Mayhem Solutions

M207. *Proposed by Edward J. Barbeau, University of Toronto, Toronto, ON.*

At noon, Iphigenia set off on a bike ride from her home in Saskatoon, maintaining a leisurely pace of 20 km/h on the pleasantly level terrain. Later, her mother noticed that she had forgotten her lunch, and sent Electra off on her bike to meet her; Electra maintained a steady pace of 30 km/h. But then the sky darkened and the storm clouds gathered. So, exactly a half hour after Electra left, Orestes was sent off to meet the others with rain gear. Orestes rode at a steady pace of 40 km/h. All three followed the same route. As it happened, the three siblings met at exactly the same time. What time was that?

Solution by Titu Zvonaru, Comǎneşti, Romania.

When Orestes departed, Electra had ridden 15 km. The gap between Electra and Orestes decreased by 10 km for every hour; hence, they met after one and a half hours. During this time, Orestes rode $40 \times 1.5 = 60$ km, and Iphigenia rode $60/20 = 3$ h. All three met at 3:00 pm.

Also solved by John DeLeon, student, Angelo State University, San Angelo, TX; and Jean-David Houle, Cégep de Drummondville, Drummondville, QC.

M208. *Proposed by K.R.S. Sastry, Bangalore, India.*

Determine all distinct triangles having one side of length 6, with the other two sides being integers, and the perimeter numerically equal to the area.

Solution by Titu Zvonaru, Comǎneşti, Romania.

Let a, b, c denote the sides of the triangle, and let $s = \frac{1}{2}(a+b+c)$ denote its semiperimeter. Without loss of generality, we may assume that $a = 6$ and $b \le c$.

By the given assumption and Heron's Formula, we have

$$\sqrt{s(s-a)(s-b)(s-c)} \;=\; 2s\,,$$

which is successively equivalent to

$$\begin{aligned}
(s-a)(s-b)(s-c) &= 4s\,,\\
(b+c-a)(c+a-b)(a+b-c) &= 16(a+b+c)\,,\\
(b+c-6)\bigl(36-(c-b)^2\bigr) &= 16\bigl((b+c-6)+12\bigr)\,,\\
(b+c-6)\bigl(20-(c-b)^2\bigr) &= 16\cdot 12 \;=\; 192\,.
\end{aligned}$$

Since $b+c-6$ cannot be negative (by the Triangle Inequality), both factors on the left side above must be positive. Therefore, $20-(c-b)^2 \ge 1$, and hence, $0 \le c-b \le 4$.

If $c - b \in \{0,\ 1,\ 3\}$, then $20 - (c - b)^2 \in \{20,\ 19,\ 11\}$; but none of these numbers divide 192, a contradiction.

If $c - b = 2$, then we have $b + c - 6 = 12$, which leads to the solution $(a, b, c) = (6, 8, 10)$.

If $c - b = 4$, then we have $b + c - 6 = 48$, which leads to the solution $(a, b, c) = (6, 25, 29)$.

Editor's comments: This is a special case of a more general problem of determining *all* triangles with integer sides, each of which has its perimeter numerically equal to its area. This more general problem was proposed as E2420 in the *American Mathematical Monthly* [1973, 691; 1974, 662-663] by Edward T.H. Wang. The answer is that there are exactly five such triangles: $(6, 8, 10)$, $(5, 12, 13)$, $(9, 10, 17)$, $(7, 15, 20)$, and $(6, 25, 29)$. Actually, this problem has appeared and reappeared many times in the literature. The earliest solution appears to be due to B. Yates in 1865 (!). Interested readers can find all the information about this problem in the references cited above.

M209. *Proposed by Mihály Bencze, Brasov, Romania.*

Prove that $3x^2 + 4y^2$ and $4x^2 + 3y^2$ cannot be simultaneously perfect squares for all x, y positive integers.

Solution by the proposer.

Suppose that $3x^2 + 4y^2$ and $4x^2 + 3y^2$ are perfect squares for some positive integers x and y. Let $d = (x, y)$; then $x = da$ and $y = db$ with $(a, b) = 1$. Thus, $3x^2 + 4y^2 = d^2(3a^2 + 4b^2)$ and $4x^2 + 3y^2 = d^2(4a^2 + 3b^2)$. Therefore, $3a^2 + 4b^2 = m^2$ and $4a^2 + 3b^2 = n^2$, for some positive integers m and n. Then, $m^2 + n^2 = 7(a^2 + b^2)$, which implies that $7 \mid (m^2 + n^2)$; hence, $7 \mid m$ and $7 \mid n$. [*Ed.:* Use congruences modulo 7.] Therefore, $7 \mid (a^2 + b^2)$, which implies that $7 \mid a$ and $7 \mid b$. This is a contradiction since $(a, b) = 1$. Therefore, $3x^2 + 4y^2$ and $4x^2 + 3y^2$ cannot both be perfect squares.

M210. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

A 9×9 grid is subdivided into nine 3×3 smaller grids, called boxes. Each row and each column of the 9×9 grid, and each 3×3 box, must contain each of the digits 1 through 9.

Complete the grid on the right.

4				9			8	
			5			7		
6	2	3	7				4	
	4	9					7	3
7	6					9	2	
	3				2	4	1	5
		2			6			
	1			5				7

Solution by Titu Zvonaru, Comănești, Romania.

