MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

Mayhem Problems

Veuillez nous transmettre vos solutions aux problèmes du présent numéro avant le **premier février 2007**. *Les solutions reçues après cette date ne seront prises en compte que s'il nous reste du temps avant la publication des solutions.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais.

La rédaction souhaite remercier Jean-Marc Terrier et Martin Goldstein, de l'Université de Montréal, d'avoir traduit les problèmes.

M257. *Proposé par Fabio Zucca, Politecnico di Milano, Milano, Italie.*

Pour un nombre entier positif donné k, on considère l'ensemble des points $\{(x,y)\}$ d'un réseau où x et y sont des entiers tels que $0 \leq x \leq 2k+1$ and $0 \leq y \leq 2k+1$. On choisit deux points de cet ensemble au hasard. Tous les points ont la même probabilité d'être choisis et les points peuvent ne pas être distincts. Trouver la probabilité pour que l'aire du triangle (peut-être dégénéré) formé par ces points et le point $(0,0)$ soit un entier (peut-être 0).

M258. *Proposé par Edward T.H. Wang, Université Wilfrid Laurier, Waterloo, ON.*

Soit c, d et n des entiers tels que $n = c^2 + d^2$. Montrer que $n = (a^2+b^2)/5$ pour des entiers a et b.

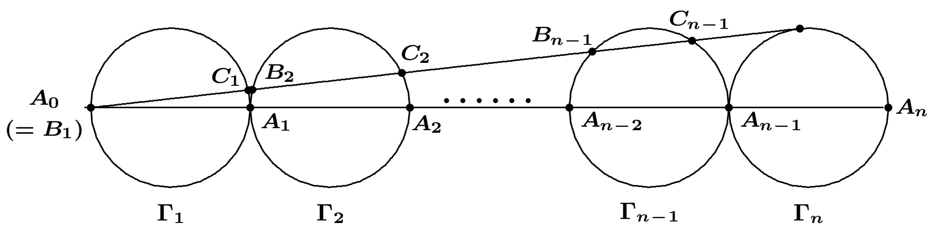
M259. *Proposé par l'Équipe de Mayhem.*

On forme le nombre n par juxtaposition des chiffres de 2^{2006} et de 5^{2006}. Combien de chiffres le nombre n comporte-t-il ?

M260. *Proposé par Bruce Shawyer, Université Memorial de Terre-Neuve, St. John's, NL.*

Sur une droite, on donne dans l'ordre les points A_0, A_1, ..., A_n, tous espacés de $2r$. Pour $1 \le k \le n$, soit Γ_k le cercle de diamètre $A_{k-1}A_k$. La droite passant par A_0 tangente à Γ_n coupe le cercle Γ_k aux points B_k et C_k, pour $1 \le k \le n-1$.

Déterminer la longueur des segments B_kC_k pour $1 \le k \le n-1$.



M261. *Proposé par Bruce Shawyer, Université Memorial de Terre-Neuve, St. John's, NL.*

Dans un rectangle $ABCD$, on a $AB = \frac{1}{2}BC$. A l'extérieur du rectangle, on dessine le triangle DCF, où l'angle $DFC = 30°$ et ADF est un segment de droite. Soit E le point milieu de AD.

Déterminer la mesure de l'angle EBF.

M262. *Proposé par Yakub N. Aliyev, Baku State University, Baku, Azerbaijan.*

Trouver toutes les fonctions $f : \mathbb{R} \to \mathbb{R}$ pour lesquelles $f(1) = 1$ et telles que $f(x+y) = 3^y f(x) + 2^x f(y)$ pour tous les nombres réels x et y.

....

M257. *Proposed by Fabio Zucca, Politecnico di Milano, Milano, Italy.*

For a given positive integer k, consider the set of lattice points $\{(x,y)\}$ where x and y are integers such that $0 \le x \le 2k+1$ and $0 \le y \le 2k+1$. Two points are selected at random from this set. All points have the same probability of being selected and the points need not be distinct. Find the probability that the area of the triangle (possibly degenerate) formed by these two points and the point $(0,0)$ is an integer (possibly 0).

