Hexagons and Inequalities

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The popular author Lewis Carroll is also famous for his problems. Problem 71 in [2] pages 18 and 108–109 states,

In a given Triangle place a Hexagon having its opposite sides equal and parallel, and three of them lying along the sides of the Triangle, and such that its diagonals intersect in a given Point.

The problem has been generalized for the case when the given point is not inside the triangle [5]. We will look at another way to modify the problem, where the main diagonals of our hexagons will each be parallel to one of the sides.

**Problem 1.** Let $M$ be a given point in the plane of triangle $ABC$. Construct the lines $A_1B_2$, $B_1C_2$, and $C_1A_2$ meeting at $M$ such that for $i = 1$ and $i = 2$, $A_i$ lies on $BC$, $B_i$ lies on $CA$, $C_i$ lies on $AB$, and moreover, $A_1B_2 || A_2B_1$, $B_1C_2 || B_2C_1$, $C_1A_2 || C_2A_1$ (as in Figure 1).

![Figure 1](linked-image-url)  
**Figure 1:** Problem 1.

![Figure 2](linked-image-url)  
**Figure 2:** Solution to Problem 1.

**Solution.** Analysis. Let lines $C_2A_1$, $A_2B_1$, and $B_2C_1$ intersect at $X$, $Y$, and $Z$ as in Figure 2, and let $A_0$, $B_0$, and $C_0$ be the mid-points of $A_1A_2$, $B_1B_2$, and $C_1C_2$, respectively. Because the lines joining these mid-points are the mid-lines of trapezoids, we therefore have

$$A_0B_0 || A_1B_2 || A_2B_1, \quad B_0C_0 || B_1C_2 || B_2C_1, \quad C_0A_0 || C_1A_2 || C_2A_1.$$

Thus, triangles $A_1ZB_2$ and $A_0C_0B_0$ have corresponding sides parallel; by Desargues' Theorem, $A_1A_0$, $B_2B_0$, and $ZC_0$ are concurrent. In other words, $C$ lies on $ZC_0$. But, $M$ also lies on $ZC_0$ (since $ZC_2MC_1$ is a parallelogram).
This means that $CM$ passes through $C_0$. Similarly, $AM$ passes through $A_0$ and $BM$ passes through $B_0$.

**Construction.** We are to construct the points $A_i$, $B_i$, and $C_i$. Extend the lines $AM$, $BM$, and $CM$ to intersect the sides of $\triangle ABC$ at $A_0$, $B_0$, and $C_0$, respectively. Next construct the parallel to $A_0C_0$ through $M$, which intersects $BA$ and $BC$ at $C_1$ and $A_2$, respectively. Analogously, draw the parallel through $M$ to $B_0A_0$ (and to $B_0C_0$) to find $A_1$ and $B_2$ (and $B_1$ and $C_2$). Desargues' Theorem then implies that the joins of appropriate pairs of these points form parallel lines as desired.

Although the figures and discussion apply to the case where $M$ is inside the triangle, the construction for other locations of $M$ will be essentially the same.

**Problem 2.** Construct the point $M$ for which the hexagon $A_1A_2B_1B_2C_1C_2$ of Problem 1 is inscribed in a circle.

![Figure 3: Problem 2.](image1)

![Figure 4: Solution to Problem 2.](image2)

**Solution.** *Analysis.* We assume that $A_1A_2B_1B_2C_1C_2$ is cyclic (Figure 3.) From $C_1A_2 || C_2A_1$ we deduce that line segments $C_1C_2 = A_1A_2$. Since the secants from $B$ satisfy $BC_2 = BA_1 \cdot BA_2$, these two equations give us $BC_2 = BA_1$. Analogously, $AC_1 = AB_2$ and $CB_1 = CA_2$. Recalling that $A_0$, $B_0$, and $C_0$ are the mid-points of $A_1A_2$, $B_1B_2$, and $C_1C_2$, respectively, we finally obtain

$$BC_0 = BA_0, \quad AC_0 = AB_0, \quad CB_0 = CA_0.$$  

It is then easy to prove that $A_0$, $B_0$, and $C_0$ are the points where the incircle of $\triangle ABC$ touches the sides.

*Construction.* Construct the incircle of triangle $ABC$ and label the points $A_0$, $B_0$, and $C_0$ where the incircle touches the sides $BC$, $CA$, and $AB$, respectively. (See Figure 4.) Next, draw the lines $AA_0$, $BB_0$, $CC_0$, which intersect at the desired point $M$. The point $M$ is, of course, the **Gergonne point** of $\triangle ABC$ (see [3], page 13).

