Mayhem Solutions

M213. Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.

Set $S = (2 + 1)(2^2 + 1)(2^4 + 1)(2^8 + 1) \cdots (2^{1024} + 1) + 1$. Evaluate $S^{\text{m}}$ without using a calculator.

Solution by John DeLeon, Angelo State University, San Angelo, TX.

Multiply $S$ by $(2 - 1).$ Then

\[ S = (2 - 1)(2 + 1)(2^2 + 1)(2^4 + 1) \cdots (2^{1024} + 1) + 1 \]
\[ = (2^2 - 1)(2^2 + 1)(2^4 + 1) \cdots (2^{1024} + 1) + 1 \]
\[ = (2^4 - 1)(2^4 + 1) \cdots (2^{1024} + 1) + 1 \]
\[ \vdots \]
\[ = (2^{1024} - 1)(2^{1024} + 1) + 1 \]
\[ = (2^{2048} - 1) + 1 = 2^{2048}. \]

Therefore, $S^{\text{m}} = 2^{\frac{2048}{2}} = 2^2 = 4.$

Also solved by JAMES T. BRUENING, Southeast Missouri State University, Cape Girardeau, MO, USA; ALPER CAY, Uzman Private School, Kayseri, Turkey; JOSE LUIS DIAZ-BARRERO, Universitat Politècnica de Catalunya, Barcelona, Spain; SAMUEL GÓMEZ MORENO, Universidad de Jaén, Jaén, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; GUSTAVO KRIMKER, Universidad CAECE, Buenos Aires, Argentina; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; and VEDULA N. MURTY, Dover, PA, USA.

M214. Proposed by Babis Stergiou, Chalkida, Greece.

Two equilateral triangles $ABC$ and $CDE$ are on the same side of line $BCD$. If $BE$ intersects $AC$ at $K$ and $DA$ intersects $CE$ at $L$, prove that $KL$ is parallel to $BD$.

Solution by Missouri State University Problem Solving Group, Springfield, MO, USA.

Since $\angle BCA = \angle ECD = 60^\circ$, we see that $\angle ACE = 60^\circ$, from which it follows that $\angle BCE = \angle ACD$. Since $AC = BC$ and $CD = CE$, we see that $\triangle ACD$ and $\triangle BCE$ are congruent (SAS). Thus, $\angle CAD = \angle CBE$, and we see that $\triangle ACL$ is congruent to $\triangle BCK$ (AAS). From this, we have $CK = CL$. Therefore, $\triangle CKL$ is isosceles. Since $\angle ACE = 60^\circ$, we see that $\angle CKL = \angle CLK = 60^\circ = \angle BCA$. Hence, $KL$ is parallel to $BD$.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; and GUSTAVO KRIMKER, Universidad CAECE, Buenos Aires, Argentina. There was one incorrect solution received.
M215. Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John’s, NL.

Find a rational number \( s \) such that \( s^2 + 5 \) and \( s^2 - 5 \) are both squares of rational numbers.

Solution by Samuel Gómez Moreno, Universidad de Jaén, Jaén, Spain.

Let \( p_1 \) and \( p_2 \) be positive rational numbers such that \( p_1^2 = s^2 + 5 \) and \( p_2^2 = s^2 - 5 \). Then \((p_1 + p_2)(p_1 - p_2) = p_1^2 - p_2^2 = (s^2 + 5) - (s^2 - 5) = 10\). If we set \( a = p_1 + p_2 \) and \( b = p_1 - p_2 = 10/a \).

The solution to this linear system of equations is \( p_1 = (a^2 + 10)/2a \) and \( p_2 = (a^2 - 10)/2a \). We may then express \( s^2 \) in terms of \( a \):

\[
    s^2 = p_1^2 - 5 = \left(\frac{a^2 + 10}{2a}\right) - 5 = \frac{100 + a^4}{4a^2}.
\]

This implies that \( 100 + a^4 = (2as)^2 \) is the square of a rational number.

Clearly, \( 100 + a^4 > (a^2)^2 \). Hence, we may write \( 100 + a^4 = (a^2 + b)^2 \) with \( b > 0 \). Then \( 100 = 2a^2b + b^2 \), which yields \( a = \sqrt{(100 - b^2)/(2b)} \).