M258. *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.*

Let c, d, and n be integers such that $n = c^2 + d^2$. Prove that $n = (a^2+b^2)/5$ for some integers a and b.

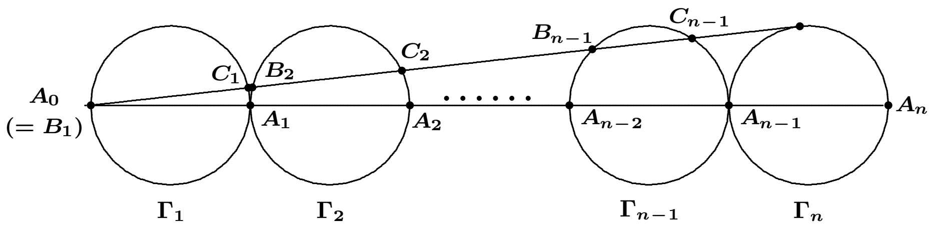
M259. *Proposed by the Mayhem Staff.*

The number n is formed by concatenating the strings of digits formed by the numbers 2^{2006} and 5^{2006}. How many digits does n have?

M260. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Points A_0, A_1, ..., A_n lie on a line, in that order, spaced a uniform distance $2r$ apart. For $1 \le k \le n$, let Γ_k be the circle with $A_{k-1}A_k$ as diameter. The line through A_0 tangent to Γ_n intersects the circle Γ_k at the points B_k and C_k, for $1 \le k \le n-1$.

Determine the length of the line segment B_kC_k for $1 \le k \le n-1$.



M261. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Rectangle $ABCD$ has $AB = \frac{1}{2}BC$. On the outside of the rectangle, draw $\triangle DCF$, where $\angle DFC = 30^\circ$ and ADF is a straight line segment. Let E be the mid-point of AD.

Determine the measure of $\angle EBF$.

M262. *Proposed by Yakub N. Aliyev, Baku State University, Baku, Azerbaijan.*

Find all functions $f : \mathbb{R} \to \mathbb{R}$ for which $f(1) = 1$ and, for all real numbers x and y, we have $f(x+y) = 3^y f(x) + 2^x f(y)$.

Mayhem Solutions

M207. *Proposed by Edward J. Barbeau, University of Toronto, Toronto, ON.*

At noon, Iphigenia set off on a bike ride from her home in Saskatoon, maintaining a leisurely pace of 20 km/h on the pleasantly level terrain. Later, her mother noticed that she had forgotten her lunch, and sent Electra off on her bike to meet her; Electra maintained a steady pace of 30 km/h. But then the sky darkened and the storm clouds gathered. So, exactly a half hour after Electra left, Orestes was sent off to meet the others with rain gear. Orestes rode at a steady pace of 40 km/h. All three followed the same route. As it happened, the three siblings met at exactly the same time. What time was that?

Solution by Titu Zvonaru, Comănești, Romania.

When Orestes departed, Electra had ridden 15 km. The gap between Electra and Orestes decreased by 10 km for every hour; hence, they met after one and a half hours. During this time, Orestes rode $40 \times 1.5 = 60$ km, and Iphigenia rode $60/20 = 3$ h. All three met at 3:00 pm.

Also solved by John DeLeon, student, Angelo State University, San Angelo, TX; and Jean-David Houle, Cégep de Drummondville, Drummondville, QC.

M208. *Proposed by K.R.S. Sastry, Bangalore, India.*

Determine all distinct triangles having one side of length 6, with the other two sides being integers, and the perimeter numerically equal to the area.

Solution by Titu Zvonaru, Comănești, Romania.

Let a, b, c denote the sides of the triangle, and let $s = \frac{1}{2}(a+b+c)$ denote its semiperimeter. Without loss of generality, we may assume that $a = 6$ and $b \leq c$.

By the given assumption and Heron's Formula, we have

$$\sqrt{s(s-a)(s-b)(s-c)} \;=\; 2s\,,$$

which is successively equivalent to

$$\begin{aligned} (s-a)(s-b)(s-c) &= 4s\,, \\ (b+c-a)(c+a-b)(a+b-c) &= 16(a+b+c)\,, \\ (b+c-6)\big(36-(c-b)^2\big) &= 16\big((b+c-6)+12\big)\,, \\ (b+c-6)\big(20-(c-b)^2\big) &= 16\cdot 12 \;=\; 192\,. \end{aligned}$$

Since $b+c-6$ cannot be negative (by the Triangle Inequality), both factors on the left side above must be positive. Therefore, $20-(c-b)^2 \geq 1$, and hence, $0 \leq c-b \leq 4$.