**Problem 3.** With points $X$, $Y$, and $Z$ defined as in the solution to Problem 1, construct the point $M$ for which the hexagon $AYCXBZ$ is inscribed in a circle.
Solution. Analysis. We suppose that $AYCXBZ$ is cyclic. Then, for the angles, we have (as in Figure 5)

$$
\angle XAB = \angle XYB = \alpha_1, \quad \angle CAX = \angle CZX = \alpha_2,
\angle YBC = \angle YZC = \beta_1, \quad \angle ABY = \angle AXY = \beta_2,
\angle ZCA = \angle ZXA = \gamma_1, \quad \angle BCZ = \angle BYZ = \gamma_2.
$$

Using the given parallel lines, we also have

$$
\angle A_0B_0B = \alpha_1, \quad \angle B_0C_0C = \beta_1, \quad \angle C_0A_0A = \gamma_1,
\angle C_0A_0 = \alpha_2, \quad \angle AA_0B_0 = \beta_2, \quad \angle BB_0C_0 = \gamma_2.
$$

From the equality $\gamma_1 = \angle C_0A_0A = \angle ZCA = \angle C_0CA$, we deduce that the points $A, C, A_0,$ and $C_0$ are concyclic. Thus, $\gamma_2 = \alpha_1$. Similarly, $\beta_1 = \alpha_2$ and $\beta_2 = \gamma_1$. But, we also know that

$$
\angle ABC + \angle BCA + \angle CAB = \alpha_1 + \alpha_2 + \beta_1 + \beta_2 + \gamma_1 + \gamma_2 = 180^\circ.
$$

Then $\alpha_1 + \beta_1 + \beta_2 = \gamma_2 + \alpha_2 + \gamma_1 = 90^\circ$. That is, considering $\triangle AA_0C$, we see that $\angle A_0AC = \alpha_1 + \beta_1 + \beta_2 = 90^\circ$. Consequently, the cevian $AA_0$ is perpendicular to the base $BC$ and similarly, $BB_0 \perp AC$ and $CC_0 \perp AB$. It follows that $M$ is the orthocentre of $\triangle ABC$.

Construction. Draw the altitudes $AA_0$, $BB_0$, and $CC_0$. Their intersection point is the desired point $M$. For the proof that $M$ satisfies all the requirements, simply read the preceding analysis in reverse.
Remarks. We could also ask for our hexagons to be circumscribed about a circle. Other properties of hexagons with sides parallel to diagonals were investigated by S.I. Zetel in Problems 185–187 in [6], Ch. IV, pages 118–120.

We turn now to investigating the areas of triangles associated with the hexagons of Problem 1—hexagons that are inscribed in triangles and have diagonals that intersect in a point and are parallel to the nontriangular sides. Areas will be denoted by square brackets. We are interested in relationships among the following nine areas (as displayed in Figure 6):

\[
\begin{align*}
T_1 &= [MC_1B_2], & T_2 &= [MA_1C_2], & T_3 &= [MB_1A_2], \\
S_1 &= [MA_1A_2], & S_2 &= [MB_1B_2], & S_3 &= [MC_1C_2], \\
P_1 &= [AB_2C_1], & P_2 &= [BC_2A_1], & P_3 &= [CA_2B_1].
\end{align*}
\]

![Figure 6: Definition of \(T_i\), \(S_i\), and \(P_i\).](image1)

**Figure 6:** Definition of \(T_i\), \(S_i\), and \(P_i\).

**Figure 7: Lemma 1.**

**Lemma 1.** \(T_iT_j = S_i^2\), for \(\{i, j, k\} = \{1, 2, 3\}\).

**Proof:** From the assumptions \(A_1C_2 \parallel A_2M\) and \(A_1M \parallel A_2B_1\), we get \(\angle C_2A_1M = \angle A_2MA_1 = \angle MA_2B_1\), whose measure has been denoted by \(\alpha\) in Figure 7. Let \(a = A_1C_2, b = A_1M, c = A_2M, d = A_2B_1\), as in the figure. Then, by finding the areas \(T_2, T_3,\) and \(S_1\) and substituting them into the proposed equation \(T_2T_3 = S_1^2\), we obtain

\[
\frac{1}{2} ab \sin \alpha \cdot \frac{1}{2} cd \sin \alpha = \frac{1}{4} b^2c^2 \sin^2 \alpha \quad \text{if and only if} \quad ad = bc,
\]

which is true because of the similarity of triangles \(A_1C_2M\) and \(A_2MB_1\). Similarly, we get \(T_1T_3 = S_2^2\) and \(T_1T_2 = S_3^2\). \(\blacksquare\)

