Problem of the Month

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This month, a variation on a classic problem:

**Problem** (2005 Small c Contest) In the road-map shown in the diagram, each line segment represents a street which can only be travelled along in either the rightwards or upwards direction. How many paths are there from point A to point B?

This is an neat twist on a classic problem whose text is identical, but whose diagram is a complete grid, with no "hole" in it:

To solve this old chestnut, the insight is to notice that every possible path from A to B consists of exactly 6 moves to the right and 4 moves up. Also, each possible ordering of 6 moves to the right and 4 moves up gives a different path. If we use "R" to denote a move to the right and "U" to denote a move up, then possible paths include URUURRRRR, RURURURURRR, and so on. Thus, we have transformed the problem of counting paths into one of arranging letters (6 Rs and 4 Us).

How many arrangements of 6 Rs and 4 Us are there? Since we are arranging 10 objects, 6 of one type which are identical and 4 of another type which are identical, there are \( \frac{10!}{6!4!} = \frac{10 	imes 9 	imes 8}{4 	imes 3 	imes 2 	imes 1} = 210 \) arrangements, and hence, 210 possible paths in this classic problem. (Alternatively, we could think of choosing 6 of the 10 places in which to put the Rs, leaving the remaining 4 places for the Us; there are \( \binom{10}{6} = 210 \) such ways.)

How does this help with this month's Problem? At the very least, this gives us an upper bound for the number of paths in the problem that we want to solve—there certainly cannot be any more paths with some streets in the middle removed. It also gives us an idea of how to start. Here is a first approach that models the idea above.
**Solution 1:** It is difficult to get a handle on how to count these paths until we break up the paths into four categories. Consider the intersections in the diagram labelled \( W, X, Y, \) and \( Z. \)

Any path from \( A \) to \( B \) must pass through EXACTLY one of the four points \( W, X, Y, \) or \( Z. \) (You may need to stare at the diagram for a little while to convince yourself of this.) Why does this help? It helps because we can break up the paths from \( A \) to \( B \) into four disjoint categories—those from \( A \) to \( B \) passing through each of \( W, X, Y, \) and \( Z. \) We count the paths in this way by counting the number of paths from \( A \) to the particular intermediate point, and then from that point to \( B. \)

To get from \( A \) to \( W, \) we make 1 move to the right and 4 moves up. There are \( \binom{5}{1} = 5 \) such paths. To get from \( W \) to \( B, \) we make 5 moves to the right. There is only 1 such path. Thus, there are \( 5 \times 1 = 5 \) paths from \( A \) to \( B \) through \( W. \)

To get from \( A \) to \( X, \) we make 2 moves to the right and 3 moves up. There are \( \binom{5}{2} = 10 \) such paths. To get from \( X \) to \( B, \) we make 4 moves to the right and 1 move up. There are \( \binom{5}{4} = 5 \) such paths. Thus, there are \( 10 \times 5 = 50 \) paths from \( A \) to \( B \) through \( X. \)

To get from \( A \) to \( Y, \) we make 5 moves to the right and 1 move up. There are \( \binom{6}{5} = 6 \) such paths. To get from \( Y \) to \( B, \) we make 1 move to the right and 3 moves up. There are \( \binom{4}{1} = 4 \) such paths. Thus, there are \( 6 \times 4 = 24 \) paths from \( A \) to \( B \) through \( Y. \)

To get from \( A \) to \( Z, \) we make 6 moves to the right. There is only 1 such path. To get from \( Z \) to \( B, \) we make 4 moves up. There is only 1 such path. Thus, there is \( 1 \times 1 = 1 \) path from \( A \) to \( B \) through \( Z. \)

Therefore, in total there are \( 5 + 50 + 24 + 1 = 80 \) paths from \( A \) to \( B. \)

That's a pretty insightful method. Taking a problem that seems difficult to get a handle on and dividing it up into separate problems that are relatively easy to deal with is always a good idea—figuring out how to divide it up was the real trick here.
Have I led you far enough down the, er, path? Unfortunately, this relation to the "classic" problem and its method of solution may have blinded us to an easier solution.

**Solution 2:** If we choose any intersection, how many paths are there to get to it? Well, to get to any intersection, we must come from directly to the left or directly below. Thus, the total number of paths to any intersection is the sum of the number of paths to the intersection immediately to the left and the number of paths immediately below.

As an example, let's consider the bottom left corner. To get to the intersection directly above $A$, there is only one path. To get to the intersection directly to the right of $A$, there is only one path. To get to the intersection 1 move up and 1 move right of $A$, there are 2 paths—1 each through each of the two previously mentioned intersections.

We can follow this line of reasoning to fill in the number of paths to any intersection on the grid (using a slightly enlarged version of the grid so that we can fit the numbers on it!).

![Grid with numbers](image)

Hence, there are 80 paths from $A$ to $B$.

Now that was much easier. And it's much easier to generalize—if we had a much bigger grid with a whole bunch of "holes" in the middle, we could generalize this very nicely. This is something that is always worth thinking about when solving a problem—will our method generalize to more complicated situations?