If $c - b \in \{0,\ 1,\ 3\}$, then $20 - (c - b)^2 \in \{20,\ 19,\ 11\}$; but none of these numbers divide 192, a contradiction.

If $c - b = 2$, then we have $b + c - 6 = 12$, which leads to the solution $(a, b, c) = (6, 8, 10)$.

If $c - b = 4$, then we have $b + c - 6 = 48$, which leads to the solution $(a, b, c) = (6, 25, 29)$.

Editor's comments: This is a special case of a more general problem of determining *all* triangles with integer sides, each of which has its perimeter numerically equal to its area. This more general problem was proposed as E2420 in the *American Mathematical Monthly* [1973, 691; 1974, 662–663] by Edward T.H. Wang. The answer is that there are exactly five such triangles: $(6, 8, 10)$, $(5, 12, 13)$, $(9, 10, 17)$, $(7, 15, 20)$, and $(6, 25, 29)$. Actually, this problem has appeared and reappeared many times in the literature. The earliest solution appears to be due to B. Yates in 1865 (!). Interested readers can find all the information about this problem in the references cited above.

M209. *Proposed by Mihály Bencze, Brasov, Romania.*

Prove that $3x^2 + 4y^2$ and $4x^2 + 3y^2$ cannot be simultaneously perfect squares for all x, y positive integers.

Solution by the proposer.

Suppose that $3x^2 + 4y^2$ and $4x^2 + 3y^2$ are perfect squares for some positive integers x and y. Let $d = (x, y)$; then $x = da$ and $y = db$ with $(a, b) = 1$. Thus, $3x^2 + 4y^2 = d^2(3a^2 + 4b^2)$ and $4x^2 + 3y^2 = d^2(4a^2 + 3b^2)$. Therefore, $3a^2 + 4b^2 = m^2$ and $4a^2 + 3b^2 = n^2$, for some positive integers m and n. Then, $m^2 + n^2 = 7(a^2 + b^2)$, which implies that $7 \mid (m^2 + n^2)$; hence, $7 \mid m$ and $7 \mid n$. [*Ed.:* Use congruences modulo 7.] Therefore, $7 \mid (a^2 + b^2)$, which implies that $7 \mid a$ and $7 \mid b$. This is a contradiction since $(a, b) = 1$. Therefore, $3x^2 + 4y^2$ and $4x^2 + 3y^2$ cannot both be perfect squares.

M210. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

A 9×9 grid is subdivided into nine 3×3 smaller grids, called boxes. Each row and each column of the 9×9 grid, and each 3×3 box, must contain each of the digits 1 through 9.

Complete the grid on the right.

4				9			8	
			5			7		
6	2	3	7				4	
	4	9					7	3
7	6					9	2	
	3				2	4	1	5
		2			6			
	1			5				7

Solution by Titu Zvonaru, Comănești, Romania.

Let a_{ij}, $i = 1, 2, \ldots, 9$, $j = 1, 2, \ldots, 9$ represent the cells of the grid, where i is the row number number and j the column number. One way to complete the grid is the following:

$a_{93} = 4$, $a_{73} = 6$, $a_{51} = 3$, $a_{41} = 2$, $a_{21} = 1$,
$a_{81} = 5$, $a_{82} = 7$, $a_{13} = 7$, $a_{12} = 5$, $a_{22} = 9$,
$a_{23} = 8$, $a_{52} = 8$, $a_{97} = 2$, $a_{98} = 6$, $a_{39} = 9$,
$a_{88} = 9$, $a_{87} = 3$, $a_{89} = 8$, $a_{75} = 7$, $a_{56} = 7$,
$a_{47} = 8$, $a_{28} = 3$, $a_{58} = 5$, $a_{37} = 5$, $a_{63} = 5$,
$a_{53} = 1$, $a_{46} = 5$, $a_{69} = 1$, $a_{17} = 1$, $a_{59} = 4$,
$a_{57} = 6$, $a_{54} = 9$, $a_{55} = 2$, $a_{14} = 2$, $a_{16} = 3$,
$a_{19} = 6$, $a_{29} = 2$, $a_{25} = 6$, $a_{44} = 6$, $a_{45} = 1$,
$a_{36} = 1$, $a_{35} = 8$, $a_{96} = 9$, $a_{66} = 8$, $a_{26} = 4$,
$a_{71} = 9$, $a_{91} = 8$, $a_{94} = 3$, $a_{65} = 3$, $a_{64} = 4$,
$a_{85} = 4$, $a_{84} = 1$, $a_{74} = 8$.

