Contributor Profiles: D.J. Smeenk

Derk Jan Smeenk was born on the third of October, 1919, in the village of Gorssel, in the eastern part of the Netherlands. He was the second son of a farmer.

He attended secondary school in Deventer, where his mathematics instructor was Dr. Oene Bottema, who became his mentor. In 1938 he began his studies of mathematics and physics at the universities of Utrecht and Leiden, respectively. His studies were interrupted by the Second World War, but he finished them in 1948.

From 1946 to 1949, Smeenk was employed at the Technical University of Delft, but then he decided to pursue a career in secondary education. Eventually, in 1961, he became the headmaster of a secondary school in Zaltbommel, a position which he held until his retirement in 1981. During his period as headmaster, he renewed his studies, and in 1973 he obtained his doctorate degree under the supervision of Bottema, who had in the meantime been appointed as Professor at the Technical University of Delft.

In addition, Smeenk has devoted considerable time to his church and community. For example, he was involved for ten years in the restoration of the Sint-Maarten church of Zaltbommel, one of the most important medieval churches of the Netherlands. He wrote a book about the church and was honoured by his municipality for doing so.

He has been a member of the Rotary Club of Zaltbommel for more than 30 years. For his dedication to the principles and ideals of the rotarians, he was appointed to be a Paul Harris Fellow by the Rotary Club of Walsrode, Germany.

In 1948, Smeenk married Tine Zeevat. They have three children and three grandchildren.

Over the years, Smeenk has had a number of hobbies, which include gardening, photography, and historical research. Indeed, he has published more than twenty articles about persons and affairs of historical significance in the Zaltbommel area in the second half of the nineteenth century.

Since his retirement, Smeenk has found more time to pursue his passion for geometry. His name first appeared in the pages of ***Crux Mathematicorum*** in 1982, and has been appearing regularly ever since. We sincerely hope that Smeenk will continue to share his interesting problems with the rest of the world through the pages of ***CRUX with MAYHEM***.

In this issue, Smeenk has proposed problem 3175.

SKOLIAD No. 96

Robert Bilinski

Please send your solutions to the problems in this edition by ***April 1, 2007***. A copy of MATHEMATICAL MAYHEM Vol. 6 will be presented to one pre-university reader who sends in solutions before the deadline. The decision of the editor is final.

Our items this issue come from the 2005 W.J. Blundon Mathematics Contest, for which I thank Don Rideout, Memorial University. I would also like to thank Bruce Shawyer, who recently retired from Memorial University; Bruce has been helping with ***CRUX with MAYHEM*** as a whole and with the Skoliad in particular for a long time.

22e Concours de Mathématiques W.J. Blundon
Commandité par la SMC et le département de mathématiques de l'Université Mémorial, 23 Février 2005

1. Une automobile monte une colline avec une vitesse moyenne de 30 km/h et descend la même distance avec une vitesse moyenne de 60 km/h. Quelle est la vitesse moyenne pour le trajet ?

2. Soit P un point à l'intérieur du rectangle $ABCD$. Si $PA = 9$, $PB = 4$ et $PC = 6$, trouver PD.

3. Trouver l'aire de la région au-dessus de l'axe des x et sous le graphique de $x^2 + (y+1)^2 = 2$.

4. Un carré est inscrit dans un triangle équilateral. Trouver le ratio de l'aire du carré avec l'aire du triangle.

5. Trouver le nombre de solutions de l'équation $2x + 5y = 2005$ pour lesquelles x et y sont tous les deux des entiers positifs.

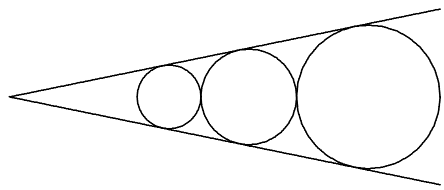
6. Pour quelles valeurs de a l'équation $4x^2 + 4ax + a + 6 = 0$ a des solutions réelles ?

7. Ace court avec une vitesse constante et Flash court x fois plus vite, $x > 1$. Flash donne Ace une longueur d'avance de y mètres, et, à un signal donné, ils partent dans la même direction. Trouver la distance que Flash doit courir pour rattraper Ace.

8. Montrer que $3^n - 2n - 1$ est divisible par 4 pour tout entier positif n.

9. Si le polynôme $P(x) = x^3 - x^2 + x - 2$ a les trois zéros a, b et c, trouver $a^3 + b^3 + c^3$.

10. Un cercle de rayon 2 est tangent aux deux côtés d'un angle. Un cercle de rayon 3 est tangent au premier cercle et aux deux côtés de l'angle. Un troisième cercle est tangent au second cercle et aux deux côtés de l'angle. Trouver le rayon du troisième cercle.



The 22nd W.J. Blundon Mathematics Contest
Sponsored by the CMS and the Mathematics Department at Memorial University, February 23, 2005

1. An automobile went up a hill at an average speed of 30 km/hr and down the same distance at an average speed of 60 km/hr. What was the average speed for the trip?

2. Let P be a point in the interior of rectangle $ABCD$. If $PA = 9$, $PB = 4$, and $PC = 6$, find PD.

3. Find the area of the region above the x–axis and below the graph of $x^2 + (y + 1)^2 = 2$.

4. A square is inscribed in an equilateral triangle. Find the ratio of the area of the square to the area of the triangle.

5. Find the number of solutions to the equation $2x + 5y = 2005$ for which both x and y are positive integers.

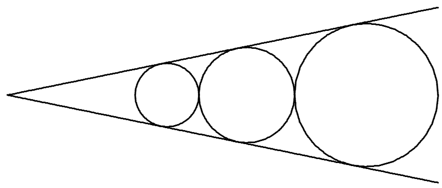
6. For what values of a does the equation $4x^2 + 4ax + a + 6 = 0$ have real solutions?

7. Ace runs with constant speed and Flash runs x times as fast, $x > 1$. Flash gives Ace a head start of y metres, and, at a given signal, they start off in the same direction. Find the distance Flash must run to catch Ace.

8. Show that $3^n - 2n - 1$ is divisible by 4 for any positive integer n.

9. If the polynomial $P(x) = x^3 - x^2 + x - 2$ has the three zeroes a, b, and c, find $a^3 + b^3 + c^3$.

10. A circle of radius 2 is tangent to both sides of an angle. A circle of radius 3 is tangent to the first circle and both sides of the angle. A third circle is tangent to the second circle and both sides of the angle. Find the radius of the third circle.



Next we give the solutions to the 2003-04 Concours Montmorency for secondary 4 and 5 students in the Laval region of Quebec [2006 : 3–5].

Concours Montmorency 2003-04

Sec V, novembre 2003

1. Un magicien vous propose de calculer le carré de n'importe quel nombre entre 50 et 59 inclusivement.

Vous lui proposez de calculer 57^2.

Il répond : «Abracadabra : $5^2 = 25$ et $25 + 7 = 32$».

Il continue : «Abracadabra : $7^2 = 49$». (Rem : si le deuxième chiffre avait été 2, il aurait écrit $2^2 = 04$).

Il ajoute finalement : «Le carré est 3249».

Il a raison ! Justifiez-le algébriquement.

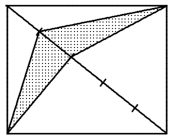
Solution officielle.

Soit u le chiffre des unités du nombre entre 50 et 59, autrement dit $(5u)_{10} = 50 + u$. Puisque l'on a

$$(50 + u)^2 \;=\; 2500 + 100u + u^2 \;=\; u^2 + 100(5^2 + u)\,,$$

la méthode du magicien est bonne.

2. Considérons un rectangle de largeur 8 et de longueur 10. En séparant la diagonale en cinq parties égales, calculer l'aire de la zone hachurée.



Solution officielle.

Si on relie toutes les marques de la diagonale avec les 2 sommets, on obtient 10 triangles ayant la même aire. En effet, les hauteurs sont égales par symétrie et les bases sont égales par construction. Ainsi, la zone hachurée est le cinquième du rectangle et son aire est $8 \times 10 \times \frac{1}{5} = 16$.

3. (a) Montrer que, pour toute paire de nombres réels $a > 0$ et $b > 0$, on a : $\frac{a}{b} + \frac{b}{a} \geq 2$.

(b) Déduire, grâce au résultat obtenu en (a), que pour $x > 0$, $y > 0$ et $z > 0$, on a toujours $(x + y + z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) > 8$.

Solution officielle.

(a) On part du fait que le carré de tout nombre est positif. Ainsi, en développant, $(a - b)^2 \geq 0$ est équivalent à $a^2 + b^2 \geq 2ab$. Puisque a et b sont strictement positifs, on peut diviser par ab sans changer le sens de l'inégalité, d'où $\frac{a}{b} + \frac{b}{a} \geq 2$.

(b) On développe le produit :

$$(x + y + z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = 3 + \frac{x}{y} + \frac{y}{x} + \frac{x}{z} + \frac{z}{x} + \frac{y}{z} + \frac{z}{y} \geq 9 > 8\,.$$

On utilise ainsi trois fois le résultat du (a).

4. On veut partager un terrain rectangulaire de 4000 m^2 à l'aide de deux lignes droites parallèles aux côtés, en quatre petits terrains rectangulaires A, B, C et D.

Est-il possible de le faire de telle manière que l'aire de $A = 2000$ m^2, l'aire de $B = 1000$ m^2, l'aire de $C = 600$ m^2 et l'aire de $D = 400$ m^2 ?

A	B
C	D

Solution officielle.

Non, pour qu'un tel partage soit réalisable, il faudrait que le rapport $\frac{\text{Aire de } A}{\text{Aire de } C}$ soit égal au rapport $\frac{\text{Aire de } B}{\text{Aire de } D}$. Or :

$$\frac{\text{Aire de } A}{\text{Aire de } C} = \frac{10}{3} \neq \frac{5}{2} = \frac{\text{Aire de } B}{\text{Aire de } D}\,.$$

5. Durant une panne d'électricité, deux chandelles de même longueur sont allumées à 18:00 h. La première chandelle se consume en 6 heures et la seconde, en 8 heures. À une certaine heure, on éteint les deux chandelles et on observe que la première est exactement deux fois plus courte que la seconde.

À quelle heure exactement, a-t-on éteint les deux chandelles ?

Solution officielle.

On pose ℓ la longueur des deux bougies. Les hauteurs h_1 et h_2 des deux chandelles sont obtenues par les équation suivantes :

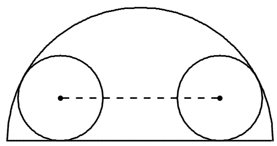
$$h_1 \;=\; \ell\left(1 - \frac{t}{6}\right) \qquad \text{et} \qquad h_2 \;=\; \ell\left(1 - \frac{t}{8}\right)\,,$$

où t est le temps écoulé depuis 18:00 h. Il ne reste plus qu'à résoudre l'équation $2h_1 = h_2$. En simplifiant les ℓ et en isolant le t, on obtient $t = \frac{24}{5}$.

On a donc éteint les bougies exactement 4 heures et 48 minutes après les avoir allumées. Les deux chandelles ont été éteintes à 22:48 h.

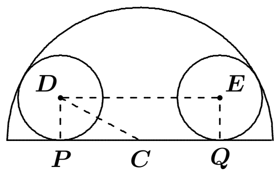
6. Two circles of radius 8 are placed inside a semi-circle of radius 25. The two circles are each tangent to the diameter and to the semicircle.

What is the distance between the centres of the two circles?



I. *Solution by Vedula N. Murty, Dover, PA, USA.*

Let C be the centre of the large semi-circle. Let D and E be the centres of the small circles, and let P and Q be their respective points of tangency with the diameter of the semi-circle (see diagram). Let a be the length of CP (which is also the length of CQ, by symmetry).