Let a_{ij}, $i = 1, 2, \ldots, 9$, $j = 1, 2, \ldots, 9$ represent the cells of the grid, where i is the row number number and j the column number. One way to complete the grid is the following:

$a_{93} = 4$, $a_{73} = 6$, $a_{51} = 3$, $a_{41} = 2$, $a_{21} = 1$, $a_{81} = 5$, $a_{82} = 7$, $a_{13} = 7$, $a_{12} = 5$, $a_{22} = 9$, $a_{23} = 8$, $a_{52} = 8$, $a_{97} = 2$, $a_{98} = 6$, $a_{39} = 9$, $a_{88} = 9$, $a_{87} = 3$, $a_{89} = 8$, $a_{75} = 7$, $a_{56} = 7$, $a_{47} = 8$, $a_{28} = 3$, $a_{58} = 5$, $a_{37} = 5$, $a_{63} = 5$, $a_{53} = 1$, $a_{46} = 5$, $a_{69} = 1$, $a_{17} = 1$, $a_{59} = 4$, $a_{57} = 6$, $a_{54} = 9$, $a_{55} = 2$, $a_{14} = 2$, $a_{16} = 3$, $a_{19} = 6$, $a_{29} = 2$, $a_{25} = 6$, $a_{44} = 6$, $a_{45} = 1$, $a_{36} = 1$, $a_{35} = 8$, $a_{96} = 9$, $a_{66} = 8$, $a_{26} = 4$, $a_{71} = 9$, $a_{91} = 8$, $a_{94} = 3$, $a_{65} = 3$, $a_{64} = 4$, $a_{85} = 4$, $a_{84} = 1$, $a_{74} = 8$.

4	5	7	2	9	3	1	8	6
1	9	8	5	6	4	7	3	2
6	2	3	7	8	1	5	4	9
2	4	9	6	1	5	8	7	3
3	8	1	9	2	7	6	5	4
7	6	5	4	3	8	9	2	1
9	3	6	8	7	2	4	1	5
5	7	2	1	4	6	3	9	8
8	1	4	3	5	9	2	6	7

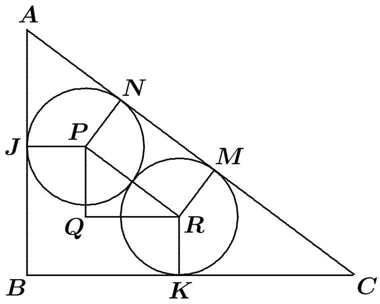
Also solved by Natalia Desy, student, Palembang, Indonesia; Isabel Díaz-Barrero and José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain; Jean-David Houle, Cégep de Drummondville, Drummondville, QC; and John DeLeon, Michelle Ellenburg, Morgan Lynch, Halley Newman, Christopher Odom, Mandy Rodgers, Josh Trejo, Tim Wilson, students, Angelo State University, San Angelo, TX, USA. Most solvers simply provided the completed grid.

M211. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Two circles of radius r are externally tangent. They are also internally tangent to the sides of a right triangle of sides 3, 4, and 5, with the hypotenuse of the triangle being tangent to both circles. Determine r.

Solution by Natalia Desy, student, Palembang, Indonesia.

Let ABC be the 3–4–5 right triangle in question, let r be the radius of each of the two circles, and let their points of tangency with triangle ABC be M, N, J, and K, as shown in the diagram.



By design, we have $AB = 3$, $BC = 4$, $CA = 5$. Let $x = AN = AJ$ and $y = CM = CK$. Then $QR = 4 - r - y$, $PQ = 3 - r - x$, and $PR = 2r = 5 - x - y$.

Since triangle PQR is similar to triangle ABC, we have

$$\frac{4 - r - y}{2r} = \frac{4}{5}, \quad \text{or} \quad 5y = 20 - 13r \qquad (1)$$

$$\text{and} \quad \frac{3 - r - x}{2r} = \frac{3}{5}, \quad \text{or} \quad 5x = 15 - 11r\,. \qquad (2)$$

Now, multiplying $2r = 5 - x - y$ by 5 gives us $10r = 25 - 5x - 5y$. Using equations (1) and (2) in this yields

$$\begin{aligned} 25 - (15 - 11r) - (20 - 135) &= 10r \\ 14r &= 10 \\ r &= \tfrac{5}{7}\,. \end{aligned}$$

Also solved by Titu Zvonaru, Comănești, Romania.

M212. *Proposed by Robert Bilinski, Collège Montmorency, Laval, QC.*

In the computer program Excel, the columns are labelled with letters. The first 26 columns are labelled with the letters A to Z. The 27^{th} column is labelled AA; the 28^{th} column is labelled AB.

(a) What is the number of the column labelled DXA?

(b) What label appears on the 2005^{th} column?

Solution by Titu Zvonaru, Comănești, Romania; and Michelle Ellenburg and Christopher Odom, students, Angelo State University, San Angelo, TX.

(a) Let A, B, $\dots$, Z be equivalent to 1, 2, $\dots$, 26, in base 26. But we cannot have a digit equal to 0; hence $Z = A0$. The label DXA is in base 26. When we convert it to the decimal system, we get

26^2	26^1	26^0
D	X	A
4	24	1

Simplifying, we multiply and sum the values to get

$$4 \cdot 26^2 + 24 \cdot 26^1 + 1 \cdot 26^0 = 3329\,.$$

Therefore, the number of the column labelled DXA is 3329.

(b) The number 2005 can be expressed as $2 \cdot 26^2 + 25 \cdot 26^1 + 3$, which converts to base 26 as follows:

26^2	26^1	26^0
2	25	3
B	Y	C

Therefore, the label for the 2005^{th}column is BYC.

Also solved by Natalia Desy, student, Palembang, Indonesia; John DeLeon, student, Angelo State University, San Angelo, TX.; Jean-David Houle, Cégep de Drummondville, Drummondville, QC; and Mandy Rodgers and Joshua Trejo, students, Angelo State University, San Angelo, TX.