4	5	7	2	9	3	1	8	6
1	9	8	5	6	4	7	3	2
6	2	3	7	8	1	5	4	9
2	4	9	6	1	5	8	7	3
3	8	1	9	2	7	6	5	4
7	6	5	4	3	8	9	2	1
9	3	6	8	7	2	4	1	5
5	7	2	1	4	6	3	9	8
8	1	4	3	5	9	2	6	7

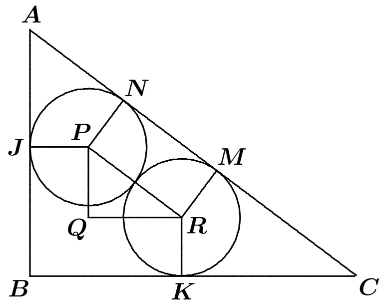
Also solved by Natalia Desy, student, Palembang, Indonesia; Isabel Díaz-Barrero and José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain; Jean-David Houle, Cégep de Drummondville, Drummondville, QC; and John DeLeon, Michelle Ellenburg, Morgan Lynch, Halley Newman, Christopher Odom, Mandy Rodgers, Josh Trejo, Tim Wilson, students, Angelo State University, San Angelo, TX, USA. Most solvers simply provided the completed grid.

M211. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Two circles of radius r are externally tangent. They are also internally tangent to the sides of a right triangle of sides 3, 4, and 5, with the hypotenuse of the triangle being tangent to both circles. Determine r.

Solution by Natalia Desy, student, Palembang, Indonesia.

Let ABC be the 3–4–5 right triangle in question, let r be the radius of each of the two circles, and let their points of tangency with triangle ABC be M, N, J, and K, as shown in the diagram.



By design, we have $AB = 3$, $BC = 4$, $CA = 5$. Let $x = AN = AJ$ and $y = CM = CK$. Then $QR = 4 - r - y$, $PQ = 3 - r - x$, and $PR = 2r = 5 - x - y$.

Since triangle PQR is similar to triangle ABC, we have

$$\frac{4 - r - y}{2r} = \frac{4}{5}, \quad \text{or} \quad 5y = 20 - 13r \tag{1}$$

$$\text{and} \quad \frac{3 - r - x}{2r} = \frac{3}{5}, \quad \text{or} \quad 5x = 15 - 11r\,. \tag{2}$$

Now, multiplying $2r = 5 - x - y$ by 5 gives us $10r = 25 - 5x - 5y$. Using equations (1) and (2) in this yields

$$\begin{aligned} 25 - (15 - 11r) - (20 - 135) &= 10r \\ 14r &= 10 \\ r &= \tfrac{5}{7} \,. \end{aligned}$$

Also solved by Titu Zvonaru, Comăneşti, Romania.

M212. *Proposed by Robert Bilinski, Collège Montmorency, Laval, QC.*

In the computer program Excel, the columns are labelled with letters. The first 26 columns are labelled with the letters A to Z. The 27^{th} column is labelled AA; the 28^{th} column is labelled AB.

(a) What is the number of the column labelled DXA?

(b) What label appears on the 2005^{th} column?

Solution by Titu Zvonaru, Comăneşti, Romania; and Michelle Ellenburg and Christopher Odom, students, Angelo State University, San Angelo, TX.