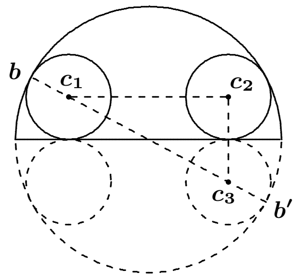


The distance between the centres of the two circles is the length of PQ, which equals $2a$. To find a, we note that the right triangle DPC has a hypotenuse of length $25 - 8 = 17$, with $DP = 8$ and $CP = a$. Using Pythagoras' Theorem, we easily see that $a = 15$.

Hence, the distance between the centres of the small circles is 30.

II. *Solution officielle.*

On commence par compléter le demi-cercle en lui juxtaposant son image miroir tel qu'illustré. Ensuite, on remarque que le segment bb' constitue un diamètre du grand cercle, sa longueur est donc 50. De plus, comme tous les petits cercles ont un rayon de longueur 8, on a que :



$$bc_1 \;=\; 8\,,\quad c_3b \;=\; 8\,,\quad c_2c_3 \;=\; 16\,,\quad c_1c_3 \;=\; 50 - 2\times 8 \;=\; 34\,.$$

Finalement, par Pythagore, on trouve que $\overline{c_1c_2} = \sqrt{34^2 - 16^2} = 30$.

7. Évaluer le très long produit suivant :

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{2003^2}\right).$$

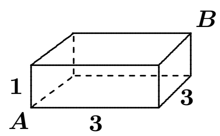
(Indice : $\left(1-\frac{1}{n^2}\right)=\frac{n^2-1}{n^2}=\cdots$)

Solution officielle.

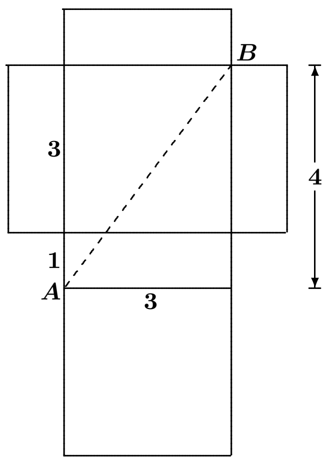
Premièrement on remarque que $1-\frac{1}{n^2}=\frac{n^2-1}{n^2}=\frac{(n-1)(n+1)}{n^2}$. On peut alors réécrire le produit et simplifier :

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{2003^2}\right)=\frac{1}{2}\cdot\frac{2004}{2003}=\frac{1002}{2003}.$$

8. Un parallélépipède rectangle a une base 3×3 et une hauteur 1. Trouver la longueur minimale d'un chemin qu'une araignée pourrait suivre, le long de la surface, pour se rendre du sommet A au sommet opposé B.



Solution officielle.



Il suffit ensuite d'appliquer Pythagore pour obtenir la longueur recherchée : $\sqrt{3^2+4^2}=5$.

That brings us to the end of another issue. Continue sending in your contests and solutions.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

Mayhem Problems

Veuillez nous transmettre vos solutions aux problèmes du présent numéro avant le **premier février 2007**. *Les solutions reçues après cette date ne seront prises en compte que s'il nous reste du temps avant la publication des solutions.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais.

La rédaction souhaite remercier Jean-Marc Terrier et Martin Goldstein, de l'Université de Montréal, d'avoir traduit les problèmes.

M257. *Proposé par Fabio Zucca, Politecnico di Milano, Milano, Italie.*

Pour un nombre entier positif donné k, on considère l'ensemble des points $\{(x,y)\}$ d'un réseau où x et y sont des entiers tels que $0 \leq x \leq 2k+1$ and $0 \leq y \leq 2k+1$. On choisit deux points de cet ensemble au hasard. Tous les points ont la même probabilité d'être choisis et les points peuvent ne pas être distincts. Trouver la probabilité pour que l'aire du triangle (peut-être dégénéré) formé par ces points et le point $(0,0)$ soit un entier (peut-être 0).

M258. *Proposé par Edward T.H. Wang, Université Wilfrid Laurier, Waterloo, ON.*

Soit c, d et n des entiers tels que $n = c^2 + d^2$. Montrer que $n = (a^2 + b^2)/5$ pour des entiers a et b.

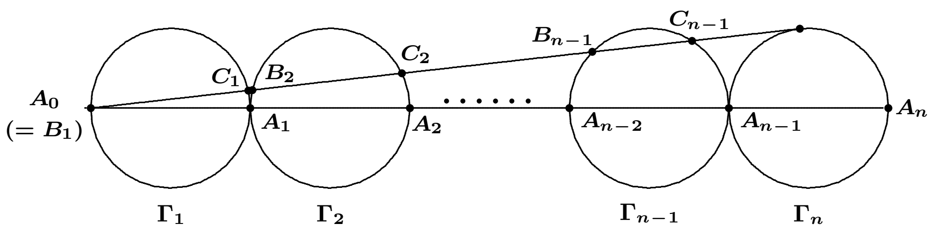
M259. *Proposé par l'Équipe de Mayhem.*

On forme le nombre n par juxtaposition des chiffres de 2^{2006} et de 5^{2006}. Combien de chiffres le nombre n comporte-t-il ?

M260. *Proposé par Bruce Shawyer, Université Memorial de Terre-Neuve, St. John's, NL.*

Sur une droite, on donne dans l'ordre les points A_0, A_1, $\ldots$, A_n, tous espacés de $2r$. Pour $1 \leq k \leq n$, soit Γ_k le cercle de diamètre $A_{k-1}A_k$. La droite passant par A_0 tangente à Γ_n coupe le cercle Γ_k aux points B_k et C_k, pour $1 \leq k \leq n-1$.

Déterminer la longueur des segments B_kC_k pour $1 \leq k \leq n-1$.



M261. *Proposé par Bruce Shawyer, Université Memorial de Terre-Neuve, St. John's, NL.*

Dans un rectangle $ABCD$, on a $AB = \frac{1}{2}BC$. A l'extérieur du rectangle, on dessine le triangle DCF, où l'angle $DFC = 30^\circ$ et ADF est un segment de droite. Soit E le point milieu de AD.

Déterminer la mesure de l'angle EBF.

M262. *Proposé par Yakub N. Aliyev, Baku State University, Baku, Azerbaijan.*

Trouver toutes les fonctions $f : \mathbb{R} \to \mathbb{R}$ pour lesquelles $f(1) = 1$ et telles que $f(x+y) = 3^y f(x) + 2^x f(y)$ pour tous les nombres réels x et y.

. . . .

M257. *Proposed by Fabio Zucca, Politecnico di Milano, Milano, Italy.*

For a given positive integer k, consider the set of lattice points $\{(x,y)\}$ where x and y are integers such that $0 \leq x \leq 2k+1$ and $0 \leq y \leq 2k+1$. Two points are selected at random from this set. All points have the same probability of being selected and the points need not be distinct. Find the probability that the area of the triangle (possibly degenerate) formed by these two points and the point $(0,0)$ is an integer (possibly 0).

M258. *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.*

Let c, d, and n be integers such that $n = c^2 + d^2$. Prove that $n = (a^2+b^2)/5$ for some integers a and b.

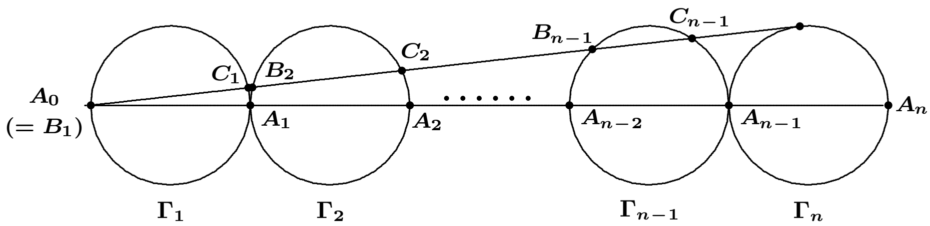
M259. *Proposed by the Mayhem Staff.*

The number n is formed by concatenating the strings of digits formed by the numbers 2^{2006} and 5^{2006}. How many digits does n have?

M260. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Points A_0, A_1, ..., A_n lie on a line, in that order, spaced a uniform distance $2r$ apart. For $1 \leq k \leq n$, let Γ_k be the circle with $A_{k-1}A_k$ as diameter. The line through A_0 tangent to Γ_n intersects the circle Γ_k at the points B_k and C_k, for $1 \leq k \leq n-1$.

Determine the length of the line segment B_kC_k for $1 \leq k \leq n-1$.



M261. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Rectangle $ABCD$ has $AB = \frac{1}{2}BC$. On the outside of the rectangle, draw $\triangle DCF$, where $\angle DFC = 30^\circ$ and ADF is a straight line segment. Let E be the mid-point of AD.

Determine the measure of $\angle EBF$.

M262. *Proposed by Yakub N. Aliyev, Baku State University, Baku, Azerbaijan.*

Find all functions $f : \mathbb{R} \to \mathbb{R}$ for which $f(1) = 1$ and, for all real numbers x and y, we have $f(x+y) = 3^y f(x) + 2^x f(y)$.

Mayhem Solutions

M207. *Proposed by Edward J. Barbeau, University of Toronto, Toronto, ON.*

At noon, Iphigenia set off on a bike ride from her home in Saskatoon, maintaining a leisurely pace of 20 km/h on the pleasantly level terrain. Later, her mother noticed that she had forgotten her lunch, and sent Electra off on her bike to meet her; Electra maintained a steady pace of 30 km/h. But then the sky darkened and the storm clouds gathered. So, exactly a half hour after Electra left, Orestes was sent off to meet the others with rain gear. Orestes rode at a steady pace of 40 km/h. All three followed the same route. As it happened, the three siblings met at exactly the same time. What time was that?

Solution by Titu Zvonaru, Comăneşti, Romania.

When Orestes departed, Electra had ridden 15 km. The gap between Electra and Orestes decreased by 10 km for every hour; hence, they met after one and a half hours. During this time, Orestes rode $40 \times 1.5 = 60$ km, and Iphigenia rode $60/20 = 3$ h. All three met at 3:00 pm.

Also solved by John DeLeon, student, Angelo State University, San Angelo, TX; and Jean-David Houle, Cégep de Drummondville, Drummondville, QC.

M208. *Proposed by K.R.S. Sastry, Bangalore, India.*

Determine all distinct triangles having one side of length 6, with the other two sides being integers, and the perimeter numerically equal to the area.

Solution by Titu Zvonaru, Comăneşti, Romania.

Let a, b, c denote the sides of the triangle, and let $s = \frac{1}{2}(a+b+c)$ denote its semiperimeter. Without loss of generality, we may assume that $a = 6$ and $b \le c$.

By the given assumption and Heron's Formula, we have

$$\sqrt{s(s-a)(s-b)(s-c)} \;=\; 2s\,,$$

which is successively equivalent to

$$\begin{aligned}(s-a)(s-b)(s-c) &= 4s\,,\\ (b+c-a)(c+a-b)(a+b-c) &= 16(a+b+c)\,,\\ (b+c-6)\big(36-(c-b)^2\big) &= 16\big((b+c-6)+12\big)\,,\\ (b+c-6)(20-(c-b)^2) &= 16\cdot 12 \;=\; 192\,.\end{aligned}$$

Since $b+c-6$ cannot be negative (by the Triangle Inequality), both factors on the left side above must be positive. Therefore, $20-(c-b)^2 \ge 1$, and hence, $0 \le c-b \le 4$.

If $c - b \in \{0,\ 1,\ 3\}$, then $20 - (c - b)^2 \in \{20,\ 19,\ 11\}$; but none of these numbers divide 192, a contradiction.

