq\; \exp\left(\sqrt{\sum_{k=1}^{n} \ln^2 a_k}\right)\,.$$

. . . .

3182. Remplacement. *Proposé par Arkady Alt, San Jose, CA, USA.*

Soit a, b et c des nombres réels positifs quelconques, et soit p un nombre réel tel que $0 < p < 1$.

(a) Montrer que

$$\frac{a}{(b+c)^p} + \frac{b}{(c+a)^p} + \frac{c}{(a+b)^p} \;\geq\; \frac{1}{2^p}\left(a^{1-p} + b^{1-p} + c^{1-p}\right)\,.$$

(b) Montrer que, si $p = 1/3$, alors

$$\frac{a}{(a+b)^p} + \frac{b}{(b+c)^p} + \frac{c}{(c+a)^p} \;\geq\; \frac{1}{2^p}\left(a^{1-p} + b^{1-p} + c^{1-p}\right)\,.$$

(c)★ Confirmer ou infirmer que

$$\frac{a}{\sqrt{a+b}} + \frac{b}{\sqrt{b+c}} + \frac{c}{\sqrt{c+a}} \;\geq\; \frac{1}{\sqrt{2}}\left(\sqrt{a} + \sqrt{b} + \sqrt{c}\right)\,.$$

3185. Remplacement. *Proposé par Shaun White, étudiant, École secondaire Vincent Massey, Windsor, ON.*

Désignons par a_n le chiffre des unités de $(4n)^{(3n)^{(2n)^n}}$. Trouver tous les entiers positifs n tels que $\sum_{i=1}^{n} a_i \geq 4n$.

3198. Remplacement. *Proposé par Michel Bataille, Rouen, France.*

Soit $p = 2n + 1$ un nombre premier, et soit s un entier quelconque tel que $1 \leq s \leq n$. Montrer que :

(a) $4^s \sum_{k=0}^{n-s} \binom{2s+2k-1}{2s-1} \equiv 1 \pmod{p}$,

(b) $4^s \sum_{k=0}^{n-s} \binom{2s+2k}{2s-1} \equiv -1 \pmod{p}$.

3201. *Proposé par G.P. Henderson, Garden Hill, Campbellcroft, ON en mémoire de Murray S. Klamkin.*

Etant donné des nombres entiers positifs m et n, on considère les polynômes réels unitaires $P(x) = \sum_{i=0}^{m} a_i x^i$ et $Q(x) = \sum_{j=0}^{n} b_j x^j$ avec des coefficients non négatifs. La question est de savoir si P et Q peuvent satisfaire la condition

$$P(x)Q(x) = \sum_{k=0}^{m+n} x^k .$$

(a) Montrer que si m et n sont tous deux impairs, il n'y a pas de tels polynômes.

(b) Montrer que si $m = n$, il n'y a pas de tels polynômes.

(c) Montrer que pour tout m, il y a une infinité de valeurs de n pour lesquelles de tels polynômes existent.

(d) Montrer que dans toute paire de tels polynômes, les coefficients sont 0 ou 1.

(Comparer au problème 266 dans Edward J. Barbeau, Murray S. Klamkin, and William O.J. Moser, *Five Hundred Mathematical Challenges*, où $m = n = 5$.)

3202. *Proposé par D.J. Smeenk, Zaltbommel, Pays-Bas.*

Soit Γ un cercle de rayon r, A un point quelconque sur Γ, et t la tangente à Γ en A. Soit B et C deux points sur t, de part et d'autre de A, tels que $AB = mr$ et $AC = nr$, m et n deux nombres réels positifs. Montrer que $\cot \angle APB + \cot \angle APC$ a une valeur constante pour tous les points P sur Γ, distincts de A, et déterminer cette valeur en fonction de m and n.

3203. *Proposé par D.J. Smeenk, Zaltbommel, Pays-Bas.*

Soit AB le diamètre d'un demi-cercle Γ. Soit D un point quelconque sur la tangente à Γ en B et situé du même côté de AB que Γ, et soit C le point milieu de BD. Soit K et L les deuxièmes points d'intersection respectifs des segments AC et AD avec Γ. Si M et N sont les projections respectives de A et B sur KL, montrer que $ML = LK = KN$.

3204. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit A, $J \in M_{n\times n}(\mathbb{R})$, où J est la matrice dont tous les éléments sont égaux à 1, et soit $b \in \mathbb{R}$. Posons $B = bJ$ et, pour $k = 1, 2, \ldots, n$, désignons par A_k la matrice obtenue à partir de A en remplaçant k par b. Montrer que

$$\det(A+B)\det(A-B) \;=\; (\det A)^2 - \left(\sum_{k=1}^{n} \det A_k\right)^2 .$$

3205. *Proposé par Mihály Bencze et Marian Dinca, Brasov, Roumanie.*

Soit $A_1A_2\cdots A_n$ un polygone convexe, et soit P un point intérieur arbitraire du polygone. Pour $k = 1, 2, \ldots, n$, soit G_k le centre de gravité du polygone $A_1A_2\cdots A_{k-1}A_{k+1}\cdots A_n$ (le polygone obtenu en enlevant le sommet A_k de $A_1A_2\cdots A_n$). Si B_k désigne la réflexion de G_k par rapport à P, montrer que les droites A_iB_i sont concourantes pour $i = 1, 2, \ldots, n$.

3206. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit n un nombre positif et x un nombre réel. Montrer que

$$\lfloor x\rfloor^2 + \left\lfloor x+\frac{1}{n}\right\rfloor^2 + \cdots + \left\lfloor x+\frac{n-1}{n}\right\rfloor^2 \;=\; \lfloor nx\rfloor^2$$

si et seulement si $\lfloor nx\rfloor = \lfloor x\rfloor$, où $\lfloor\cdot\rfloor$ désigne la fonction «partie entière».

3207. *Proposé par Peter Y. Woo, Biola University, La Mirada, CA, USA.*

Soit $APQC$ un quadrilatère convexe dont les côtés AP, PQ et QC sont respectivement tangents à un arc circulaire mineur ABC aux points A, B et C. Soit E la projection de B sur AC. Le demi-cercle de diamètre PQ coupe AC en H et en K, avec H entre A et K.

Sans trigonométrie, montrer que BE est une bissectrice de l'angle PEQ et que PH est une bissectrice de l'angle APB.

3208. *Proposé par Shaun White, étudiant, École secondaire Vincent Massey, Windsor, ON.*

Trouver le plus grand entier k tel que pour tous les nombres réels positifs a, b et c, on ait

$$(a^3+3)(b^3+6)(c^3+12) \;>\; k(a+b+c)^3 .$$

3209. *Proposé par Vasile Cîrtoaje, Université de Ploiesti, Roumanie.*

Soit f une fonction convexe sur un intervalle I. Pour $i = 1, 2, \ldots, n$, soit $a_i \in I$. On définit $a = \frac{1}{n}\sum_{i=1}^{n} a_i$. Montrer que

$$\frac{n(n-2)}{2} f(a) + \sum_{i=1}^{n} f(a_i) \ \geq \ \frac{n}{2(n-1)} \sum_{i \neq j} f\left(a + \frac{a_i - a_j}{n}\right) .$$

3210. *Proposé par Vasile Cîrtoaje, Université de Ploiesti, Roumanie.*

Montrer que, pour tous les nombres réels $a_1, a_2, \ldots, a_n \in \left[\frac{1}{\sqrt{2}}, \sqrt{2}\right]$, on a

$$\sum_{i=1}^{n} \frac{3}{a_i + 2a_{i+1}} \ \geq \ \sum_{i=1}^{n} \frac{2}{a_i + a_{i+1}} ,$$

où les indices sont calculés modulo n.