By inspection, testing over the natural numbers 1, 2, …, 9, we find that for \( b = 8 \) we get \( a = 3/2 \); whence, \( s = 41/12 \).

Also solved by JAMES T. BRUENING, Southeast Missouri State University, Cape Girardeau, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and VEDULA N. MURTY, Dover, PA, USA.

Bruening attached a note regarding this problem. It was solved by Fibonacci as part of a mathematical tournament at the court of Frederick II. Not only can it be found in books (for example, Eves’ An Introduction to the History of Mathematics, pp. 263, 284), but a version of it also appeared recently (March 2005) as problem #771 in the Problems Section of the College Mathematics Journal (CMJ), of which Bruening is a co-editor. According to the solution of that CMJ problem, there are an infinite number of solutions to this problem.

M216. Proposed by K.R.S. Sastry, Bangalore, India.

A Heron triangle has integer sides and area. Two sides of a Heron triangle are 442 and 649. If its area is 132396, find its perimeter.

Solution by Vedula N. Murty, Dover, PA, USA.

Let \( A \) denote the angle between the two sides whose lengths are 442 and 669. Then \( \frac{1}{2}(442)(649) \sin A = 132396 \). This gives \( \sin A = 12/13 \). Hence, \( \cos A = \pm5/13 \). Let \( a \) denote the length of the unknown side. Then we have \( a^2 = (442)^2 + (649)^2 - 2(442)(649) \cos A \). Substituting \( 5/13 \) for \( \cos A \), we get a non-integer value for \( a \), which is not allowed if the triangle is a Heron triangle. Using \( \cos A = -5/13 \), we obtain \( a = 915 \). Thus, the perimeter is equal to \( 442 + 669 + 915 = 2006 \).

Also solved by JAMES T. BRUENING, Southeast Missouri State University, Cape Girardeau, MO, USA; ALPER CAY, Uzman Private School, Kayseri, Turkey; ESTHER MARIA GARCIA-CABALLERO, Universidad de Jaén, Jaén, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA.
\textbf{M217. Proposed by Bill Sands, University of Calgary, Calgary, AB.}

Let \( a, b, c \) be integers such that 2005 divides both \( ab + 9b + 81 \) and \( bc + 9c + 81 \). Prove that 2005 also divides \( ca + 9a + 81 \).

\textit{Solution by the Mayhem Staff.}

Let \( w = ca + 9a + 81 \). We want to prove that 2005 divides \( w \). Since 2005 divides \( ab + 9b + 81 \), we have

\[
ab + 9b + 81 = 2005k
\]

for some integer \( k \). The prime divisors of 2005 are 5 and 401, neither of which divides 81. It then follows from (1) that neither 5 nor 401 divides \( b \). On the other hand, both 5 and 401 divide \( bw \), since

\[
bw = abc + 9ab + 81b
\]

which is divisible by 2005 (using the information given in the problem). Therefore, both 5 and 401 must divide \( w \). Thus, 2005 divides \( w \).

There were 2 incomplete solutions received.

\textbf{M218. Proposed by Neven Jurić, Zagreb, Croatia.}

Compute the sum

\[
\sum_{k=1}^{99} \frac{1}{k\sqrt{k+1} + (k+1)\sqrt{k}}.
\]

\textit{Solution by Esther María García-Caballero, Universidad de Jaén, Jaén, Spain.}

By rationalizing the denominators, we obtain a telescoping series:

\[
\sum_{k=1}^{99} \frac{1}{k\sqrt{k+1} + (k+1)\sqrt{k}} = \sum_{k=1}^{99} \frac{1}{(k+1)\sqrt{k} - k\sqrt{k+1}} = \sum_{k=1}^{99} \frac{1}{(k+1)^2k - k^2(k+1)} = \sum_{k=1}^{99} \frac{1}{k(k+1)} = \frac{1}{1} - \frac{1}{\sqrt{100}} = 9.
\]

Also solved by ALPER CAY, Uzman Private School, Kayseri, Turkey; JOSÉ LUIS DÍAZ-BARRERO, Universitat Politècnica de Catalunya, Barcelona, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; GUSTAVO KRIMKER, Universidad CAECE, Buenos Aires, Argentina; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; and VEDULA N. MURTY, Dover, PA, USA.