If $c - b = 2$, then we have $b + c - 6 = 12$, which leads to the solution $(a, b, c) = (6, 8, 10)$.

If $c - b = 4$, then we have $b + c - 6 = 48$, which leads to the solution $(a, b, c) = (6, 25, 29)$.

Editor's comments: This is a special case of a more general problem of determining *all* triangles with integer sides, each of which has its perimeter numerically equal to its area. This more general problem was proposed as E2420 in the *American Mathematical Monthly* [1973, 691; 1974, 662-663] by Edward T.H. Wang. The answer is that there are exactly five such triangles: $(6, 8, 10)$, $(5, 12, 13)$, $(9, 10, 17)$, $(7, 15, 20)$, and $(6, 25, 29)$. Actually, this problem has appeared and reappeared many times in the literature. The earliest solution appears to be due to B. Yates in 1865 (!). Interested readers can find all the information about this problem in the references cited above.

M209. *Proposed by Mihály Bencze, Brasov, Romania.*

Prove that $3x^2 + 4y^2$ and $4x^2 + 3y^2$ cannot be simultaneously perfect squares for all x, y positive integers.

Solution by the proposer.

Suppose that $3x^2 + 4y^2$ and $4x^2 + 3y^2$ are perfect squares for some positive integers x and y. Let $d = (x, y)$; then $x = da$ and $y = db$ with $(a, b) = 1$. Thus, $3x^2 + 4y^2 = d^2(3a^2 + 4b^2)$ and $4x^2 + 3y^2 = d^2(4a^2 + 3b^2)$. Therefore, $3a^2 + 4b^2 = m^2$ and $4a^2 + 3b^2 = n^2$, for some positive integers m and n. Then, $m^2 + n^2 = 7(a^2 + b^2)$, which implies that $7 \mid (m^2 + n^2)$; hence, $7 \mid m$ and $7 \mid n$. [*Ed.:* Use congruences modulo 7.] Therefore, $7 \mid (a^2 + b^2)$, which implies that $7 \mid a$ and $7 \mid b$. This is a contradiction since $(a, b) = 1$. Therefore, $3x^2 + 4y^2$ and $4x^2 + 3y^2$ cannot both be perfect squares.

M210. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

A 9×9 grid is subdivided into nine 3×3 smaller grids, called boxes. Each row and each column of the 9×9 grid, and each 3×3 box, must contain each of the digits 1 through 9.

Complete the grid on the right.

4				9			8	
			5			7		
6	2	3	7				4	
	4	9					7	3
7	6					9	2	
	3				2	4	1	5
		2			6			
	1			5				7

Solution by Titu Zvonaru, Comăneşti, Romania.

Let a_{ij}, $i = 1, 2, \ldots, 9$, $j = 1, 2, \ldots, 9$ represent the cells of the grid, where i is the row number number and j the column number. One way to complete the grid is the following:

$a_{93} = 4$, $a_{73} = 6$, $a_{51} = 3$, $a_{41} = 2$, $a_{21} = 1$,
$a_{81} = 5$, $a_{82} = 7$, $a_{13} = 7$, $a_{12} = 5$, $a_{22} = 9$,
$a_{23} = 8$, $a_{52} = 8$, $a_{97} = 2$, $a_{98} = 6$, $a_{39} = 9$,
$a_{88} = 9$, $a_{87} = 3$, $a_{89} = 8$, $a_{75} = 7$, $a_{56} = 7$,
$a_{47} = 8$, $a_{28} = 3$, $a_{58} = 5$, $a_{37} = 5$, $a_{63} = 5$,
$a_{53} = 1$, $a_{46} = 5$, $a_{69} = 1$, $a_{17} = 1$, $a_{59} = 4$,
$a_{57} = 6$, $a_{54} = 9$, $a_{55} = 2$, $a_{14} = 2$, $a_{16} = 3$,
$a_{19} = 6$, $a_{29} = 2$, $a_{25} = 6$, $a_{44} = 6$, $a_{45} = 1$,
$a_{36} = 1$, $a_{35} = 8$, $a_{96} = 9$, $a_{66} = 8$, $a_{26} = 4$,
$a_{71} = 9$, $a_{91} = 8$, $a_{94} = 3$, $a_{65} = 3$, $a_{64} = 4$,
$a_{85} = 4$, $a_{84} = 1$, $a_{74} = 8$.

4	5	7	2	9	3	1	8	6
1	9	8	5	6	4	7	3	2
6	2	3	7	8	1	5	4	9
2	4	9	6	1	5	8	7	3
3	8	1	9	2	7	6	5	4
7	6	5	4	3	8	9	2	1
9	3	6	8	7	2	4	1	5
5	7	2	1	4	6	3	9	8
8	1	4	3	5	9	2	6	7

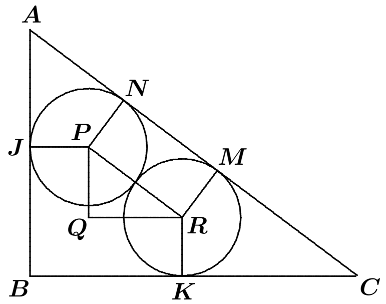
Also solved by Natalia Desy, student, Palembang, Indonesia; Isabel Díaz-Barrero and José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain; Jean-David Houle, Cégep de Drummondville, Drummondville, QC; and John DeLeon, Michelle Ellenburg, Morgan Lynch, Halley Newman, Christopher Odom, Mandy Rodgers, Josh Trejo, Tim Wilson, students, Angelo State University, San Angelo, TX, USA. Most solvers simply provided the completed grid.

M211. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Two circles of radius r are externally tangent. They are also internally tangent to the sides of a right triangle of sides 3, 4, and 5, with the hypotenuse of the triangle being tangent to both circles. Determine r.

Solution by Natalia Desy, student, Palembang, Indonesia.

Let ABC be the 3–4–5 right triangle in question, let r be the radius of each of the two circles, and let their points of tangency with triangle ABC be M, N, J, and K, as shown in the diagram.



By design, we have $AB = 3$, $BC = 4$, $CA = 5$. Let $x = AN = AJ$ and $y = CM = CK$. Then $QR = 4 - r - y$, $PQ = 3 - r - x$, and $PR = 2r = 5 - x - y$.

Since triangle PQR is similar to triangle ABC, we have

$$\frac{4-r-y}{2r} = \frac{4}{5}\,, \qquad \text{or} \qquad 5y = 20 - 13r \tag{1}$$

$$\text{and} \qquad \frac{3-r-x}{2r} = \frac{3}{5}\,, \qquad \text{or} \qquad 5x = 15 - 11r\,. \tag{2}$$

Now, multiplying $2r = 5 - x - y$ by 5 gives us $10r = 25 - 5x - 5y$. Using equations (1) and (2) in this yields

$$\begin{aligned} 25 - (15 - 11r) - (20 - 135) &= 10r \\ 14r &= 10 \\ r &= \tfrac{5}{7}\,. \end{aligned}$$

Also solved by Titu Zvonaru, Comănești, Romania.

M212. *Proposed by Robert Bilinski, Collège Montmorency, Laval, QC.*

In the computer program Excel, the columns are labelled with letters. The first 26 columns are labelled with the letters A to Z. The 27^{th} column is labelled AA; the 28^{th} column is labelled AB.

(a) What is the number of the column labelled DXA?

(b) What label appears on the 2005^{th} column?

Solution by Titu Zvonaru, Comănești, Romania; and Michelle Ellenburg and Christopher Odom, students, Angelo State University, San Angelo, TX.

(a) Let A, B, …, Z be equivalent to 1, 2, …, 26, in base 26. But we cannot have a digit equal to 0; hence $Z = A0$. The label DXA is in base 26. When we convert it to the decimal system, we get

26^2	26^1	26^0
D	X	A
4	24	1

Simplifying, we multiply and sum the values to get

$$4 \cdot 26^2 + 24 \cdot 26^1 + 1 \cdot 26^0 = 3329\,.$$

Therefore, the number of the column labelled DXA is 3329.

(b) The number 2005 can be expressed as $2 \cdot 26^2 + 25 \cdot 26^1 + 3$, which converts to base 26 as follows:

26^2	26^1	26^0
2	25	3
B	Y	C

Therefore, the label for the 2005^{th}column is BYC.

Also solved by Natalia Desy, student, Palembang, Indonesia; John DeLeon, student, Angelo State University, San Angelo, TX.; Jean-David Houle, Cégep de Drummondville, Drummondville, QC; and Mandy Rodgers and Joshua Trejo, students, Angelo State University, San Angelo, TX.

Problem of the Month

Ian VanderBurgh

This month, we will consider two problems with the same theme.

Problem 1 (2006 Gauss Contest (Grade 7)). A triangle can be formed having side lengths 4, 5, and 8. It is impossible however, to construct a triangle with side lengths 4, 5, and 10. Using the side lengths 2, 3, 5, 7, and 11, how many different triangles *with exactly two equal sides* can be formed?

So what is this all about? Just what are those mysterious first two sentences trying to tell us? Why can we make a triangle with certain side lengths and not with others? We could solve this problem in an intuitive way, but let's try to be more systematic.

With certain potential sets of side lengths, it makes more sense that a triangle cannot be formed than with other potential sets. For instance, it seems highly unlikely that we should be able to make a triangle with sides of length 1, 2, and 1000. With lengths that are closer together, it is not quite as clear (as in the case with sides of length 4, 5, and 10). What is the technical reason here?

This is an example of something called the *triangle inequality*, which says that in any triangle, the length of each side must be less than the sum of the lengths of the other two sides.

More technically, if $\triangle ABC$ has side lengths $AB = c$, $AC = b$, and $BC = a$, then we must have $c < a + b$, $b < a + c$, and $a < b + c$. How can we justify these facts? The easiest way is actually quite simple. Consider the two points A and B. What is the shortest path between A and B? Yes, you in the back... Yes, it is the straight line segment AB, whose length is c. Any other path from A to B is longer. In particular, going from A to B via C (a distance $AC + CB = b + a$) is longer, which means that $b + a > c$. We can obtain the inequalities $a + c > b$ and $b + c > a$ in a similar way.

At this stage, we could launch into the solution to Problem 1. However, let's hold off to make one more observation. Suppose that the lengths of the sides of $\triangle ABC$ satisfy $0 < a \leq b \leq c$. How many of the three inequalities $c < a + b$, $b < a + c$, and $a < b + c$ actually contain "useful" information? Since $b \geq a$ and $c > 0$, we get $b + c > a$ automatically. Similarly, $a + c > b$ automatically. Thus, only one of the three inequalities is worth considering, namely the one that says that the sum of the lengths of the two shorter sides is greater than the length of the longest side.

Solution to Problem 1: Consider a triangle with two equal sides. We write the side lengths as a, a, and b. Certainly $a + b > a$, which accounts for two of the three inequalities. The third states that $a + a > b$, or $2a > b$. In other words, if we are given the lengths a, a, and b, we need only check whether $b < 2a$ to determine whether the triangle inequality is satisfied by these lengths.

In this problem, the possible values for a and b are 2, 3, 5, 7, and 11. For each possible value of b, let's count the number of possible values for a with $2a > b$, remembering that a cannot equal b.

If $b = 2$, then a can be 3, 5, 7, or 11.
If $b = 3$, then a can be 2, 5, 7, or 11.
If $b = 5$, then a can be 3, 7, or 11. (a cannot be 2.)
If $b = 7$, then a can be 5 or 11. (a cannot be 2 or 3.)
If $b = 11$, then a can be 7. (a cannot be 2, 3, or 5.)

Adding up the possibilities, we discover that there are 14 different triangles that can be formed.