3211. *Proposé par un proposeur anonyme.*

Soit $ABCD$ un quadrilatère inscrit dans un cercle Γ. On suppose de plus que $ABCD$ possède lui-même un cercle inscrit. Soit EF le diamètre de Γ perpendiculaire à BD, E situé du même côté de BD que A. BD coupe EF disons en M et AC en S.

Montrer que $AS : SC = EM : MF$.

[*Ed :* Ce problème est apparu sur le site internet central de Math de l'Université de Regina, mais le nom du proposeur a disparu par la suite.]

3212. *Proposé par José Luis Díaz-Barrero et Francisco Palacios Quiñonero, Université Polytechnique de Catalogne, Barcelone, Espagne.*

Soit $a_1, a_2, \ldots, a_n$ des nombres réels tels que $a_k \geq 1$ pour $1 \leq k \leq n$. Montrer que

$$\prod_{k=1}^{n} a_k^{\left(\frac{2k}{n(n+1)}\right)^{1/2}} \ \leq \ \exp\left(\sqrt{\sum_{k=1}^{n} \ln^2 a_k}\right) .$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Yakub N. Aliyev, Baku State University, Baku, Azerbaijan has brought to our attention a difficulty with the featured solution to 3075 [2006 : 469]. There is a claim made that $a^x - b^x$ is strictly increasing for $a > b > 0$ and $x \geq 0$; this claim is not correct. The problem moderator, upon review of the featured solution, has informed me that in each of the cases where $a^x - b^x$ is used in the solution, it is indeed strictly increasing. We apologize for any confusion this may have generated.

3101. [2006 : 44, 47] *Proposed by K.R.S. Sastry, Bangalore, India.*

The two distinct cevians AP and AQ of $\triangle ABC$ satisfy the equation $AQ^2 = AP^2 + |AC - AB|^2$.

(a) If $BP = CQ$, show that AP bisects $\angle BAC$.

(b)★ If AP bisects $\angle BAC$, prove or disprove that $BP = CQ$.

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA; and Joel Schlosberg, Bayside, NY, USA.

If $b = c$, statement (a) of the problem is incorrect. As a counterexample, choose any cevians AP and AQ such that $\angle BAP = \angle CAQ \neq \frac{1}{2}\angle BAC$. Then AP and AQ are distinct, since $\angle BAP + \angle CAQ \neq \angle BAC$. Since $\triangle ABC$ is symmetrical under reflection in the bisector of $\angle BAC$, and since this reflection interchanges P and Q, we see that $AP = AQ$. Therefore, $AQ^2 = AP^2 = AP^2 + (b-c)^2$ and $BP = CQ$.

Hence, we will assume that $b \neq c$. Let $x = CQ = PB$. Then, by the Law of Cosines, we have

$$\begin{aligned} AQ^2 &= b^2 + x^2 - 2bx\cos C = b^2 + x^2 - \frac{x}{a}(a^2+b^2-c^2) \\ \text{and}\quad AP^2 &= c^2 + x^2 - 2cx\cos B = c^2 + x^2 - \frac{x}{a}(a^2+c^2-b^2)\,. \end{aligned}$$

Therefore,

$$(b-c)^2 = AQ^2 - AP^2 = \left(1 - \frac{2x}{a}\right)(b^2-c^2)\,,$$

which simplifies to $a(b-c) = (a-2x)(b+c)$. Hence, $\dfrac{b}{c} = \dfrac{a-x}{x} = \dfrac{CP}{PB}$, and the proof is complete.

We show next that part (b) is not true (but almost true). Let $x = CQ$. Since $BP = \dfrac{ac}{b+c}$, we get

$$AP^2 = c^2 + BP^2 - 2cBP\cos B = c^2 + \left(\frac{ac}{b+c}\right)^2 - \frac{c(a^2+c^2-b^2)}{b+c}\,.$$

Also,

$$AQ^2 \;=\; x^2+b^2-\left(\frac{a^2+b^2-c^2}{a}\right)x\,.$$

Substituting these into $AQ^2 = AP^2 + (b-c)^2$, we get

$$x^2-\left(\frac{a^2+b^2-c^2}{a}\right)x+\frac{c((b^2-c^2)(b+c)+a^2b)}{(b+c)^2}=0\,,$$

which factors into

$$\left(x+\frac{ac}{b+c}\right)\left(x-\frac{a^2+b^2-c^2}{a}+\frac{ac}{b+c}\right) \;=\; 0\,.$$

Hence, $x = BP$ or $x = 2b\cos C - BP$. Note that these two solutions yield two positions for Q that are symmetric with respect to the altitude on BC. For a concrete example, if $a = 5$, $b = 4$, and $c = 3$, then $BP = 15/7$ and $CQ = 15/7$ or $CQ = 149/35$.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, University of Beirut, Beirut, Lebanon; MICHEL BATAILLE, Rouen, France; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comăneşti, Romania; and the proposer.

3102. [2006 : 44, 47] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

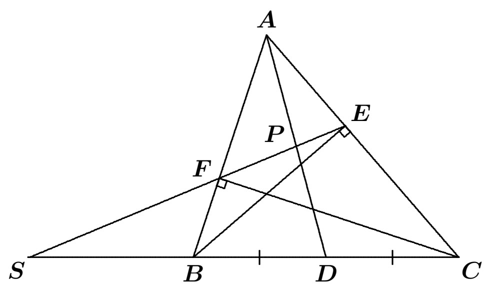
Let D be the mid-point of the side BC of $\triangle ABC$. Let E and F be the projections of B onto AC and C onto AB, respectively. Let P be the point of intersection of AD and EF. Show that, if $AD = \frac{\sqrt{3}}{2}BC$, then P is the mid-point of AD.

I. *Composite of similar solutions by Geoffrey A. Kandall, Hamden, CT, USA; and Titu Zvonaru, Comăneşti, Romania, expanded by the editor.*

We will employ directed lengths, so as to take account of all possible cases. (A triangle satisfying the hypotheses of the problem need not be acute; it may have an obtuse angle at B or at C.) Our proof will establish both the required implication and its converse.

First we will prove the following (cf. [2000 : 205]):

$$\frac{DP}{PA} \;=\; \frac{1}{2}\left(\frac{BF}{FA}+\frac{CE}{EA}\right)\,. \tag{1}$$



Let S be the point at which the line EF meets the line BC. (If these lines are parallel, then take S to be at infinity.) Let $x = \frac{BF}{FA}$ and $y = \frac{CE}{EA}$. By Menelaus' Theorem applied in $\triangle ABC$, we have

$$\frac{BF}{FA} \cdot \frac{AE}{EC} \cdot \frac{CS}{SB} = -1 ;$$

that is, $x\,\frac{CS}{BS} = y$. Writing $CS = BS - BC$, we obtain

$$x\,\frac{BC}{BS} = x - y . \tag{2}$$

Now we apply Menelaus' Theorem in $\triangle ABD$ to get

$$\frac{BF}{FA} \cdot \frac{AP}{PD} \cdot \frac{DS}{SB} = -1 ;$$

that is, $\frac{DP}{PA} = x\,\frac{DS}{BS}$. Writing $DS = DB + BS$ and using (2), we get

$$\begin{aligned} \frac{DP}{PA} &= x\,\frac{DB + BS}{BS} = x\,\frac{BC}{BS}\frac{DB}{BC} + x = (x - y)\frac{DB}{BC} + x \\ &= x\,\frac{DB + BC}{BC} - y\,\frac{DB}{BC} = x\,\frac{DC}{BC} + y\,\frac{BD}{BC} . \end{aligned}$$

Since D is the mid-point of BC, we have $\frac{DC}{BC} = \frac{BD}{BC} = \frac{1}{2}$, and (1) follows immediately.