(a) Let A, B, ..., Z be equivalent to 1, 2, ..., 26, in base 26. But we cannot have a digit equal to 0; hence $Z = A0$. The label DXA is in base 26. When we convert it to the decimal system, we get

26^2	26^1	26^0
D	X	A
4	24	1

Simplifying, we multiply and sum the values to get

$$4 \cdot 26^2 + 24 \cdot 26^1 + 1 \cdot 26^0 = 3329 \,.$$

Therefore, the number of the column labelled DXA is 3329.

(b) The number 2005 can be expressed as $2 \cdot 26^2 + 25 \cdot 26^1 + 3$, which converts to base 26 as follows:

26^2	26^1	26^0
2	25	3
B	Y	C

Therefore, the label for the 2005^{th}column is BYC.

Also solved by Natalia Desy, student, Palembang, Indonesia; John DeLeon, student, Angelo State University, San Angelo, TX.; Jean-David Houle, Cégep de Drummondville, Drummondville, QC; and Mandy Rodgers and Joshua Trejo, students, Angelo State University, San Angelo, TX.

Problem of the Month

Ian VanderBurgh

This month, we will consider two problems with the same theme.

Problem 1 (2006 Gauss Contest (Grade 7)). A triangle can be formed having side lengths 4, 5, and 8. It is impossible however, to construct a triangle with side lengths 4, 5, and 10. Using the side lengths 2, 3, 5, 7, and 11, how many different triangles *with exactly two equal sides* can be formed?

So what is this all about? Just what are those mysterious first two sentences trying to tell us? Why can we make a triangle with certain side lengths and not with others? We could solve this problem in an intuitive way, but let's try to be more systematic.

With certain potential sets of side lengths, it makes more sense that a triangle cannot be formed than with other potential sets. For instance, it seems highly unlikely that we should be able to make a triangle with sides of length 1, 2, and 1000. With lengths that are closer together, it is not quite as clear (as in the case with sides of length 4, 5, and 10). What is the technical reason here?

This is an example of something called the *triangle inequality*, which says that in any triangle, the length of each side must be less than the sum of the lengths of the other two sides.

More technically, if $\triangle ABC$ has side lengths $AB = c$, $AC = b$, and $BC = a$, then we must have $c < a + b$, $b < a + c$, and $a < b + c$. How can we justify these facts? The easiest way is actually quite simple. Consider the two points A and B. What is the shortest path between A and B? Yes, you in the back... Yes, it is the straight line segment AB, whose length is c. Any other path from A to B is longer. In particular, going from A to B via C (a distance $AC + CB = b + a$) is longer, which means that $b + a > c$. We can obtain the inequalities $a + c > b$ and $b + c > a$ in a similar way.

At this stage, we could launch into the solution to Problem 1. However, let's hold off to make one more observation. Suppose that the lengths of the sides of $\triangle ABC$ satisfy $0 < a \leq b \leq c$. How many of the three inequalities $c < a + b$, $b < a + c$, and $a < b + c$ actually contain "useful" information? Since $b \geq a$ and $c > 0$, we get $b + c > a$ automatically. Similarly, $a + c > b$ automatically. Thus, only one of the three inequalities is worth considering, namely the one that says that the sum of the lengths of the two shorter sides is greater than the length of the longest side.

Solution to Problem 1: Consider a triangle with two equal sides. We write the side lengths as a, a, and b. Certainly $a + b > a$, which accounts for two of the three inequalities. The third states that $a + a > b$, or $2a > b$. In other words, if we are given the lengths a, a, and b, we need only check whether $b < 2a$ to determine whether the triangle inequality is satisfied by these lengths.

In this problem, the possible values for a and b are 2, 3, 5, 7, and 11. For each possible value of b, let's count the number of possible values for a with $2a > b$, remembering that a cannot equal b.

If $b = 2$, then a can be 3, 5, 7, or 11.
If $b = 3$, then a can be 2, 5, 7, or 11.
If $b = 5$, then a can be 3, 7, or 11. (a cannot be 2.)
If $b = 7$, then a can be 5 or 11. (a cannot be 2 or 3.)
If $b = 11$, then a can be 7. (a cannot be 2, 3, or 5.)

Adding up the possibilities, we discover that there are 14 different triangles that can be formed.