Wait! "Different" triangles? Yes, no two of them are congruent, since "side-side-side" is a valid check for congruency. "Can be formed"? Trickier, but still fine here—try to justify this on your own.

Here is a second problem which "sticks" to the same topic.

Problem 2 (2001 Gauss Contest (Grade 7)). A triangle can be formed having side lengths 4, 5, and 8. It is impossible, however, to construct a triangle with side lengths 4, 5, and 10. Ron has eight sticks, each having an integer length. He observes that he cannot form a triangle using any three of these sticks as side lengths. What is the shortest possible length of the longest of the eight sticks?

It looks like we have to use the pesky triangle inequality again. This problem also asks us to find the "minimum possible value of a maximum"—a standard type of problem, both in mathematics and in real life. It is like trying to determine the "worst case scenario".

Solution to Problem 2: Suppose that we write the lengths of the sticks in order as $a \leq b \leq c \leq d \leq \cdots$. Since a, b, and c do not form a triangle, we cannot have $c < a + b$; hence, $c \geq a + b$. To make c as small as possible given fixed a and b, we want $c = a + b$. Similarly, since b, c, and d cannot form a triangle, we want $d = b + c$, and so on.

Of course, to make everything as small as possible, we should start with a and b as small as possible. Since the smallest positive integer is 1, we choose $a = b = 1$. Using the idea of giving each new stick the sum of the lengths of the two previous sticks, we get lengths 1, 1, 2, 3, 5, 8, 13, and 21. This ensures that no triangle can be formed from any set of three sticks. Why? At each stage, the new longest stick will have a length equal to the sum of the two previous largest lengths and, therefore, at least as large as the sum of any two of the previous lengths.

Thus, the lengths are 1, 1, 2, 3, 5, 8, 13, and 21, and so the eighth stick has length 21. (Fibonacci strikes again!)

We see that the triangle inequality, though perhaps difficult to put one's finger on, can be useful. In fact, it appears throughout mathematics in many different guises—so always be on the lookout for it!

Pólya's Paragon

Playing Games with Mathematics (Part II)

John Grant McLoughlin

The September 2006 Paragon consisted of five challenges that were presented as the opening of a two-part Paragon. Here in Part II, the challenges are repeated and immediately followed by some insights into the mathematical principles beneath these challenges.

1. Sim

Six dots are drawn on a piece of paper to form the vertices of a hexagon. Two players are each assigned a colour. The players take turns joining any two of the dots with a line segment, using their assigned colours. The loser is the player who completes a triangle with three of the original six dots as its vertices and with all three edges the same colour.

Challenge: Prove that there must always be a loser (and a winner).

An outline of a proof by contradiction is provided here. Assume that it is possible to have a tie. Let the original dots be A, B, C, D, E, and F. Consider any one of these dots. Suppose we choose A. Note that there are five segments that can be drawn from A. Select two colours, say blue and red. At least three of the segments from A must be one of these colours, say blue. Again it does not matter which three segments. Suppose that AB, AC, and AE are blue. It follows that none of BC, BE, or CE are blue because otherwise a blue triangle would be formed, thus creating a loser. Aha! That makes triangle BCE a red triangle. Therefore, we have a proof by contradiction that a tie is impossible.

Readers interested in reading more about this and other combinatorial problems may refer to sources such as [4].

2. 31

This mental math game involves a running total which starts at zero. Each player has the choice to add 1, 2, 3, 4, 5, or 6 to the total. Players alternate turns. The winner is the player who is able to bring the total to 31.

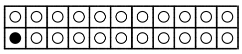
Challenge: Determine a winning strategy. You may choose to play first or second.

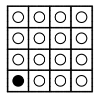
This is the easiest of the five challenges. Working backwards from 31, you can determine that a player who gets to 24 should win the game. Continuing backwards, we find that 17, 10, and 3 are winning positions. The winning strategy is to begin with 3 and ensure that you get each of 10, 17, and 24 on your way to 31. A misstep will allow your opponent to claim a winning position.

3. Chomp

Counters are placed in a rectangular grid such that one counter appears in each small rectangle. The counter in the bottom left-hand corner is a different colour than the others. Players take turns selecting one counter. If the counter selected occupies the bottom left-hand corner of a rectangle on the grid, all the counters in that rectangle are removed. The object is to force your opponent to select the differently coloured counter (the one in the bottom left-hand corner).

Challenge: Suppose that you play two games of Chomp in which the boards are $2 \times n$ and $k \times k$, examples of which are shown. Determine a winning strategy in each case. You may choose to play first or second.





This game appears in various sources. My reference is [5], which recommends [2] for further reading on the game. (Coincidentally, a review of the collection [2] will soon appear in ***CRUX with MAYHEM***.) We now turn our attention to the winning strategies for the pictured boards. The boards shown above are among those for which winning strategies do exist for the first player.

The solution for the $2 \times n$ board is to remove the single square in the top right-hand corner, thus, producing a symmetric situation in which the first player can copy the moves of the second player until the second player is forced to take the unwanted counter. The $k \times k$ example uses a similar idea: remove the entire figure except for the squares along the left edge and the bottom edge, thus leaving an L-shape of unit width. Observe that the first player can again engage in a copying strategy to ensure victory in the game.

Those of you who enjoy Chomp might want to consider yet another challenge posed in [5]: determine the unique opening move for a winning strategy with a 3×5 board.

4. Fifteen Finesse

The numbers 1, 2, 3, 4, 5, 6, 7, 8, and 9 are available for use in this game. Each number can be used only once. Two players alternate turns selecting one of the available numbers. To win the game, a player must obtain exactly three numbers that sum to 15. (Neither a pair of numbers, such as 7 and 8, nor a set of four numbers, such as 1, 3, 5, and 6, constitutes a winning combination.) The game ends in a draw if no player is able to acquire three numbers that sum to 15.

Challenge: Explain the underlying structure of the game, and suggest strategies that may help to win a game.

Does this game sound vaguely familiar? My experience has been that people play this game joyfully as they win, lose, or draw. Can you identify another game with an objective of three-in-a-row that you may win, lose, or draw while learning the game? Indeed, Tic-Tac-Toe! Where did we get three-in-a-row from the description of the game? Here we have an example of an isomorphism. That is, Fifteen Finesse is isomorphic to playing Tic-Tac-Toe on a magic square board. One such board is provided here for consideration:

8	1	6
3	5	7
4	9	2

This game can be an embarrassment! I suspect that if you recorded the sequence of number selections in early games and transposed them onto the board, you would not be so proud of some of your moves. It's fun to see how one played the game prior to having knowledge of the underlying structure. This game was introduced to me twenty years ago in [3], which has recently been reprinted by the MAA and is to be reviewed in ***CRUX with MAYHEM***.

5. A Polynomial in Transition

Consider the polynomial $x^2 + 10x + 20$. Under the conditions below, is it possible to convert this polynomial to $x^2 + 20x + 10$? Justify your answer.

Conditions:

(i) On each step you may only change the constant term or the coefficient of x (but not both).

(ii) The change must be an increase of 1 or a decrease of 1.

(iii) The change must NOT produce a polynomial that can be factored into the form $(x+m)(x+n)$ where m and n are integers. For example, you could not begin by reducing 10 to 9, since $x^2+9x+20 = (x+5)(x+4)$.

This challenge, drawn to my attention in a problem solving course in 1988 by Ed Barbeau (see [1]), can be solved readily with an insight. Suppose that we consider the polynomial to be of the form $x^2 + bx + c$. Consider the value $b - c$. This value begins at -10 and must end at 10 while changing by exactly ± 1 at each step. Therefore, there must be a point at which $b - c = 1$ if the transformation is possible. We would have a reducible form, however, using $(x + c)(x + 1)$. Therefore, it is impossible to complete the process.

References

[1] Edward J. Barbeau, *Polynomials*, Springer, New York, 1989.

[2] E.R. Berlekamp, J.H. Conway, and R.K. Guy, *Winning Ways for your Mathematical Plays*, A.K. Peters, 2004.

[3] Martin Gardner, *aha! Insight*, W.H. Freeman, San Francisco, 1978.

[4] Richard Hoshino and John Grant McLoughlin, *Combinatorial Explorations*, Canadian Mathematical Society, Ottawa, 2005.

[5] Ian Stewart, *Math Hysteria*, Oxford University Press, Oxford, 2004.

THE OLYMPIAD CORNER

No. 256

R.E. Woodrow

In this issue we present the problems of the three rounds of the Iranian Mathematical Olympiad 2002. Thanks go to Andy Liu, Canadian Team Leader to the IMO 2003 in Japan, for collecting the contests for our use.

IRANIAN MATHEMATICAL OLYMPIAD 2002

First Round

Time: 2×4.5 hours

1. Find all permutations $(a_1, \ldots, a_n)$ of $(1, \ldots, n)$ which have the property that $i+1$ divides $2(a_1 + \cdots + a_i)$ for every i, $1 \leq i \leq n$.

2. A rectangle is partitioned into small rectangles so that the edges of the small rectangles are parallel to the edges of the first rectangle. We call a point *a cross point* if it belongs to four different small rectangles. We call a segment *maximal* if there is no other segment containing it.

Show that the number of maximal segments plus the number of cross points is 3 less than the number of small rectangles.

3. In the convex quadrilateral $ABCD$, we have $\angle ABC = \angle ADC = 135^\circ$. There are two points M and N on the rays AB and AD, respectively, such that $\angle MCD = \angle NCB = 90^\circ$. The circumcircles of AMN and ABD intersect at A and K. Prove that $AK \perp KC$.

4. Let A and B be two fixed points in the plane. Let $ABCD$ be a convex quadrilateral such that $AB = BC$, $AD = DC$, and $\angle ADC = 90^\circ$. Prove that there is a fixed point P such that, for every such quadrilateral $ABCD$ on the same side of the line AB, the line DC passes through P.

5. Let δ be a symbol such that $\delta \neq 0$ and $\delta^2 = 0$. Define

$$\begin{array}{rcl} \mathbb{R}[\delta] & = & \{a + b\delta \mid a\,,\, b \in \mathbb{R}\} \\ a + b\delta = c + d\delta & \Longleftrightarrow & a = c \text{ and } b = d\,, \\ (a+b\delta) + (c+d\delta) & = & (a+c) + (b+d)\delta\,, \\ (a+b\delta)\cdot(c+d\delta) & = & ac + (ad+bc)\delta\,. \end{array}$$

Let $P(x)$ be a polynomial with real coefficients. Show that $P(x)$ has a multiple root in $\mathbb{R}$ if and only if $P(x)$ has a non-real root in $\mathbb{R}[\delta]$.

6. Let G be a simple graph with 100 edges on 20 vertices. We can choose a pair of disjoint edges in 4050 ways. Prove that G is regular.

Second Round

Time: 2×4.5 hours

1. The sequence $\{a_n\}$ is defined by $a_0 = 2$, $a_1 = 1$, and $a_{n+1} = a_n + a_{n-1}$ for $n \geq 1$. Show that if p is a prime factor of $a_{2k} - 2$, then p is a factor of $a_{2k+1} - 1$.

2. Let A be a point outside the circle Ω. The tangents from A to Ω touch Ω at B and C. A tangent L to Ω intersects AB and AC at P and Q, respectively. The line parallel to AC passing through P meets BC at R. Prove that as L varies, QR passes through a fixed point.

3. An ant moves on a straight path on the surface of a cube. If the ant reaches an edge, it goes on in such a way that if the cube were opened to make the adjacent faces coplanar, the path would become a straight line. If the ant reaches a vertex, it returns on the same path.

(a) Show that for every starting point of the ant, there are infinitely many directions for the ant to move in a periodic path.

(b) Show that if the ant starts on a fixed face, the periodicity of the path depends only on the direction (not the starting point).