With the usual notation for sides and angles in triangle ABC, we find, using (1), that

$$\begin{aligned} \frac{DP}{PA} &= \frac{1}{2}\left(\frac{a\cos B}{b\cos A} + \frac{a\cos C}{c\cos A}\right) = \frac{1}{2} \cdot \frac{2ac\cos B + 2ab\cos C}{2bc\cos A} \\ &= \frac{1}{2} \cdot \frac{\left(a^2 + c^2 - b^2\right) + \left(a^2 + b^2 - c^2\right)}{b^2 + c^2 - a^2} = \frac{a^2}{b^2 + c^2 - a^2} . \end{aligned}$$

We now use the well-known relation $4AD^2 = 2b^2 + 2c^2 - a^2$ to obtain

$$\frac{DP}{PA} = \frac{2a^2}{4AD^2 - a^2} ,$$

from which we see that $DP/PA = 1$ if and only if $AD = \left(\sqrt{3}/2\right)a$.

II. *Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.*

Let $a = BC$, $b = CA$, and $c = AB$. Since $4AD^2 = 2b^2 + 2c^2 - a^2$, the given condition $AD = \left(\sqrt{3}/2\right)a$ is equivalent to $b^2 + c^2 = 2a^2$. Let Q be the mid-point of AD. We will prove that Q, E, F are collinear, which implies that Q is the point P defined in the problem.

Let $\triangle ABC$ be the coordinate triangle of the homogeneous barycentric coordinate system, with $A = (1:0:0)$, $B = (0:1:0)$, and $C = (0:0:1)$.

Then $D = (0 : 1 : 1)$, $Q = (2 : 1 : 1)$, $E = (b - c\cos A : 0 : c\cos A)$, and $F = (c - b\cos A : b\cos A : 0)$. Hence,

$$\begin{aligned}\det\begin{bmatrix} Q \\ E \\ F \end{bmatrix} &= \det\begin{bmatrix} 2 & 1 & 1 \\ b - c\cos A & 0 & c\cos A \\ c - b\cos A & b\cos A & 0 \end{bmatrix} \\ &= (b^2 + c^2 - 4bc\cos A)\cos A \\ &= \left[2(b^2 + c^2 - 2bc\cos A) - b^2 - c^2\right]\cos A \\ &= (2a^2 - b^2 - c^2)\cos A \;=\; 0\,,\end{aligned}$$

which means that Q, E, F are collinear.

Also solved by MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; TAICHI MAEKAWA, Takatsuki City, Osaka, Japan; JOEL SCHLOSBERG, Bayside, NY, USA; and the proposer. There was one incorrect solution.

3103. [2006 : 44, 47] *Proposed by Michel Bataille, Rouen, France.*

Let ABC be an acute-angled triangle with circumcentre O. Let the lines AO, BO, and CO meet the circles BCO, CAO, and ABO for the second time at A', B', and C', respectively. Let $|XYZ|$ denote the perimeter and $[XYZ]$ the area of the triangle XYZ. Prove that

(a) $\dfrac{BC}{|BCA'|} + \dfrac{CA}{|CAB'|} + \dfrac{AB}{|ABC'|} \;=\; 1$;

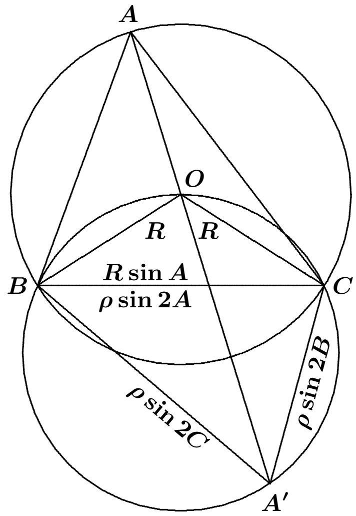
(b) $[BCO]\cdot[BCA'] + [CAO]\cdot[CAB'] + [ABO]\cdot[ABC'] \;=\; [ABC]^2$.

Solution by D.J. Smeenk, Zaltbommel, the Netherlands.

Since $\triangle ABC$ is acute-angled, O is in the interior and $\angle COA = 2B$; then $\angle A'OC = 180^\circ - 2B$. Similarly, $\angle A'OB = 180^\circ - 2C$. These two angles sum to $\angle BOC = 2A$. Because $OBA'C$ is cyclic, the angles at O and at A' are supplementary, $\angle A'BC = \angle A'OC$, and $\angle A'CB = \angle A'OB$. Thus, the angles of $\triangle A'BC$ are

$$\begin{aligned}\angle BA'C &= 180^\circ - 2A\,,\\ \angle A'BC &= 180^\circ - 2B\,,\\ \angle A'CB &= 180^\circ - 2C\,.\end{aligned}$$

We denote the circumradius of $\triangle A'BC$ by ρ and apply the Law of Sines to get



$$\frac{BC}{|BCA'|} = \frac{2\rho\sin 2A}{2\rho(\sin 2A+\sin 2B+\sin 2C)} = \frac{\sin 2A}{\sin 2A+\sin 2B+\sin 2C}\,.$$

Result (a) now follows immediately.

For part (b), the Law of Sines applied to the common side BC of $\triangle ABC$ and $\triangle A'BC$ (with circumradii R and ρ, respectively) implies that $2R\sin A = 2\rho\sin 2A$, or

$$\rho = \frac{R}{2\cos A}\,. \qquad (1)$$

We note that

$$\begin{aligned}[BCO]\cdot[BCA'] &= \left(\tfrac{1}{2}BO\cdot BC\cos A\right)\left(\tfrac{1}{2}BC\cdot BA'\sin 2B\right)\\ &= R^2\sin A\cos A\cdot 2\rho^2\sin 2A\sin 2B\sin 2C\,.\end{aligned}$$

Inserting the value of ρ from (1) in this last equation gives us

$$[BCO]\cdot[BCA'] = \tfrac{1}{2}R^4\sin 2A\sin 2B\sin 2C\tan A\,.$$

Of course, the analogous equations hold for the other pairs of areas. Since $\tan A+\tan B+\tan C = \tan A\tan B\tan C$, we find that

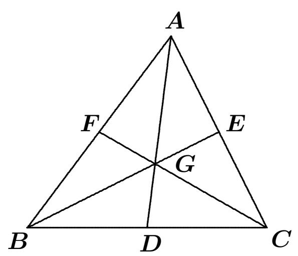
$$\begin{aligned}\sum_{\text{cyclic}}[BCO]\cdot[BCA'] &= \tfrac{1}{2}R^4\sin 2A\sin 2B\sin 2C(\tan A+\tan B+\tan C)\\ &= \tfrac{1}{2}R^4\sin 2A\sin 2B\sin 2C\tan A\tan B\tan C\\ &= (2R^2\sin A\sin B\sin C)^2 = [ABC]^2\,.\end{aligned}$$

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; JOHN G. HEUVER, Grande Prairie, AB; WALTHER JANOUS, Ursulinen-gymnasium, Innsbruck, Austria; JOEL SCHLOSBERG, Bayside, NY, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comăneşti, Romania; and the proposer.