Wait! "Different" triangles? Yes, no two of them are congruent, since "side-side-side" is a valid check for congruency. "Can be formed"? Trickier, but still fine here—try to justify this on your own.

Here is a second problem which "sticks" to the same topic.

Problem 2 (2001 Gauss Contest (Grade 7)). A triangle can be formed having side lengths 4, 5, and 8. It is impossible, however, to construct a triangle with side lengths 4, 5, and 10. Ron has eight sticks, each having an integer length. He observes that he cannot form a triangle using any three of these sticks as side lengths. What is the shortest possible length of the longest of the eight sticks?

It looks like we have to use the pesky triangle inequality again. This problem also asks us to find the "minimum possible value of a maximum"—a standard type of problem, both in mathematics and in real life. It is like trying to determine the "worst case scenario".

Solution to Problem 2: Suppose that we write the lengths of the sticks in order as $a \le b \le c \le d \le \cdots$. Since a, b, and c do not form a triangle, we cannot have $c < a + b$; hence, $c \ge a + b$. To make c as small as possible given fixed a and b, we want $c = a + b$. Similarly, since b, c, and d cannot form a triangle, we want $d = b + c$, and so on.

Of course, to make everything as small as possible, we should start with a and b as small as possible. Since the smallest positive integer is 1, we choose $a = b = 1$. Using the idea of giving each new stick the sum of the lengths of the two previous sticks, we get lengths 1, 1, 2, 3, 5, 8, 13, and 21. This ensures that no triangle can be formed from any set of three sticks. Why? At each stage, the new longest stick will have a length equal to the sum of the two previous largest lengths and, therefore, at least as large as the sum of any two of the previous lengths.

Thus, the lengths are 1, 1, 2, 3, 5, 8, 13, and 21, and so the eighth stick has length 21. (Fibonacci strikes again!)

We see that the triangle inequality, though perhaps difficult to put one's finger on, can be useful. In fact, it appears throughout mathematics in many different guises—so always be on the lookout for it!

Pólya's Paragon

Playing Games with Mathematics (Part II)

John Grant McLoughlin

The September 2006 Paragon consisted of five challenges that were presented as the opening of a two-part Paragon. Here in Part II, the challenges are repeated and immediately followed by some insights into the mathematical principles beneath these challenges.

1. Sim

Six dots are drawn on a piece of paper to form the vertices of a hexagon. Two players are each assigned a colour. The players take turns joining any two of the dots with a line segment, using their assigned colours. The loser is the player who completes a triangle with three of the original six dots as its vertices and with all three edges the same colour.

Challenge: Prove that there must always be a loser (and a winner).

An outline of a proof by contradiction is provided here. Assume that it is possible to have a tie. Let the original dots be A, B, C, D, E, and F. Consider any one of these dots. Suppose we choose A. Note that there are five segments that can be drawn from A. Select two colours, say blue and red. At least three of the segments from A must be one of these colours, say blue. Again it does not matter which three segments. Suppose that AB, AC, and AE are blue. It follows that none of BC, BE, or CE are blue because otherwise a blue triangle would be formed, thus creating a loser. Aha! That makes triangle BCE a red triangle. Therefore, we have a proof by contradiction that a tie is impossible.

Readers interested in reading more about this and other combinatorial problems may refer to sources such as [4].

2. 31

This mental math game involves a running total which starts at zero. Each player has the choice to add 1, 2, 3, 4, 5, or 6 to the total. Players alternate turns. The winner is the player who is able to bring the total to 31.

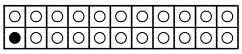
Challenge: Determine a winning strategy. You may choose to play first or second.

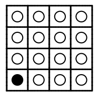
This is the easiest of the five challenges. Working backwards from 31, you can determine that a player who gets to 24 should win the game. Continuing backwards, we find that 17, 10, and 3 are winning positions. The winning strategy is to begin with 3 and ensure that you get each of 10, 17, and 24 on your way to 31. A misstep will allow your opponent to claim a winning position.

3. Chomp

Counters are placed in a rectangular grid such that one counter appears in each small rectangle. The counter in the bottom left-hand corner is a different colour than the others. Players take turns selecting one counter. If the counter selected occupies the bottom left-hand corner of a rectangle on the grid, all the counters in that rectangle are removed. The object is to force your opponent to select the differently coloured counter (the one in the bottom left-hand corner).