4. Find the smallest positive integer n for which the following condition holds: For every finite set of points in the plane, if, for every n points in this set, there exist two lines covering all n points, then there exist two lines covering all points in the set.

5. Let I be the incentre of triangle ABC. Assume that the incircle touches AB and AC at X and Y, respectively. The line through X and I meets the incircle at M. Let X' be the point of intersection of AB and CM. Point L is on the segment $X'C$ such that $X'L = CM$. Prove that A, L, and Y are collinear if and only if $AB = AC$.

6. Let a, b, and c be positive real numbers such that $a^2 + b^2 + c^2 + abc = 4$. Prove that $a + b + c \leq 3$.

Third Round

Time: 2×4.5 hours

1. Find all real polynomials $P(x)$ such that $P(a) \in \mathbb{Z}$ implies that $a \in \mathbb{Z}$.

2. Let E be a fixed ellipse. Let B_1 be an arbitrary point outside E. The tangent from B_1 to E touches E at a point C_1. Let B_2 be a point on the line of B_1C_1 such that $B_1C_1 = C_1B_2$. For each positive integer i, define B_{i+1} in terms of B_i in this manner. Prove that the sequence $\{B_i\}$ is bounded in the plane.

3. In a triangle ABC, define C_a to be the circle tangent to AB, to AC, and to the incircle of the triangle ABC, and let r_a be the radius of C_a. Define r_b and r_c in the same way. Prove that $r_a + r_b + r_c \geq 4r$, where r is the inradius of the triangle ABC.

4. Let n and k be integers such that $2 \leq k \leq n$. Let $\mathcal{F}$ be a subset of $P(\{1, \ldots, n\})$ with the property that, for every F, $G \in \mathcal{F}$, there exists an integer t such that $1 \leq t \leq n$ and $\{t, t+1, \ldots, t+k-1\} \subseteq F \cap G$. Prove that $|\mathcal{F}| \leq 2^{n-k}$.

5. For every real number x define $\langle x \rangle = \min(\{x\}, \{1-x\})$, where $\{x\}$ denotes the fractional part of x. Prove that, for every irrational number α and every positive real number ε, there exists a positive integer n such that $\langle n^2\alpha \rangle < \varepsilon$.

We next give an alternative solution to problem 4 of the Hong Kong (China) Contest, for which we published a solution in the December 2005 number of the *Corner*.

4. [2004 : 84; 2005 : 522–523] *Hong Kong (China) Olympiad 1999.*
Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that, for all x, $y \in \mathbb{R}$,

$$f\big(x + yf(x)\big) = f(x) + xf(y)\,.$$

Alternate solution by B.J. Venkatachala, Indian Institute of Science, Bangalore, India.

The function which is identically 0 clearly satisfies the given condition,

$$f\big(x + yf(x)\big) = f(x) + xf(y)\,. \tag{1}$$

Now let f be any other function satisfying this condition. We will show that f must be both additive and multiplicative, which implies that $f(x) = x$ for all $x \in \mathbb{R}$.

Taking $x = 1$ and $y = 0$ in (1), we get $f(0) = 0$. If $f(x) = 0$ for some x, then

$$0 = f(x) = f\big(x + yf(x)\big) = f(x) + xf(y) = xf(y)\,.$$

Choosing y such that $f(y) \neq 0$, we see that $x = 0$. Thus, $f(x) = 0$ implies $x = 0$.

Putting $x = 1$, we get $f\big(1 + yf(1)\big) = f(1) + f(y)$, for all $y \in \mathbb{R}$. If $f(1) \neq 1$, we may choose $y = 1/\big(1 - f(1)\big)$. This gives $1 + yf(1) = y$; hence, we obtain $f(y) = f\big(1 + yf(1)\big) = f(1) + f(y)$ forcing $f(1) = 0$. This leads to the absurdity that $1 = 0$. Hence, $f(1) = 1$. Taking $x = 1$ in (1), we obtain $f(1 + y) = 1 + f(y)$ for all $y \in \mathbb{R}$.

Take any $x \neq 0$. Then $f(x) \neq 0$. Setting $y = 1/f(x)$ in (1), we get

$$f(x+1) \;=\; f(x) + xf\left(\frac{1}{f(x)}\right) .$$

We conclude that $f\left(\frac{1}{f(x)}\right) = \frac{1}{x}$ for all $x \neq 0$. Replacing y in (1) by $y/f(x)$ with $x \neq 0$, we get

$$f(x+y) \;=\; f(x) + xf\left(\frac{y}{f(x)}\right) , \tag{2}$$

valid for all $x \neq 0$ and $y \in \mathbb{R}$. Replacing x by $1/f(x)$ in (2) gives

$$f\left(\frac{1+yf(x)}{f(x)}\right) \;=\; \frac{1}{x} + \frac{1}{f(x)}f(yx) , \tag{3}$$

which is again valid for all $x \neq 0$ and $y \in \mathbb{R}$. Replacing y in (2) by $1 + yf(x)$ and using (3), we obtain

$$f\big(x+1+yf(x)\big) \;=\; f(x) + xf\left(\frac{1+yf(x)}{f(x)}\right) \;=\; f(x) + 1 + \frac{x}{f(x)}f(yx) .$$

Since $f(x+1) = f(x) + 1$ for all $x \in \mathbb{R}$, this simplifies to

$$f\big(x+yf(x)\big) \;=\; f(x) + \frac{x}{f(x)}f(yx) .$$

We then use (1) to obtain $xf(y) = \frac{x}{f(x)}f(yx)$. Since $x \neq 0$, we get $f(xy) = f(x)f(y)$. This last equation is valid for $x = 0$, since $f(0) = 0$. Thus, $f(xy) = f(x)f(y)$ for all x, $y \in \mathbb{R}$. Using this in (2), we get additivity:

$$f(x+y) \;=\; f(x) + xf(y)f\big(1/f(x)\big) \;=\; f(x) + f(y) ,$$

for all x, $y \in \mathbb{R}$. Thus, f is both additive and multiplicative. Since f is not the zero function, it follows that $f(x) = x$ for all $x \in \mathbb{R}$.

We now turn to solutions from our readers to problems of the 2nd Czech-Polish-Slovak Mathematical Competition, written in Zwardoń, Poland, June 2002 and given in [2005 : 152–153].

4. An integer $n > 1$ and a prime p are such that n divides $p-1$, and p divides $n^3 - 1$. Show that $4p - 3$ is the square of an integer.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

Since n divides $p-1$, we deduce that $p \equiv 1 \pmod{n}$ and $n \leq p-1$. It follows that $n - 1 < p$. Since p is prime, we have $\gcd(n-1, p) = 1$.

Since p divides $n^3 - 1 = (n-1)(n^2+n+1)$, it follows from Gauss' Theorem that p divides n^2+n+1. Let $n^2+n+1 = kp$, where k is a positive integer. Then $k \equiv kp = n^2+n+1 \equiv 1 \pmod{n}$. Moreover,

$$k \;=\; \frac{n^2+n+1}{p} \;\leq\; \frac{n^2+n+1}{n+1} \;<\; n+1\,.$$

Therefore, $k=1$ and $p = n^2+n+1$. Then

$$4p-3 \;=\; 4(n^2+n+1)-3 \;=\; (2n+1)^2\,,$$

and we are done.

5. In an acute-angled triangle ABC with circumcentre O, points P and Q lying respectively on sides AC and BC are such that

$$\frac{AP}{PQ} \;=\; \frac{BC}{AB} \quad\text{and}\quad \frac{BQ}{PQ} \;=\; \frac{AC}{AB}\,.$$

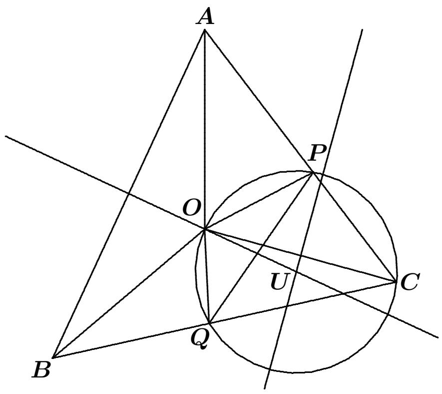
Show that the points O, P, Q, and C are concyclic.

Solution by Michel Bataille, Rouen, France.

We will use standard notation for the sides, angles, and circumradius of $\triangle ABC$. Define $k = AP/a$. Using the given equations, we get

$$k \;=\; \frac{AP}{a} \;=\; \frac{BQ}{b} \;=\; \frac{PQ}{c}\,.$$

Then $CP = b - AP = b - ka$ and $CQ = a - BQ = a - kb$.



The Law of Cosines gives

$$\begin{aligned} k^2c^2 &= PQ^2 \;=\; (a-kb)^2 + (b-ka)^2 - 2(a-kb)(b-ka)\cos C \\ &= (1+k^2)(a^2+b^2-2ab\cos C) - 4kab + 2k(a^2+b^2)\cos C \\ &= (1+k^2)c^2 - 2k\bigl(2ab - (a^2+b^2)\cos C\bigr)\,, \end{aligned}$$

and hence,

$$c^2 \;=\; 2k\bigl(2ab - (a^2+b^2)\cos C\bigr)\,.$$

But, using the Law of Sines, we have

$$\begin{aligned}
&2ab - (a^2+b^2)\cos C\\
&\qquad = \; 8R^2\sin A\sin B - 4R^2\cos C(\sin^2 A + \sin^2 B)\\
&\qquad = \; 4R^2\bigl[\cos(A-B) - \cos(A+B) - \cos C(\sin^2 A + \sin^2 B)\bigr]\\
&\qquad = \; 4R^2\sin^2 C\cos(A-B) \;=\; c^2\cos(A-B)\,,
\end{aligned}$$

where we have used the identity

$$\begin{aligned}
\cos(A-B)\cos(A+B) &= \cos^2 A(1-\sin^2 B) - \sin^2 B(1-\cos^2 A)\\
&= \cos^2 A - \sin^2 B\,.
\end{aligned}$$

It follows that $k = \dfrac{1}{2\cos(A-B)}$.

Since $\angle AOC = 2B$ and $OA = OC$, we have $\angle OAP = 90^\circ - B$. Thus,

$$\begin{aligned}
OP^2 &= OA^2 + AP^2 - 2OA\cdot AP\cos(90^\circ - B)\\
&= R^2 + k^2a^2 - 2kRa\sin B\\
&= 4R^2k^2\left(\frac{1}{4k^2} + \sin^2 A - \frac{1}{k}\sin A\sin B\right)\\
&= 4R^2k^2\bigl[\cos^2(A-B) + \sin^2 A - 2\cos(A-B)\sin A\sin B\bigr]\\
&= 4R^2k^2\bigl[\cos^2(A-B) + \sin^2 A\\
&\qquad\qquad - \cos(A-B)\bigl(\cos(A-B) - \cos(A+B)\bigr)\bigr]\\
&= 4R^2k^2(\sin^2 A + \cos^2 A - \sin^2 B) \;=\; 4R^2k^2\cos^2 B\,.
\end{aligned}$$

Therefore, $OP = 2kR\cos B$. Similarly, $OQ = 2kR\cos A$. Now,

$$\begin{aligned}
&OP\cdot CQ + OQ\cdot CP\\
&\quad = \; 4kR^2(\cos B\sin A - k\sin B\cos B + \cos A\sin B - k\sin A\cos A)\\
&\quad = \; 4kR^2\bigl[\sin(A+B) - k\sin(A+B)\cos(A-B)\bigr]\\
&\quad = \; 4kR^2\left(\sin C - \tfrac{1}{2}\sin C\right) \;=\; 2kR^2\sin C \;=\; R\cdot kc \;=\; OC\cdot PQ\,,
\end{aligned}$$

where we have used the identity

$$\sin(A+B)\cos(A-B) \;=\; \sin A\cos A + \sin B\cos B\,.$$

It follows from Ptolemy's Theorem that O, P, Q, C are concyclic.