3104. [2006 : 45, 47] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

In $\triangle ABC$, let D, E, and F be the mid-points of the sides BC, CA, and AB, respectively. Show that, if $AD = \frac{\sqrt{3}}{2}BC$, then $\angle BEC = \angle AFC$.

Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA.



We strengthen the problem from "if" to "if and only if".

Let G be the centroid. Using Apollonius' Theorem, we get

$$BE^2 = \tfrac{1}{2}\left(c^2+a^2\right)-\left(\tfrac{1}{2}b\right)^2 = \tfrac{1}{2}c^2+\tfrac{1}{2}a^2-\tfrac{1}{4}b^2 = \tfrac{1}{4}\left(2c^2+2a^2-b^2\right) .$$

Therefore, we have $BG\cdot BE = \frac{2}{3}BE^2 = \frac{1}{6}\left(2c^2+2a^2-b^2\right)$. Note also that $BF\cdot BA = \frac{1}{2}c^2$.

Now the following statements are equivalent:

- $\angle BEC = \angle AFC$;
- quadrilateral $AEGF$ is cyclic;
- $BG\cdot BE = BF\cdot BA$;
- $\frac{1}{2}c^2 = \frac{1}{6}(2c^2+2a^2-b^2$, that is, $b^2+c^2 = 2a^2$;
- $2AD^2 = 2a^2-\frac{1}{2}a^2$ (using Apollonius' Theorem);
- $AD = \left(\sqrt{3}/2\right)a$.

Thus, $\angle BEC = AFC$ if and only if $AD = \left(\sqrt{3}/2\right)BC$.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie, AB; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GEOFFREY A. KANDALL, Hamden, CT, USA; KEE-WAI LAU, Hong Kong, China; VEDULA N. MURTY, Dover, PA, USA; JOEL SCHLOSBERG, Bayside, NY, USA; BOB SERKEY, Leonia, NJ, USA; TAICHI MAEKAWA, Takatsuki City, Osaka, Japan; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comăneşti, Romania; and the proposer.

3105. [2006 : 45, 48] *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

Let a, b, c, d be positive real numbers.

(a) Prove that the following inequality holds for $0 \le x \le \left(5-\sqrt{17}\right)/2$ and also for $x=1$:

$$\sum_{\text{cyclic}} \frac{a}{a+(3-x)b+xc} \;\ge\; 1 .$$

(b)★ Prove the above inequality for $0 \le x \le 1$.

Solution to (a) *by the proposer, expanded by the editor.*

Let $y = 3-x$. Then, by straightforward computations, we find that

$$\frac{a}{a+by+cx}+\frac{c}{c+dy+ax} \;=\; \frac{A}{A+B} ,$$

where A and B are defined as follows:

$$\begin{aligned} A &= a(c+dy+ax)+c(a+by+cx) \\ &= (a^2+c^2)x+(ad+bc)y+2ac\,, \\ B &= (a+by+cx)(c+dy+ax)-A \\ &= (ab+cd)xy-ac(1-x^2)+bdy^2\,. \end{aligned}$$

Similarly, $\dfrac{b}{b+cy+dx}+\dfrac{d}{d+ay+bx}=\dfrac{C}{C+D}$, where

$$\begin{aligned} C &= (b^2+d^2)x+(ab+cd)y+2bd \\ \text{and}\quad D &= (ad+bc)xy-bd(1-x^2)+acy^2\,. \end{aligned}$$

Thus, the given inequality can be written as

$$\frac{A}{A+B}+\frac{C}{C+D} \;\geq\; 1\,,$$

which is equivalent to

$$AC \;\geq\; BD\,. \tag{1}$$

Let $E=2ac(x+1)+(ad+bc)y$ and $F=2bd(x+1)+(ab+cd)y$. Since

$$\begin{aligned} A-E &= (a^2+c^2-2ac)x \;=\; (a-c)^2x \;\geq\; 0 \\ \text{and}\quad C-F &= (b^2+d^2-2bd)y \;=\; (b-d)^2y \;\geq\; 0\,, \end{aligned}$$

we have $A\geq E$ and $C\geq F$. Hence, to prove (1), it suffices to prove that

$$EF \;\geq\; BD\,. \tag{2}$$

Making the substitutions $p=ac$, $q=bd$, $r=ad+bc$, and $s=ab+cd$, we have $E=2p(x+1)+ry$, $F=2q(x+1)+sy$, $B=sxy-p(1-x^2)+qy^2$, and $D=rxy-q(1-x^2)+py^2$. Now

$$\begin{aligned} EF &= 4pq(x+1)^2+rsy^2+2(ps+qr)(x+1)y \\ \text{and}\quad BD &= rsx^2y^2+pq(1-x^2)^2+pqy^4-(pr+qs)(1-x^2)xy \\ &\quad +(ps+qr)xy^3-(p^2+q^2)(1-x^2)y^2\,. \end{aligned}$$

By direct but tedious manipulations using $y=3-x$, we get

$$\begin{aligned} 2(x+1)y-xy^3 &= y(2(x+1)-x(3-x)^2) \\ &= y(2-7x+6x^2-x^3) \;=\; (1-x)(2-5x+x^2)y \end{aligned}$$

and

$$\begin{aligned} &(1-x^2)^2+y^4-4(x+1)^2 \\ &\quad= y^4+(x+1)^2((1-x)^2-4) \;=\; y^4+(x+1)^2(x^2-2x-3) \\ &\quad= y^4+(x+1)^2(x+1)(x-3) \;=\; ((3-x)^3-(x+1)^3)y \\ &\quad= (26-30x+6x^2-2x^3)y \;=\; 2(13-15x+3x^2-x^3)y \\ &\quad= (1-x)(13-2x+x^2)y\,. \end{aligned}$$

Hence, (2) is equivalent to

$$\begin{aligned}(p^2+q^2+rs)(1-x^2)y^2 &+ (pr+qs)(1-x^2)xy \\ &+ (ps+qr)(1-x)(2-5x+x^2)y \\ &\geq 2pq(1-x)(13-2x+x^2)y\,. \qquad (3)\end{aligned}$$

If $x=1$, equality holds in (3). If $x \neq 1$, then dividing by $(1-x)y$ gives

$$\begin{aligned}&(p^2+q^2+rs)(1+x)y+(pr+qs)(1+x)x \\ &+(ps+qr)(2-5x+x^2) \;\geq\; 2pq(1-x)(13-2x+x^2)\,. \qquad (4)\end{aligned}$$

Since $0 \leq x \leq \frac{1}{2}(5-\sqrt{17})$, we obtain $2-5x+x^2 \geq 0$. We also have $r=ad+bc \geq 2\sqrt{pq}$ and $s=ab+cd \geq \sqrt{pq}$. Thus,

$$\begin{aligned}p^2+q^2+rs &\geq 2pq+4pq = 6pq\,,\\ pr+qs &\geq 2(p+q)\sqrt{pq} \geq 4pq\,,\\ \text{and}\quad ps+qr &\geq 2(p+q)\sqrt{pq} \geq 4pq\,.\end{aligned}$$

Consequently,

$$\begin{aligned}(p^2+q^2+rs)(1+x)y &+ (pr+qs)(1+x)x + (ps+qr)(2-5x+x^2)\\ &\geq 6pq(1+x)(3-x)+4pq(1+x)x+4pq(2-5x+x^2)\\ &= 2pq(9+6x-3x^2+2x+2x^2+4-10x+2x^2)\\ &= 2pq(13-2x+x^2)\,,\end{aligned}$$

establishing (4).