**Lemma 2.** For \(\{i, j, k\} = \{1, 2, 3\}\), we have \(P_i = \frac{T_i^2}{S_j + S_k - T_i}\).

**Proof:** Because \(A_1C_2 \parallel A_2C_1\), we can let \(h\) be the common height of triangles \(MA_1C_2, MC_1C_2,\) and \(MA_1C_2\). Then, \(S_1 = \frac{1}{2} \cdot h \cdot MA_2, S_3 = \frac{1}{2} \cdot h \cdot C_1M,\) and \(T_2 = \frac{1}{2} \cdot h \cdot BC_2A_1\). Since \(\triangle BC_2A_1\) is similar to \(\triangle BC_1A_2\), we see that

![Figure 8: Lemma 2.](image2)
\[
\frac{[BC_2A_1]}{[BC_1A_2]} = \frac{(C_2A_1)^2}{(C_1A_2)^2} = \left( \frac{C_2A_1}{C_1M + MA_2} \right)^2 = \left( \frac{T_2}{S_1 + S_3} \right)^2.
\]

By the definition of \( P_2 \), we therefore have
\[
\frac{P_2}{P_2 + T_2 + S_1 + S_3} = \left( \frac{T_2}{S_1 + S_3} \right)^2;
\]
whence, \( P_2 = \frac{T_2^2}{S_1 + S_3 - T_2} \), as claimed. \( \blacksquare \)

From these two lemmas we will derive a pair of inequalities.

**Inequality 1.** \( T_1 + T_2 + T_3 \geq S_1 + S_2 + S_3 \).

**Proof:** This is just the AM–GM Inequality combined with Lemma 1:
\[
T_1 + T_2 + T_3 = \frac{1}{2}(T_2 + T_3) + \frac{1}{2}(T_1 + T_3) + \frac{1}{2}(T_1 + T_2)
\]
\[
\geq \sqrt{T_2T_3} + \sqrt{T_1T_3} + \sqrt{T_1T_2} = \sqrt{S_1^2} + \sqrt{S_2^2} + \sqrt{S_3^2}.
\]

**Inequality 2.** \( T_1 + T_2 + T_3 \leq \frac{1}{3}[ABC] \).

**Proof:** We want to show that
\[
3(T_1 + T_2 + T_3) \leq T_1 + T_2 + T_3 + S_1 + S_2 + S_3 + P_1 + P_2 + P_3.
\]
By Lemma 2 this is equivalent to showing that
\[
2(T_1 + T_2 + T_3) \leq \frac{T_1^2}{S_2 + S_3 - T_1} + \frac{T_2^2}{S_1 + S_3 - T_2} + \frac{T_3^2}{S_1 + S_2 - T_3} + S_1 + S_2 + S_3.
\]

Because all three denominators for the \( P_i \) are strictly positive, we can apply a variant of the Cauchy-Schwarz Inequality, namely
\[
\frac{(T_1 + T_2 + T_3)^2}{2(S_1 + S_2 + S_3) - (T_1 + T_2 + T_3)} \leq \frac{T_1^2}{S_2 + S_3 - T_1} + \frac{T_2^2}{S_1 + S_3 - T_2} + \frac{T_3^2}{S_1 + S_2 - T_3},
\]
and reduce the proof to showing that
\[
2(T_1 + T_2 + T_3) - (S_1 + S_2 + S_3) \leq \frac{(T_1 + T_2 + T_3)^2}{2(S_1 + S_2 + S_3) - (T_1 + T_2 + T_3)}.
\]

To this end, we set \( a = S_1 + S_2 + S_3 \) and \( b = T_1 + T_2 + T_3 \), and we prove that
\[
2b - a \leq \frac{b^2}{2a - b}.
\]
This last task is equivalent to showing that
\[
0 \leq 2a^2 - 5ab + 3b^2 = (3b - 2a)(b - a).
\]
Since $b > a$ (which is Inequality 1), the result is now clear. Further inequalities of this type are to be found in [4] and [1].

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