Note: Since $AP\cdot AC = BQ\cdot BC$, the points A and B have the same power with respect to the circumcircle Γ of $\triangle CPQ$. Thus, letting U be the centre of Γ and ρ be the radius of Γ, we have $UA^2 - \rho^2 = UB^2 - \rho^2$. It follows that $UA = UB$, and U is on the perpendicular bisector of AB. This remark provides an easy construction of points P, Q satisfying the conditions of the problem: draw the perpendicular bisectors of AB and OC, which meet at U. Then the circle with centre U passing through C meets the sides AC and BC again at P and Q, respectively.

Next we move to the May 2005 number of the *Corner* and readers' solutions to problems of the Singapore Mathematical Olympiad 2002, Open Section, Part A, given in [2005 : 215–216].

1. Let $f(x)$ be a function which satisfies

$$f(29+x) \;=\; f(29-x)\,,$$

for all values of x. If $f(x)$ has exactly three real roots α, β, and γ, determine the value of $\alpha+\beta+\gamma$.

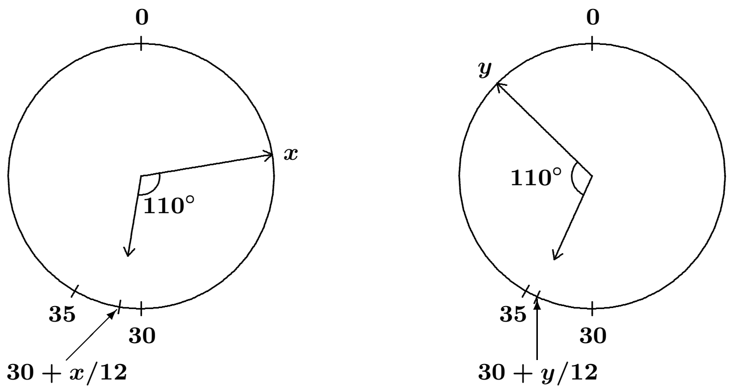
Solved by Robert Bilinski, Collège Montmorency, Laval, QC; Pierre Bornsztein, Maisons-Laffitte, France; Geoffrey A. Kandall, Hamden, CT, USA; Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina; Pavlos Maragoudakis, Pireas, Greece; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We given Krimker's solution.

Since f has exactly three real roots and f has the same value at points symmetric about 29, one of the roots must be 29. Let $\gamma = 29$. The other two roots, α and β, must be symmetric about 29; hence, $\alpha = 29 + x$ amd $\beta = 29 - x$ for some real number $x \neq 0$. Therefore,

$$\alpha+\beta+\gamma \;=\; (29+x)+(29-x)+29 \;=\; 87\,.$$

2. John left town A at x minutes past 6:00 pm and reached town B at y minutes past 6:00 pm the same day. He noticed that at both the beginning and the end of the trip, the minute hand made the same angle of 110 degrees with the hour hand on his watch. How many minutes did it take John to go from town A to town B?

Solved by Robert Bilinski, Collège Montmorency, Laval, QC; Geoffrey A. Kandall, Hamden, CT, USA; and Pavlos Maragoudakis, Pireas, Greece. We give the solution of Kandall.



It is implicit that $0 < x < y < 60$. Since an angle of $110°$ corresponds to $55/3$ minutes, we have the following equations:

$$\left(30+\frac{x}{12}\right)-x \;=\; \frac{55}{3}\,, \qquad y-\left(30+\frac{y}{12}\right) \;=\; \frac{55}{3}\,.$$

Hence, $\frac{11}{12}x = 30-\frac{55}{3}$ and $\frac{11}{12}y = 30+\frac{55}{3}$, from which we get $y-x=40$. This is the time of the trip in minutes.

3. Let $x_1 = \dfrac{1}{2002}$. For $n \geq 1$, define $nx_{n+1} = (n+1)x_n + 1$. Find x_{2002}.

Solved by Robert Bilinski, Collège Montmorency, Laval, QC; Pierre Bornsztein, Maisons-Laffitte, France; Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina; Pavlos Maragoudakis, Pireas, Greece; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give the write-up of Krimker.

The sequence $x_n = n/2002 + n - 1$ satisfies the recurrence relation of the problem. Indeed, $x_1 = 1/2002$, and

$$\begin{aligned} nx_{n+1} &= n\left(\frac{n+1}{2002}+n\right) \;=\; \frac{n(n+1)}{2002}+n^2 \\ &= (n+1)\left(\frac{n}{2002}+n-1\right)+1 \;=\; (n+1)x_n+1\,. \end{aligned}$$

Then, $x_{2002} = 2002$.

4. For integers $n \geq 1$, let $a_n = n^2 + 500$ and $d_n = \gcd(a_n, a_{n+1})$. Determine the largest value of d_n.

Solved by Pierre Bornsztein, Maisons-Laffitte, France; Pavlos Maragoudakis, Pireas, Greece; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give the solution by Wang, modified by the editor.

The answer is 2001, attained when $n = 1000$.

Since d_n divides both a_n and a_{n+1}, it follows that d_n divides

$$a_{n+1}-a_n \;=\; (n+1)^2+500-(n^2+500) \;=\; 2n+1\,.$$

Then d_n divides $n(2n+1) = 2n^2+n$, and consequently, d_n also divides $2n^2+n-2a_n = n-1000$. Since $2n+1-2(n-1000) = 2001$, we deduce that $d_n \mid 2001$.

Suppose $d_n = 2001 = 3\cdot 23\cdot 29$. Then d_n is divisible by 3, 23, and 29. Since $2n+1 \equiv 0 \pmod 3$, we have $2n \equiv -1 \equiv 2 \pmod 3$; that is,

$$n \;\equiv\; 1 \pmod 3\,. \tag{1}$$

Similarly, since $2n+1 \equiv 0 \pmod{23}$, we have $2n \equiv -1 \equiv 22$, or

$$n \;\equiv\; 11 \pmod{23}\,, \tag{2}$$

and since $2n + 1 \equiv 0 \pmod{29}$, we have $2n \equiv -1 \equiv 28$, or

$$n \equiv 14 \pmod{29} . \tag{3}$$

Applying the Chinese Remainder Theorem and using the standard method, we let $M_1 = 23 \cdot 29 = 667$, $M_2 = 3 \cdot 29 = 87$ and $M_3 = 3 \cdot 23 = 69$, and we then solve the following system of congruences:

$$\begin{aligned} 667x &\equiv 1 \pmod{3} , & (4)\\ 87x &\equiv 1 \pmod{23} , & (5)\\ 69x &\equiv 1 \pmod{29} . & (6) \end{aligned}$$

By routine methods, we find the least positive solutions of (4), (5), and (6) to be $x_1 = 1$, $x_2 = 9$, and $x_3 = 8$, respectively. Hence, a solution to the system (1), (2), and (3) is given by

$$n = 1 \cdot 667 \cdot 1 + 11 \cdot 87 \cdot 9 + 14 \cdot 69 \cdot 8 = 17008 \equiv 1000 \pmod{2001} .$$

Conversely, when $n = 1000$, we have

$$\begin{aligned} d_n &= \gcd(1000^2 + 500, 1001^2 + 500) \\ &= \gcd(500 \cdot 2001, 501 \cdot 2001) = 2001 . \end{aligned}$$

Thus, 2001 is a value for d_n (attained when $n = 1000$). There cannot be any larger value for d_n, since d_n divides 2001.

5. It is given that the polynomial $p(x) = x^3 + ax^2 + bx + c$ has three distinct positive integer roots and $p(2002) = 2001$. Let $q(x) = x^2 - 2x + 2002$. It is also given that the polynomial $p\big(q(x)\big)$ has no real roots. Determine the value of a.

Solved by Pierre Bornsztein, Maisons-Laffitte, France; Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina; and Pavlos Maragoudakis, Pireas, Greece. We give the solution by Krimker, modified by the editor.

The polynomial $q(x) = (x - 1)^2 + 2001$ takes on every value in the interval $[2001, \infty)$. Since $p\big(q(x)\big)$ has no real roots, all three roots of $p(x)$ must be less than 2001. Denoting the roots of $p(x)$ by x_1, x_2, x_3, we have

$$p(x) = (x - x_1)(x - x_2)(x - x_3) .$$

Then $p(2002) = (2002 - x_1)(2002 - x_2)(2002 - x_3) = 2001 = 3 \cdot 23 \cdot 29$. Since each factor $2002 - x_i$ is a positive integer, we must have

$$\{2002 - x_1,\ 2002 - x_2,\ 2002 - x_3\} = \{3,\ 23,\ 29\} ,$$

and hence $\{x_1, x_2, x_3\} = \{1999, 1979, 1973\}$. Then

$$a = -(x_1 + x_2 + x_3) = -5951 .$$

6. Find the largest positive integer N such that $N!$ ends with exactly twenty-five "zero" digits.

Solved by Robert Bilinski, Collège Montmorency, Laval, QC; Pierre Bornsztein, Maisons-Laffitte, France; Pavlos Maragoudakis, Pireas, Greece; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Wang's solution.

Let $f(N)$ denote the number of zeroes at the end of $N!$. Then $f(N)$ equals the number of factors of 10 that may be formed in $N!$. Since $10 = 5\times 2$ and there are clearly fewer 5s than 2s, it follows that $f(N)$ is equal to the number of factors of 5 in $1 \cdot 2 \cdot 3 \cdots N$. Then (by a well-known formula)

$$f(N) \;=\; \sum_{k=0}^{\infty} \left\lfloor \frac{N}{5^k} \right\rfloor .$$

By straightforward computations, we find that

$$f(109) \;=\; \left\lfloor \frac{109}{5} \right\rfloor + \left\lfloor \frac{109}{25} \right\rfloor \;=\; 21+4 \;=\; 25\,,$$

while

$$f(110) \;=\; \left\lfloor \frac{110}{5} \right\rfloor + \left\lfloor \frac{100}{25} \right\rfloor \;=\; 22+4 \;=\; 26\,.$$

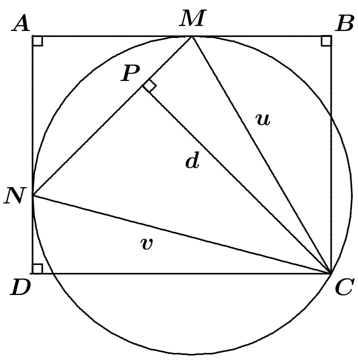
Since f is an increasing function, we see that the required value of N is 109.

7. A circle passes through the vertex C of a rectangle $ABCD$ and touches its sides AB and AD at points M and N, respectively. Suppose the distance from C to MN is 2 cm. Find the area of $ABCD$ in cm^2.

Solved by Bruce Crofoot, Thompson Rivers University, Kamloops, BC; Geoffrey A. Kandall, Hamden, CT, USA; and Pavlos Maragoudakis, Pireas, Greece. We present a composite of the solutions by Crofoot and Kandall.

The answer is $[ABCD] = 4$ cm^2.

More generally, let d be the distance from C to MN. We introduce additional notation as shown in the diagram. Since AB is tangent to the circle at M, we have



$$\angle BMC \;=\; \angle MNC \;=\; \angle PNC\,.$$

Hence, $\triangle BMC$ is similar to $\triangle PNC$. Similarly, $\triangle DNC$ is similar to $\triangle PMC$. Consequently,

$$\frac{BC}{d} \;=\; \frac{u}{v} \quad\text{and}\quad \frac{DC}{d} \;=\; \frac{v}{u}\,.$$

From these two equations, we get $BC \cdot DC = d^2$; that is, $[ABCD] = d^2$.