It follows that (3), and hence (2), hold, completing the proof.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria (part (a) only).

No solution to part (b)★ was received. Thus, it remains open. The proposer showed that, in fact, the inequality $EF \geq BD$ is true only if $x=1$ or $0 \leq x \leq \frac{1}{2}(5-\sqrt{17})$, but commented that to show $AC \geq BD$ for $0 \leq x \leq 1$ is difficult even in the case $d=0$. Using a computer, he has verified the truth of the given inequality when $\frac{1}{2}(5-\sqrt{17}) < x < 1$ for many millions of cases in which a, b, c, d are positive. Janous proposed several related problems.

3106. [2006 : 45, 48] *Proposed by Mihály Bencze, Brasov, Romania.*

Prove the following identities:

(a) $$\sum_{k=1}^{n}\sum_{i=1}^{k+1}\frac{\binom{k}{i-1}^2\binom{2k}{k}}{2^{2k}\binom{2k}{2i-2}(2i-1)} = \frac{2^{2n+1}}{\binom{2n+1}{n}}-2.$$

(b) $$\sum_{k=1}^{n}\sum_{i=1}^{k+1}\frac{\binom{k}{i-1}^2\binom{2k}{k}}{2^{2k}\binom{2k}{2i-2}i} = \frac{(2n+3)\binom{2n+2}{n+1}}{2^{2n+1}}-3.$$

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

(a) For $|t| < 1$, define

$$f(t) = \frac{\arcsin t}{\sqrt{1-t^2}} \qquad \text{and} \qquad g(t) = \sum_{k=0}^{\infty} \frac{2^{2k}}{(2k+1)\binom{2k}{k}} t^{2k+1}.$$

It is easy to verify that both $f(t)$ and $g(t)$ satisfy the following initial value problem: $(1-t^2)\frac{dy}{dt} - ty = 1$ and $y(0) = 0$. Hence, $f(t) = g(t)$.

Using the binomial series $\frac{1}{\sqrt{1-4x^2}} = \sum_{j=0}^{\infty} \binom{2j}{j} x^{2j}$, we get

$$\frac{1}{2}\arcsin(2x) = \int_0^x \frac{dz}{\sqrt{1-4z^2}} = \sum_{j=0}^{\infty} \frac{1}{2j+1}\binom{2j}{j} x^{2j+1},$$

and hence,

$$\begin{aligned} \frac{1}{2} f(2x) &= \left(\sum_{j=0}^{\infty} \binom{2j}{j} x^{2j}\right)\left(\sum_{j=0}^{\infty} \frac{1}{2j+1}\binom{2j}{j} x^{2j+1}\right) \\ &= \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} \frac{1}{2j+1}\binom{2j}{j}\binom{2k-2j}{k-j}\right) x^{2k+1}. \end{aligned} \tag{1}$$

On the other hand,

$$\frac{1}{2} g(2x) = \sum_{k=0}^{\infty} \frac{2^{4k}}{(2k+1)\binom{2k}{k}} x^{2k+1}. \tag{2}$$

Since $\frac{1}{2}f(2x) = \frac{1}{2}g(2x)$, the power series in (1) and (2) must be identical. Comparing coefficients, we find that

$$\begin{aligned} \sum_{j=0}^{k} \frac{1}{2^{2k}(2j+1)}\binom{2j}{j}\binom{2k-2j}{k-j} &= \frac{2^{2k}}{(2k+1)\binom{2k}{k}} \\ &= \frac{2^{2k+1}}{\binom{2k+1}{k}} - \frac{2^{2k-1}}{\binom{2k-1}{k-1}}. \end{aligned}$$

By telescoping, we obtain

$$\begin{aligned} \sum_{k=1}^{n}\sum_{j=0}^{k} \frac{1}{2^{2k}(2j+1)}\binom{2j}{j}\binom{2k-2j}{k-j} &= \sum_{k=1}^{n}\left(\frac{2^{2k+1}}{\binom{2k+1}{k}} - \frac{2^{2k-1}}{\binom{2k-1}{k-1}}\right) \\ &= \frac{2^{2k+1}}{\binom{2n+1}{n}} - 2, \end{aligned}$$

which is equivalent to the given identity.

(b) By the binomial series $\frac{1}{\sqrt{1-4x}} = \sum_{j=0}^{\infty} \binom{2j}{j} x^j$, we get

$$\frac{1-\sqrt{1-4x}}{2x} = \frac{1}{x}\int_0^x \frac{dz}{\sqrt{1-4z}} = \sum_{k=0}^{\infty} \frac{1}{j+1}\binom{2j}{j} x^j .$$

Hence,

$$\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} \frac{1}{j+1}\binom{2j}{j}\binom{2k-2j}{k-j}\right) x^k$$
$$= \frac{1}{2x}\left(\frac{1}{\sqrt{1-4x}}-1\right) = \sum_{j=0}^{\infty} \frac{1}{2}\binom{2j}{j} x^{j-1} = \sum_{k=0}^{\infty}\binom{2k+1}{k} x^k .$$

Comparing coefficients gives

$$\sum_{j=0}^{k} \frac{1}{2^{2k}(j+1)}\binom{2j}{j}\binom{2k-2j}{k-j} = \frac{1}{2^{2k}}\binom{2k+1}{k}$$
$$= \frac{(2k+3)\binom{2k+2}{k+1}}{2^{2k+1}} - \frac{(2k+1)\binom{2k}{k}}{2^{2n-1}} .$$

By telescoping, we conclude that

$$\sum_{k=1}^{n}\sum_{j=0}^{k} \frac{1}{2^{2k}(j+1)}\binom{2j}{j}\binom{2k-2j}{k-j}$$
$$= \sum_{k=1}^{n}\left(\frac{(2k+3)\binom{2k+2}{k+1}}{2^{2k+1}} - \frac{(2k+1)\binom{2k}{k}}{2^{2n-1}}\right)$$
$$= \frac{(2n+3)\binom{2n+2}{n+1}}{2^{2n+1}} - 3 ,$$

which is equivalent to the given identity.

Also solved by MICHEL BATAILLE, Rouen, France; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; LUÍS LOPES, Rio de Janeiro, Brazil; and the proposer.

3107. [2006 : 45, 48] *Proposed by Victor Oxman, Western Galilee College, Israel.*

Let $A_1B_1C_1$ and $A_2B_2C_2$ be two triangles with $A_1C_1 = A_2C_2$. Suppose that the interior angle bisectors A_1D_1 and A_2D_2 are equal.

(a) If the altitudes B_1H_1 and B_2H_2 are equal, show that the triangles are congruent.

(b) If the interior angle bisectors B_1E_1 and B_2E_2 are equal, show that the triangles are congruent.

Solution by Roy Barbara, University of Beirut, Beirut, Lebanon.