Challenge: Suppose that you play two games of Chomp in which the boards are $2 \times n$ and $k \times k$, examples of which are shown. Determine a winning strategy in each case. You may choose to play first or second.





This game appears in various sources. My reference is [5], which recommends [2] for further reading on the game. (Coincidentally, a review of the collection [2] will soon appear in ***CRUX with MAYHEM***.) We now turn our attention to the winning strategies for the pictured boards. The boards shown above are among those for which winning strategies do exist for the first player.

The solution for the $2 \times n$ board is to remove the single square in the top right-hand corner, thus, producing a symmetric situation in which the first player can copy the moves of the second player until the second player is forced to take the unwanted counter. The $k \times k$ example uses a similar idea: remove the entire figure except for the squares along the left edge and the bottom edge, thus leaving an L-shape of unit width. Observe that the first player can again engage in a copying strategy to ensure victory in the game.

Those of you who enjoy Chomp might want to consider yet another challenge posed in [5]: determine the unique opening move for a winning strategy with a 3×5 board.

4. Fifteen Finesse

The numbers 1, 2, 3, 4, 5, 6, 7, 8, and 9 are available for use in this game. Each number can be used only once. Two players alternate turns selecting one of the available numbers. To win the game, a player must obtain exactly three numbers that sum to 15. (Neither a pair of numbers, such as 7 and 8, nor a set of four numbers, such as 1, 3, 5, and 6, constitutes a winning combination.) The game ends in a draw if no player is able to acquire three numbers that sum to 15.

Challenge: Explain the underlying structure of the game, and suggest strategies that may help to win a game.

Does this game sound vaguely familiar? My experience has been that people play this game joyfully as they win, lose, or draw. Can you identify another game with an objective of three-in-a-row that you may win, lose, or draw while learning the game? Indeed, Tic-Tac-Toe! Where did we get three-in-a-row from the description of the game? Here we have an example of an isomorphism. That is, Fifteen Finesse is isomorphic to playing Tic-Tac-Toe on a magic square board. One such board is provided here for consideration:

8	1	6
3	5	7
4	9	2

This game can be an embarrassment! I suspect that if you recorded the sequence of number selections in early games and transposed them onto the board, you would not be so proud of some of your moves. It's fun to see how one played the game prior to having knowledge of the underlying structure. This game was introduced to me twenty years ago in [3], which has recently been reprinted by the MAA and is to be reviewed in ***CRUX with MAYHEM***.

5. A Polynomial in Transition

Consider the polynomial $x^2+10x+20$. Under the conditions below, is it possible to convert this polynomial to $x^2+20x+10$? Justify your answer.

Conditions:

(i) On each step you may only change the constant term or the coefficient of x (but not both).

(ii) The change must be an increase of 1 or a decrease of 1.

(iii) The change must NOT produce a polynomial that can be factored into the form $(x+m)(x+n)$ where m and n are integers. For example, you could not begin by reducing 10 to 9, since $x^2+9x+20=(x+5)(x+4)$.

This challenge, drawn to my attention in a problem solving course in 1988 by Ed Barbeau (see [1]), can be solved readily with an insight. Suppose that we consider the polynomial to be of the form x^2+bx+c. Consider the value $b-c$. This value begins at -10 and must end at 10 while changing by exactly ± 1 at each step. Therefore, there must be a point at which $b-c=1$ if the transformation is possible. We would have a reducible form, however, using $(x+c)(x+1)$. Therefore, it is impossible to complete the process.

References

[1] Edward J. Barbeau, *Polynomials*, Springer, New York, 1989.

[2] E.R. Berlekamp, J.H. Conway, and R.K. Guy, *Winning Ways for your Mathematical Plays*, A.K. Peters, 2004.

[3] Martin Gardner, *aha! Insight*, W.H. Freeman, San Francisco, 1978.

[4] Richard Hoshino and John Grant McLoughlin, *Combinatorial Explorations*, Canadian Mathematical Society, Ottawa, 2005.

[5] Ian Stewart, *Math Hysteria*, Oxford University Press, Oxford, 2004.