8. Let $m = 144^{\sin^2 x} + 144^{\cos^2 x}$. How many such m's are integers?

Solved by Robert Bilinski, Collège Montmorency, Laval, QC; Pierre Bornsztein, Maisons-Laffitte, France; Pavlos Maragoudakis, Pireas, Greece; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give the solution by Wang.

The answer is 122. Indeed, we prove that all the integers from 24 to 145 inclusive are attainable.

Consider the function $f(t) = 144^t + 144^{1-t}$ where $t = \sin^2 x$. Then $0 \le t \le 1$. Since $f'(t) = (\ln 144)(144^t - 144^{1-t})$, we have $f'(t) = 0$ if and only if $t = \frac{1}{2}$. Since $f'(t) > 0$ if and only if $\frac{1}{2} < t < 1$, we see that f is decreasing on $\left(0, \frac{1}{2}\right)$ and increasing on $\left(\frac{1}{2}, 1\right)$. Hence, the absolute minimum of f is $f\left(\frac{1}{2}\right) = 24$ (attained when $x = \frac{\pi}{4}$, for example), and the absolute maximum of f is $f(0) = f(1) = 145$ (attained when $x = 0$, for example).

Since f is a continuous function, the Intermediate Value Theorem then guarantees that every integer between 24 and 145 is also attainable, and our claim follows.

9. Evaluate $\sum\limits_{k=1}^{2002} \frac{k \cdot k!}{2^k} - \sum\limits_{k=1}^{2002} \frac{k!}{2^k} - \frac{2003!}{2^{2002}}$.

Solved by Pierre Bornsztein, Maisons-Laffitte, France; Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Krimker's write-up.

The expression we are trying to evaluate may be rewritten as

$$\sum_{k=1}^{n} \frac{(k-1)k!}{2^k} - \frac{(n+1)!}{2^n},$$

where $n = 2002$. We will prove by induction that this is equal to -1 for every positive integer n.

Note that the claim is true for $n = 1$. Suppose now that $n \ge 2$ and that the equality is valid for $n - 1$. We will show that it holds for n. Indeed,

$$\begin{aligned}
\sum_{k=1}^{n} \frac{(k-1) \cdot k!}{2^k} &= \sum_{k=1}^{n-1} \frac{(k-1) \cdot k!}{2^k} + \frac{n!(n-1)}{2^n} \\
&= \frac{n!}{2^{n-1}} - 1 + \frac{n!(n-1)}{2^n} \\
&= \frac{2n! + n!(n-1)}{2^n} - 1 = \frac{n!(2+n-1)}{2^n} - 1 \\
&= \frac{n!(n+1)}{2^n} - 1 = \frac{(n+1)!}{2^n} - 1\,.
\end{aligned}$$

Thus, the claim is true for all positive integers n.

10. How many ways are there to arrange 5 identical red, 5 identical blue, and 5 identical green marbles in a straight line such that every marble is adjacent to at least one marble of the same colour as itself?

Solution by Pavlos Maragoudakis, Pireas, Greece.

There are 426 ways.

Each set of 5 marbles of the same colour must remain together or else be separated into two groups, with 2 adjacent marbles and 3 adjacent marbles. Thus, we have the following cases:

(i) The three quintuplets of the same colour remain 'united'.

There are then 3 groups of marbles. We have $3! = 6$ ways to arrange them.

(ii) We choose one colour and 'break' it into the two possible parts, while we leave the other two colours 'united'.

There are then 4 groups of marbles. The two parts of the 'broken' colour should be non-adjacent among the 4 groups; thus, we have 3 choices: (1^{st},3^{rd}), or (1^{st},4^{th}), or (2^{nd},4^{th}). We have $3 \cdot 3 \cdot 2 \cdot 2 = 36$ ways.

(iii) We choose two colours and 'break' them.

There are then 5 groups of marbles. We place the two parts of the 1^{st} 'broken' colour in two non-adjacent spots, and we do the same for the 2^{nd} 'broken' colour. We have 6 choices.

1^{st}colour	2^{nd}colour
(1^{st},3^{rd})	(2^{nd},4^{th})
(1^{st},3^{rd})	(2^{nd},5^{th})
(1^{st},4^{th})	(2^{nd},5^{th})
(1^{st},4^{th})	(3^{rd},5^{th})
(1^{st},5^{th})	(2^{nd},4^{th})
(2^{nd},4^{th})	(3^{rd},5^{th})

We have $6 \cdot 3 \cdot 2 \cdot 2 \cdot 2 = 144$ ways.

(iv) We 'break' all colours.

This gives 6 groups of marbles. We have 5 choices:

1^{st}colour	2^{nd}colour	3^{rd}colour
(1^{st},3^{rd})	(2^{nd},5^{th})	(4^{th},6^{th})
(1^{st},4^{th})	(2^{nd},5^{th})	(3^{rd},6^{th})
(1^{st},5^{th})	(2^{nd},4^{th})	(3^{rd},6^{th})
(1^{st},6^{th})	(2^{nd},4^{th})	(3^{rd},5^{th})
(1^{st},4^{th})	(2^{nd},4^{th})	(3^{rd},5^{th})

We have $5 \cdot 3 \cdot 2 \cdot 2 \cdot 2 \cdot = 240$ ways.

Altogether, we have $6 + 36 + 144 + 240 = 426$ ways.

Now we turn to Part B of the Singapore Mathematical Olympiad given in [2005 : 216].

2. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be real numbers between 1001 and 2002 inclusive. Suppose $\sum\limits_{i=1}^{n} a_i^2 = \sum\limits_{i=1}^{n} b_i^2$. Prove that

$$\sum_{i=1}^{n} \frac{a_i^3}{b_i} \ \geq \ \frac{17}{10} \sum_{i=1}^{n} a_i^2 \,.$$

Determine when equality holds.

Solution by Pierre Bornsztein, Maisons-Laffitte, France, modified by the editor.

There is a misprint. The given inequality is false if $a_i = b_i$ for each i. The correct inequality is

$$\sum_{i=1}^{n} \frac{a_i^3}{b_i} \ \leq \ \frac{17}{10} \sum_{i=1}^{n} a_i^2 \,.$$

We will now prove this inequality.

For each i, we have

$$\frac{1}{2} \ = \ \frac{1001}{2002} \ \leq \ \frac{a_i}{b_i} \ \leq \ \frac{2002}{1001} \ = \ 2\,,$$

and therefore $(2a_i - b_i)(2b_i - a_i) \geq 0$; that is,

$$5a_ib_i \ \geq \ 2(a_i^2 + b_i^2)\,. \tag{7}$$

Multiplying this inequality by a_i/b_i, we get

$$5a_i^2 \ \geq \ 2\frac{a_i^3}{b_i} + 2a_ib_i\,. \tag{8}$$

From (7), we have $2a_ib_i \geq \frac{4}{5}(a_i^2 + b_i^2)$. Using this inequality in (8), we obtain

$$5a_i^2 \ \geq \ 2\frac{a_i^3}{b_i} + \frac{4}{5}(a_i^2 + b_i^2)\,,$$

which may be rewritten as

$$\frac{a_i^3}{b_i} \ \leq \ \frac{21}{10}a_i^2 - \frac{2}{5}b_i^2\,. \tag{9}$$

Note that equality occurs in (9) if and only if $b_i = 2a_i$ or $a_i = 2b_i$; that is, if and only if $(a_i, b_i) = (1001, 2002)$ or $(a_i, b_i) = (2002, 1001)$.

Summing over i in (9) and recalling that $\sum\limits_{i=1}^{n} b_i^2 = \sum\limits_{i=1}^{n} a_i^2$, we get

$$\sum_{i=1}^{n} \frac{a_i^3}{b_i} \;\le\; \frac{21}{10}\sum_{i=1}^{n} a_i^2 - \frac{2}{5}\sum_{i=1}^{n} a_i^2 \;=\; \frac{17}{10}\sum_{i=1}^{n} a_i^2\,,$$

as desired.

Equality occurs if and only if, for each i, either $(a_i, b_i) = (1001, 2002)$ or $(a_i, b_i) = (2002, 1001)$. The condition $\sum\limits_{i=1}^{n} a_i^2 = \sum\limits_{i=1}^{n} b_i^2$ can be rewritten as $1001^2 p + (n-p)2002^2 = 2002^2 p + 1001^2(n-p)$, which is $p = \frac{1}{2}n$. Thus, equality occurs if and only if n is even and $(a_i, b_i) = (1001, 2002)$ for half of the subscripts i while $(a_i, b_i) = (2002, 1001)$ for the other half.

3. Let n be a positive integer. Determine the smallest possible sum

$$a_1b_1 + a_2b_2 + \cdots + a_{2n+2}b_{2n+2}\,,$$

where a_1, a_2, ..., a_{2n+2} and b_1, b_2, ..., b_{2n+2} are rearrangements of the binomial coefficients

$$\binom{2n+1}{0},\quad \binom{2n+1}{1},\quad \ldots,\quad \binom{2n+1}{2n+1}.$$

Justify your answer.

Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Pavlos Maragoudakis, Pireas, Greece. We give Bornsztein's solution.

According to the rearrangement inequality, the sum is minimized when one of the sequences a_1, a_2, ..., a_{2n+2} and b_1, b_2, ..., b_{2n+2} is increasing and the other is decreasing. Since the binomial coefficients are increasing from $\binom{2n+1}{0}$ to $\binom{2n+1}{n+1}$ and decreasing from $\binom{2n+1}{n+1}$ to $\binom{2n+1}{2n+1}$, the minimal sum is

$$2\sum_{k=0}^{n}\binom{2n+1}{k}\binom{2n+1}{n+1+k} \;=\; 2\sum_{k=0}^{n}\binom{2n+1}{k}\binom{2n+1}{n-k}\,,$$

where the last step uses the well-known identity $\binom{m}{j} = \binom{m}{m-j}$.

Now consider a group of $2n+1$ boys and $2n+1$ girls. We want to select n persons from this group. There are clearly $\binom{4n+2}{n}$ ways to do that. On the other hand, letting k be the number of boys in the selected group, we see that the total number of ways is also $\sum\limits_{k=0}^{n}\binom{2n+1}{k}\binom{2n+1}{n-k}$. Therefore, the minimal sum is $2\binom{4n+2}{n}$.

4. Find all real-valued functions $f : \mathbb{Q} \longrightarrow \mathbb{R}$ defined on the set of all rational numbers $\mathbb{Q}$ satisfying the conditions

$$f(x+y) \;=\; f(x) + f(y) + 2xy\,,$$

for all x, y in $\mathbb{Q}$ and $f(1) = 2002$. Justify your answers.

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; and Pavlos Maragoudakis, Pireas, Greece. We present Bataille's solution.

It is readily checked that the function $r \mapsto r(r + 2001)$ is a solution. We will show that it is unique. Let f be an arbitrary solution. Denote by $\mathcal{C}$ the given condition $f(x+y) = f(x)+f(y)+2xy$, and fix $z \in \mathbb{Q}$, $z > 0$. It is easily proved that $f(nz) = n\big(f(z) + (n-1)z^2\big)$ for all $n \in \mathbb{N}$ (by induction, using $\mathcal{C}$ with $x = nz$ and $y = z$ for the inductive step). Then, for all $n \in \mathbb{N}$,

$$2002 \;=\; f(1) \;=\; f\left(n \times \frac{1}{n}\right) \;=\; n\left(f\left(\frac{1}{n}\right) + (n-1)\cdot\frac{1}{n^2}\right)\,,$$

and thus, $f\left(\frac{1}{n}\right) = \frac{1}{n}\left(2001 + \frac{1}{n}\right)$. Then, for all positive integers m and n,

$$f\left(\frac{m}{n}\right) \;=\; f\left(m \times \frac{1}{n}\right) \;=\; m\left(f\left(\frac{1}{n}\right) + (m-1)\cdot\frac{1}{n^2}\right) \;=\; \frac{m}{n}\left(\frac{m}{n} + 2001\right).$$

Thus, $f(r) = r(r + 2001)$ holds for all positive $r \in \mathbb{Q}$. Since $f(0) = 0$ (condition $\mathcal{C}$ with $x = y = 0$) and $f(-r) = 2r^2 - f(r)$ (condition $\mathcal{C}$ with $x = r$ and $y = -r$), it can be verified that $f(r) = r(r+2001)$ actually holds for all $r \in \mathbb{Q}$. This completes the proof.