In $\triangle ABC$, let a, b, c be the lengths of BC, CA, AB, respectively. Let h be the length of the altitude BH and let ℓ_a and ℓ_b be the lengths of the interior angle bisectors AD and BE, respectively. Also, let x and y be the lengths of the segments CD and BD, respectively, and let 2α be the measure of $\angle BAC$. In this solution, "increasing" means "strictly increasing".

(a) We show that $\triangle ABC$ is uniquely determined (up to a congruence) by b, h, and l_a. We assume b and h to be constant. Using $c = \dfrac{h}{\sin 2\alpha}$ and the well-known formula $\ell_a = \dfrac{2bc}{b+c}\cos\alpha$, we obtain

$$\frac{1}{\ell_a} \;=\; \frac{1}{2bh}\left(2b\sin\alpha + \frac{h}{\cos\alpha}\right)\,.$$

It is easy to see that $1/\ell_a$ is an increasing function of α. Hence, ℓ_a is an injective function of α and therefore, α is uniquely determined by b, h, and ℓ_a. Then $\triangle ABC$ is uniquely determined by b, ℓ_a, and α.

(b) We show that $\triangle ABC$ is uniquely determined (up to a congruence) by b, ℓ_a, and ℓ_b. We assume b and ℓ_a to be constant. Clearly, x is an increasing function of α. From $\ell_a = \dfrac{2bc}{b+c}\cos\alpha$, we see that $c = \dfrac{b\ell_a}{2b\cos\alpha - \ell_a}$ is an increasing function of α. From $\dfrac{y}{x} = \dfrac{c}{b}$, we see that $y = \dfrac{cx}{b}$ is an increasing function of α, and then so is $a = x + y$. Now,

$$\ell_b^2 \;=\; ac\left(1 - \frac{b^2}{(a+c)^2}\right)$$

is an increasing function of α, since each factor is. Thus, ℓ_b is an injective function of α and therefore, α is uniquely determined by b, ℓ_a, and ℓ_b. Then $\triangle ABC$ is uniquely determined by b, ℓ_a, and α.

Also solved by MICHEL BATAILLE, Rouen, France; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; JOEL SCHLOSBERG, Bayside, NY, USA (part (a) only)*; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.*

Janous pointed out that the two assertions in this problem, as well as many other similar ones, with solutions, can be found in [1] *and* [2].

References

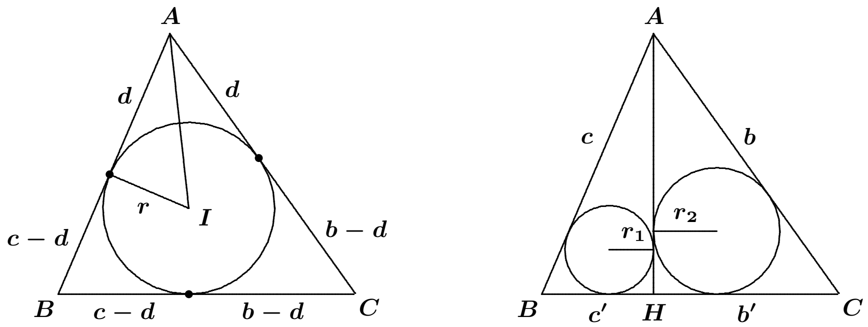
[1] Kurt Herterich, *Die Konstruktion von Dreiecken*, Ernst Klett Verlag, Stuttgart, 1986.

[2] Christo Chitov, *Geometrija na treugulnika*, Narodna Prosveta, Sofia, 1990.

3108. [2006 : 45, 48] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let ABC be a triangle in which angles B and C are both acute. Let H be the point on side BC such that $AH \perp BC$. Let r, r_1, and r_2 be the incircles of triangles ABC, ABH, and AHC, respectively. Show that $r + r_1 + r_2 - AH$ is positive, negative, or zero according as $\angle A$ is obtuse, acute, or right-angled.

Solution by Joe Howard, Portales, NM, USA.



Let d be the length of the tangent from A to the incircle. Since tangents drawn to a circle from an external point have equal length, we must have $a = (c-d) + (b-d)$; whence,

$$d \;=\; \frac{b+c-a}{2}\,.$$

Similarly, for triangles AHB and AHC, we have

$$r_1 \;=\; \frac{c'+h-c}{2}\,, \qquad r_2 \;=\; \frac{b'+h-b}{2}\,,$$

where $h = AH$, $c' = BH$, and $b' = CH$. Consequently,

$$h - r_1 - r_2 \;=\; \frac{b+c-a}{2} \;=\; d\,.$$

Note that $\angle A$ is obtuse, right, or acute according as $d < r$, $d = r$, or $d > r$. [*Editor's comment:* Readers who find a proof by picture unsatisfying should concentrate on either of the two right triangles that share their hypotenuse AI and have legs of length d and r: Fix the circle with radius r to be the incircle of a variable triangle ABC; as A moves away from the circle, d increases from 0 to infinity. Howard supplies his own alternative argument in his final paragraph.] Thus,

- $\angle A$ is obtuse if and only if $d < r$; that is, $h - r_1 - r_2 < r$,
- $\angle A$ is a right angle if and only if $d = r$; that is, $h - r_1 - r_2 = r$,
- $\angle A$ is acute if and only if $d > r$; that is, $h - r_1 - r_2 > r$.

The result follows immediately. My argument was guided by "Triangles with the Right Stuff" from *Quantum*, **8**:6 (1998), pp. 32–33. A related problem is #4322 in *School Science and Mathematics*, **92**:3 (1992), p. 167: Prove that

$$r + r_1 + r_2 \;\le\; h \sin A\,.$$

For an alternative to the proof by picture, we use $\tan\frac{A}{2} = \frac{r}{d}$. Then

$$h - r_1 - r_2 \;=\; d \;=\; r\cot\frac{A}{2}\,,$$

which is less than, equal to, or greater than r according as $\angle A$ is obtuse, right, or acute.

Also solved by MICHEL BATAILLE, Rouen, France; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GEOFFREY A. KANDALL, Hamden, CT, USA; TAICHI MAEKAWA, Takatsuki City, Osaka, Japan; VEDULA N. MURTY, Dover, PA, USA; JOEL SCHLOSBERG, Bayside, NY, USA; D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comăneşti, Romania; and the proposer.

3109. [2006 : 46, 48] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let ABC be a triangle in which angles B and C are both acute, and let a, b, c be the lengths of the sides opposite the vertices A, B, C, respectively. If h_a is the altitude from A to BC, prove that $\frac{1}{h_a^2} - \left(\frac{1}{b^2}+\frac{1}{c^2}\right)$ is positive, negative, or zero according as $\angle A$ is obtuse, acute, or right-angled.

Solution by Chip Curtis, Missouri Southern State University, Joplin, MO, USA.

Let $S = \frac{1}{h_a^2} - \left(\frac{1}{b^2}+\frac{1}{c^2}\right)$. Since $h_a = c\sin B$, we have

$$\begin{aligned}
S &= \frac{1}{c^2\sin^2 B} - \left(\frac{1}{b^2}+\frac{1}{c^2}\right) = \frac{1}{c^2\sin^2 B} - \left(\frac{\sin^2 C}{c^2\sin^2 B}+\frac{1}{c^2}\right) \\
&= \frac{1}{c^2\sin^2 B}\left(1-\sin^2 C-\sin^2 B\right) = \frac{1}{c^2\sin^2 B}\left(\cos^2 C-\sin^2 B\right) \\
&= \frac{1}{2c^2\sin^2 B}\left([1+\cos(2C)]-[1-\cos(2B)]\right) \\
&= \frac{1}{2c^2\sin^2 B}\left[\cos(2B)+\cos(2C)\right] \\
&= \frac{1}{c^2\sin^2 B}\cos(B+C)\cos(B-C) \\
&= -\frac{1}{c^2\sin^2 B}\cos A\cdot\cos(B-C)\,.
\end{aligned}$$

Since B and C are both acute, we have $-\pi/2 < B-C < \pi/2$; whence, $\cos(B-C) > 0$. Thus, the sign of S is the opposite of the sign of $\cos A$, and therefore, S is positive, negative, or zero according as $\angle A$ is obtuse, acute, or right.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, University of Beirut, Beirut, Lebanon; MICHEL BATAILLE, Rouen, France; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VEDULA N. MURTY, Dover, PA, USA; D.J. SMEENK, Zaltbommel, the Netherlands; GEORGE TSAPAKIDIS, Agrinio, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comăneşti, Romania; and the proposer.

3110. [2006 : 46, 49] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let m_b be the length of the median to side b in $\triangle ABC$, and define m_c similarly. Prove that $4a^4 + 9b^2c^2 - 16m_b^2m_c^2$ is positive, negative, or zero according as angle A is acute, obtuse, or right-angled.

Solution by Joel Schlosberg, Bayside, NY, USA.

Using the well-known formulas $m_b = \frac{1}{2}\sqrt{2a^2 - b^2 + 2c^2}$ and $m_c = \frac{1}{2}\sqrt{2a^2 + 2b^2 - c^2}$, we have

$$\begin{aligned}
&4a^4 + 9b^2c^2 - 16m_b^2m_c^2 \\
&\quad= 4a^4 + 9b^2c^2 - (2a^2 - b^2 + 2c^2)(2a^2 + 2b^2 - c^2) \\
&\quad= 4a^4 + 9b^2c^2 - 4a^4 - 2a^2\big[(2b^2 - c^2) + (2c^2 - b^2)\big] \\
&\qquad\qquad + (b^2 - 2c^2)(2b^2 - c^2) \\
&\quad= -2a^2(b^2 + c^2) + 9b^2c^2 + (2b^4 + 2c^4 - 5b^2c^2) \\
&\quad= -2a^2(b^2 + c^2) + 2(b^2 + c^2)^2 \\
&\quad= 2(b^2 + c^2)(b^2 + c^2 - a^2)\,.
\end{aligned}$$

Clearly, $2(b^2 + c^2)$ is positive, so that $4a^4 + 9b^2c^2 - 16m_b^2m_c^2$ has the same sign as $b^2 + c^2 - a^2$, which is positive, negative, or zero if $\angle BAC$ is acute, obtuse, or right, respectively.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, University of Beirut, Beirut, Lebanon; MICHEL BATAILLE, Rouen, France; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GEOFFREY A. KANDALL, Hamden, CT, USA; TAICHI MAEKAWA, Takatsuki City, Osaka, Japan; VEDULA N. MURTY, Dover, PA, USA; D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

3111. [2006 : 46, 49] *Proposed by Mihály Bencze, Brasov, Romania.*

Let a_k, b_k, and c_k be the lengths of the sides opposite the vertices A_k, B_k, and C_k, respectively, in triangle $A_kB_kC_k$, for $k = 1, 2, \ldots, n$. If r_k is the inradius of triangle $A_kB_kC_k$ and if R_k is its circumradius, prove that

$$\begin{aligned}
6\sqrt{3}\left(\prod_{k=1}^{n} r_k\right)^{\frac{1}{n}} &\le \left(\prod_{k=1}^{n} a_k\right)^{\frac{1}{n}} + \left(\prod_{k=1}^{n} b_k\right)^{\frac{1}{n}} + \left(\prod_{k=1}^{n} c_k\right)^{\frac{1}{n}} \\
&\le 3\sqrt{3}\left(\prod_{k=1}^{n} R_k\right)^{\frac{1}{n}}.
\end{aligned}$$

Solution by Michel Bataille, Rouen, France.

The right inequality may be written equivalently as

$$\left(\prod_{k=1}^{n} \frac{a_k}{R_k}\right)^{\frac{1}{n}} + \left(\prod_{k=1}^{n} \frac{b_k}{R_k}\right)^{\frac{1}{n}} + \left(\prod_{k=1}^{n} \frac{c_k}{R_k}\right)^{\frac{1}{n}} \leq 3\sqrt{3}\,. \qquad (1)$$

Using the Law of Sines and then the AM–GM Inequality, we find that the left side of (1) is equal to

$$\begin{aligned}
&\left(\prod_{k=1}^{n} 2\sin A_k\right)^{\frac{1}{n}} + \left(\prod_{k=1}^{n} 2\sin B_k\right)^{\frac{1}{n}} + \left(\prod_{k=1}^{n} 2\sin C_k\right)^{\frac{1}{n}} \\
&\qquad \leq\ 2\left(\frac{1}{n}\sum_{k=1}^{n}\sin A_k + \frac{1}{n}\sum_{k=1}^{n}\sin B_k + \frac{1}{n}\sum_{k=1}^{n}\sin C_k\right) \\
&\qquad =\ \frac{2}{n}\sum_{k=1}^{n}(\sin A_k + \sin B_k + \sin C_k)\,.
\end{aligned}$$

Now we use the well-known inequality $\sin A + \sin B + \sin C \leq 3\sqrt{3}/2$, which is valid for an arbitrary triangle with angles A, B, and C. [*Ed.:* For the convenience of any readers who are not familiar with this result, we supply a proof: since the sine function is concave on the interval $[0,\pi]$, we have $\sin A + \sin B + \sin C \leq 3\sin\left(\frac{1}{3}(A+B+C)\right) = 3\sin(\pi/3) = 3\sqrt{3}/2$.] Thus,

$$\left(\prod_{k=1}^{n} \frac{a_k}{R_k}\right)^{\frac{1}{n}} + \left(\prod_{k=1}^{n} \frac{b_k}{R_k}\right)^{\frac{1}{n}} + \left(\prod_{k=1}^{n} \frac{c_k}{R_k}\right)^{\frac{1}{n}} \leq \frac{2}{n}\left(n\cdot\frac{3\sqrt{3}}{2}\right) = 3\sqrt{3}\,.$$

Now consider the left inequality. Using the AM–GM Inequality, we get

$$\left(\prod_{k=1}^{n} a_k\right)^{\frac{1}{n}} + \left(\prod_{k=1}^{n} b_k\right)^{\frac{1}{n}} + \left(\prod_{k=1}^{n} c_k\right)^{\frac{1}{n}} \geq 3\left(\prod_{k=1}^{n} a_k b_k c_k\right)^{\frac{1}{3n}}. \qquad (2)$$

We will make use of some more standard results about triangles. Let s be the semiperimeter. Then the area of the triangle is given by both rs and $\sqrt{s(s-a)(s-b)(s-c)}$ (Heron's Formula). Therefore,

$$\begin{aligned}
r &= \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} \\
&\leq \sqrt{\frac{1}{s}\left(\frac{(s-a)+(s-b)+(s-c)}{3}\right)^3} = \sqrt{\frac{1}{s}\left(\frac{s}{3}\right)^3} = \frac{s}{3\sqrt{3}}\,.
\end{aligned}$$

(Here we have used the AM–GM Inequality again.) Thus $s \geq 3\sqrt{3}r$. We also have $abc = 4Rrs$ and $R \geq 2r$ (Euler's Inequality). Combining all these results, we obtain $abc \geq 4(2r)r(3\sqrt{3}r) = (2\sqrt{3}r)^3$, which we use in (2):

$$\left(\prod_{k=1}^{n} a_k\right)^{\frac{1}{n}} + \left(\prod_{k=1}^{n} b_k\right)^{\frac{1}{n}} + \left(\prod_{k=1}^{n} c_k\right)^{\frac{1}{n}} \geq 3\left(\prod_{k=1}^{n} (2\sqrt{3}r_k)^3\right)^{\frac{1}{3n}} = 6\sqrt{3}\left(\prod_{k=1}^{n} r_k\right)^{\frac{1}{n}}.$$

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

3112★. [2006 : 46, 49] *Proposed by Mohammed Aassila, Strasbourg, France.*

Let $MABC$ be a tetrahedron, and let M' be any point in the interior of $\triangle ABC$. Denote the area of $\triangle XYZ$ by $[XYZ]$. Prove that

$$\begin{aligned}(MM')^2 &= MA^2\,\frac{[BM'C]}{[ABC]} + MB^2\,\frac{[CM'A]}{[ABC]} + MC^2\,\frac{[AM'B]}{[ABC]} \\ &\quad - \left(AB^2\,\frac{[BM'C][CM'A]}{[ABC]^2} + BC^2\,\frac{[CM'A][AM'B]}{[ABC]^2}\right. \\ &\quad \left. + \; CA^2\,\frac{[AM'B][BM'C]}{[ABC]^2}\right).\end{aligned}$$

Comment: This result for a tetrahedron is "similar" to Stewart's Theorem for a triangle. If $M' = G$, the centroid of $\triangle ABC$, then the relation becomes

$$MG^2 = \frac{1}{3}(MA^2 + MB^2 + MC^2) - \frac{1}{9}(AB^2 + BC^2 + CA^2),$$

which is well known.

Essentially the same solution by Michel Bataille, Rouen, France; and Li Zhou, Polk Community College, Winter Haven, FL, USA.

Since M' is in the interior of $\triangle ABC$, we have $M' = \alpha A + \beta B + \gamma C$ for some positive real numbers α, β, γ such that $\alpha + \beta + \gamma = 1$. Actually (α, β, γ) are the areal coordinates of M' relative to (A, B, C) and are given by

$$\alpha = \frac{[BM'C]}{[ABC]}, \qquad \beta = \frac{[CM'A]}{[ABC]}, \qquad \gamma = \frac{[AM'B]}{[ABC]}. \tag{1}$$

Then $\overrightarrow{MM'} = \alpha\overrightarrow{MA} + \beta\overrightarrow{MB} + \gamma\overrightarrow{MC}$, and we have

$$\begin{aligned}
(MM')^2 &= \overrightarrow{MM'} \cdot \overrightarrow{MM'} \\
&= \alpha^2 MA^2 + \beta^2 MB^2 + \gamma^2 MC^2 \\
&\quad + 2\alpha\beta\overrightarrow{MA} \cdot \overrightarrow{MB} + 2\beta\gamma\overrightarrow{MB} \cdot \overrightarrow{MC} + 2\gamma\alpha\overrightarrow{MC} \cdot \overrightarrow{MA} \\
&= \alpha^2 MA^2 + \beta^2 MB^2 + \gamma^2 MC^2 \\
&\quad + \alpha\beta(MA^2 + MB^2 - AB^2) \\
&\quad + \beta\gamma(MB^2 + MC^2 - BC^2) \\
&\quad + \gamma\alpha(MC^2 + MA^2 - CA^2) \\
&= (\alpha + \beta + \gamma)(\alpha MA^2 + \beta MB^2 + \gamma MC^2) \\
&\quad - \alpha\beta AB^2 - \beta\gamma BC^2 - \gamma\alpha CA^2 \,.
\end{aligned}$$

Since $\alpha + \beta + \gamma = 1$, we conclude that

$$(MM')^2 \;=\; \alpha MA^2 + \beta MB^2 + \gamma MC^2 - (\alpha\beta AB^2 + \beta\gamma BC^2 + \gamma\alpha CA^2)\,,$$

and the proof is complete by (1) above.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and JOEL SCHLOSBERG, Bayside, NY, USA.

3113. [2006 : 47, 49; 171, 174] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

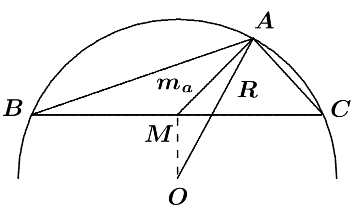
Let ABC be a triangle and let a be the length of the side opposite the vertex A. If m_a is the length of the median from A to BC, and if R is the circumradius of $\triangle ABC$, prove that $m_a - R$ is positive, negative, or zero, according as $\angle A$ is obtuse, acute, or right-angled.

Combination of similar solutions by Roy Barbara, University of Beirut, Beirut, Lebanon; and Richard I. Hess, Rancho Palos Verdes, CA, USA.

We let M be the mid-point of BC and consider the triangle OMA with sides $AM = m_a$ and $AO = R$. According to Euclid, the relative sizes of these two sides depends on the size of the opposite angles.

Case 1. A is obtuse.

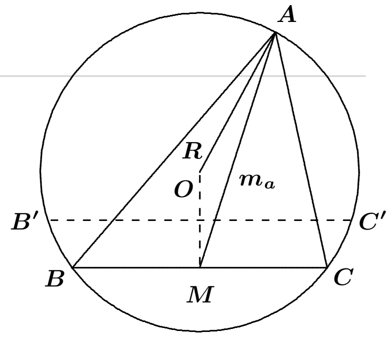
Vertex A (on the circumcircle) is separated from the circumcentre O by the chord BC. Since $OM \perp BC$, $\angle OMA$ is obtuse; whence, the opposite side R is longer than the adjacent side m_a; that is, $m_a - R < 0$ when A is obtuse, as claimed.



Case 2. $A = 90^\circ$.

Here BC is a diameter; thus, $m_a = R$, and $m_a - R = 0$.

Case 3. A is acute.



The proposal is incorrect: $m_a - R$ can be positive, zero, or negative when A is acute, as follows. Let $B'C'$ be the perpendicular bisector of OM. For A on the long arc of the circumcircle between B' and C', we have $\angle MOA > \angle OMA$; whence $m_a - R > 0$. For A at B' or at C', we get $m_a - R = 0$. Finally, when A lies on either short arc $B'B$ or $C'C$, we see that $m_a - R < 0$. Note that the proposal becomes correct for triangles ABC in which *all* angles are acute; then A will necessarily lie on the arc $B'C'$ which, as we have just seen, forces $m_a - R > 0$, as claimed in the proposal.

*Also solved by MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GEOFFREY A. KANDALL, Hamden, CT, USA; *VEDULA N. MURTY, Dover, PA, USA; *PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comănești, Romania (with two proofs for the obtuse-angle case); and the *proposer. The asterisk designates solutions that were correct, but whose analysis of the acute-angle case was incomplete. In addition VÁCLAV KONEČNÝ, Big Rapids, MI, USA provided a counterexample showing that the conclusion to the corrected proposal still was flawed. There were three incorrect submissions.*