To finish this number of the *Corner*, we give solutions by our readers to problems of the XVIII Italian Mathematical Olympiad, Cesenatico, Italy, May 2002, given in [2005 : 217].

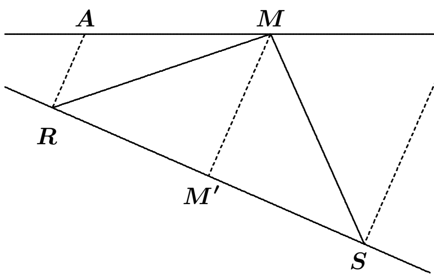
1. Find all 3–digit positive integers that are 34 times the sum of their digits.

Solved by Robert Bilinski, Collège Montmorency, Laval, QC; Pierre Bornsztein, Maisons-Laffitte, France; Geoffrey A. Kandall, Hamden, CT, USA; Pavlos Maragoudakis, Pireas, Greece; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Bilinski's write-up.

Let abc be a 3–digit positive integer with the required property (where a, b, and c are the digits). Then $100a + 10b + c = 34(a + b + c)$, which simplifies to $22a = 8b + 11c$. This equation implies that b is divisible by 11. Since b is a digit, we must have $b = 0$, and then $c = 2a$. This gives us the numbers 102, 204, 306, and 408. It can be verified that each of these is a solution.

3. Let A and B be two points of the plane, and let M be the mid-point of AB. Let r be a line, and let R and S be the projections of A and B onto r. Assuming that A, M, and R are not collinear, prove that the circumcircle of triangle AMR has the same radius as the circumcircle of BSM.

Solved by Michel Bataille, Rouen, France; and Pavlos Maragoudakis, Pireas, Greece. We present Bataille's solution.



Let M' be the projection of M onto r. Then M' is the mid-point of RS (since M is the mid-point of AB) and $MM' \perp RS$. It follows that $\triangle RMS$ is isosceles with

$$RM \;=\; MS\,. \tag{1}$$

Now, let ρ_a and ρ_b be the circumradii of $\triangle AMR$ and $\triangle BMS$, respectively. We have

$$2\rho_a \;=\; \frac{RM}{\sin(\angle RAM)}\,,\quad 2\rho_b \;=\; \frac{SM}{\sin(\angle SBM)}\,. \tag{2}$$

But $\angle RAM + \angle SBM = 180^\circ$ (since $AR \parallel BS$); hence,

$$\sin(\angle RAM) \;=\; \sin(\angle SBM)\,. \tag{3}$$

From (1), (2), and (3), we obtain $\rho_a = \rho_b$.

4. Find all values of n for which all solutions of the equation $x^3 - 3x + n = 0$ are integers.

Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Pavlos Maragoudakis, Pireas, Greece. We give Bornsztein's solution.

If the given equation has an integer root α, then $n = -\alpha^3 + 3\alpha$ is an integer too. Therefore, the values of n that we seek must all be integers.

Let $f(x) = x^3 - 3x$. The given equation is then $f(x) = -n$. Straightforward computations show that f is increasing on $(-\infty, -1]$ and $[1, +\infty)$, and decreasing on $[-1, 1]$. Moreover, $f(-1) = 2$ and $f(1) = -2$. Thus, the equation $f(x) = -n$ has three real roots if and only if $|n| \le 2$. Therefore, $n \in \{-2, -1, 0, 1, 2\}$. Furthermore, one of the integer solutions has to be -1, 0, or 1; thus, $n \in \{f(-1), f(0), f(1)\} = \{-2, 0, 2\}$. Direct checking shows that the desired values are $n = -2$ and $n = 2$.

5. Prove that, if $m = 5^n + 3^n + 1$ is prime, then 12 divides n.

Solved by Pierre Bornsztein, Maisons-Laffitte, France; Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina; Pavlos Maragoudakis, Pireas, Greece; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Krimker's solution.

By the Division Algorithm, we have $n = 12q + r$ for some integers q and r such that $0 \le r \le 11$. In the following three cases we will apply Fermat's Little Theorem and elementary congruence properties.

Case 1. r is odd; that is, $r = 2k + 1$. Then

$$m = 5^{12q+2k+1} + 3^n + 1 \equiv \left(5^2\right)^{6q}\left(5^2\right)^k 5 + 1 \equiv 5 + 1 \equiv 0 \pmod{3} .$$

Case 2. $r = 2$, $r = 6$, or $r = 10$; that is, $r = 4k + 2$ with $0 \le k \le 2$. Then

$$m = 5^n + 3^{12q+4k+2} + 1 \equiv \left(3^4\right)^{3q}\left(3^4\right)^k 3^2 + 1 \equiv 3^2 + 1 \equiv 0 \pmod{5} .$$

Case 3. $r = 4$ or $r = 8$; that is $r = 4k$ with $k = 1$ or $k = 2$. Then

$$\begin{aligned} m &= 5^{12q+4k} + 3^{12q+4k} + 1 = \left(5^6\right)^{2q}\left(5^4\right)^k + \left(3^6\right)^{2q}\left(3^4\right)^k + 1 \\ &\equiv 2^k + 4^k + 1 \equiv 0 \pmod{7} . \end{aligned}$$

Since m is prime and $m \ge 9$, none of the cases above are possible. Thus, $r = 0$, and 12 divides n.

That completes this number of the *Corner*. This is a call for solutions—readers will have noted that we are rapidly clearing our backlog and will soon be in a position to publish solutions within a year of giving the contests in ***Crux Mathematicorum***. We need your contributions of nice solutions and generalizations, preferably within 8 months of the appearance of the problem.

BOOK REVIEW

John Grant McLoughlin

Index to Mathematical Problems 1975–1979
Edited by Stanley Rabinowitz and Mark Bowron, published by MathPro Press (Chelmsford, MA), 1999.
ISBN 0-9626401-2-3, hardcover, 8.5×11, x+518 pages, US$69.95.
Reviewed by ***Edward J. Barbeau****, University of Toronto.*

While there are standard ways to track down what is known in mathematical fields, it can be daunting to determine the history of a problem and its current status. Problems come in and out of fashion. They may be passed around by word of mouth, appear on contests in different places at different times, or be posed in any number of journals having problems sections. They may be picked up and abandoned, and their solutions (if any) communicated in various ways. Accordingly, it is difficult to discover their provenance, study their development, and assess their novelty. Many collections of problems have appeared in recent decades, particularly in Eastern Europe, but often these are collections from particular contests or reflect the idiosyncracies of the compilers of the collection. A reliable guide to more recent and notorious problems is Martin Gardner, whose *Scientific American* articles and many books cover a lot of territory. However, few of these provide a thematic index to problems and none are comprehensive enough to serve as a definitive guide to the literature. We have had to rely on a few individuals with a lot of experience and prodigious memories to impose any kind of order on the corpus of problems.

One such person is Stanley Rabinowitz, who received his doctorate in convexity, combinatorics, and number theory from the Polytechnic University of New York and is a software engineer and computer consultant. In 1989, he founded *MathPro Press* to produce problem indexes as well as collections of problems, compendia of mathematical results, and books on the use of computers in the solution of mathematical problems. Rabinowitz became convinced of the necessity of a source work to which problemists could turn for material and information on the origin and status of problems they come across. But how was he to deal with the chaotic wealth of material? His solution was to list all those problems that appeared in a selection of popular journals and contests and provide tools for searchers to find their way around. In 1992, he broke new ground with his publication of the *Index to Mathematical Problems 1980–1984*, which listed and classified all the problems from standard sources published in the English language during those years. This achievement was well received by the community, which eagerly awaited its successors. Finally, in 1999, the present volume was published with the collaboration of Mark Bowron, whose day job is driving an 18–wheeler transport truck.

The core of the book is the *Subject Index* (SI), 256 pages with the texts of about 4000 problems, listed by subject according to 17 classifications with many subheadings. Each problem appears with its author and source identification. The SI is an essentially complete collection of problems from 23 journals, mostly North American, and 6 competitions (Olympiads from Canada, USA, Australia, along with the International Mathematical Olympiads, and the Putnam and Kürchák contests). Canada is represented by the *Canadian Mathematical Bulletin* (which used to have a problem section), ***Crux Mathematicorum***, the *Ontario Mathematics Gazette*, and the *Ontario Secondary School Mathematics Bulletin*.

This is supplemented by other indexes:

- *Subject classification scheme*: list of titles in the SI with page references to the SI;
- *Problem locator*: list of journals and problem numbers with page references to the SI;
- *Problem chronology*: list of journals and problems, along with references to external sources with solutions and comments;
- *Author index*, with a key to pseudonyms;
- *Title index*, based on titles or topics assigned by journal editors;
- *Journal issue checklist*: details about publishers, problems editors, along with lists of problems according to issue;
- *Unsolved problems*: eight pages with complete statements;
- Citation index: references made in the period 1975–1979 to current and previously published problems in the journals;
- *Bibliography*;
- *Keyword index*.

Many readers will be drawn to the unsolved problems. Some of these are well known, such as the Collatz or $3x + 1$ problem (CRUX problem #133 [1976 : 67]). Some look tedious, technical or deep; others are more enticing. For example:

- Suppose that each square of an $n \times n$ chessboard is coloured either black or white. A square, formed by horizontal and vertical lines of the board, will be called chromatic if its four distinct corner squares are all of the same colour. Find the smallest n such that, with any such colouring, every $n \times n$ board must contain a chromatic square. (*American Mathematical Monthly* #6211)

- Let A and B be the unique non-decreasing sequences of odd integers and even integers, respectively, such that, for all $n \geq 1$, the number of integers i satisfying $A_i = 2n - 1$ is A_n and the number of integers i satisfying $B_i = 2n$ is B_n. That is,

$$\begin{aligned} A &= \{1, 3, 3, 3, 5, 5, 5, 7, 7, 7, 9, 9, 9, 9, 9, \dots\} \\ \text{and} \quad B &= \{2, 2, 4, 4, 6, 6, 6, 6, 8, 8, 8, 8, \dots\}\,. \end{aligned}$$

  Is the difference $|A_n - B_n|$ bounded? (*Mathematics Magazine* #1073)

Then there is this one from Paul Erdős:

- Find a sequence of positive integers $1 \leq a_1 < a_2 < a_3 < \cdots$ that omits infinitely many integers from every arithmetic progression (in fact, it has density 0), but which contains all but a finite number of terms of every geometric progression. Prove also that there is a set S of real numbers which omits infinitely many terms of any arithmetic progression, but contains every geometric progression (disregarding a finite number of terms). (*Pi Mu Epsilon Journal* #389)

The scholarship underpinning this volume is impressive. The editors have performed the difficult task of classifying the problems, matched up various versions of names and pseudonyms, tracked down if and where solutions may be found, and checked the accuracy of the statement of problems and the information provided about them. But the collection scoops up indiscriminately from its sources, so there is quite a mixed bag of problems. They may be fascinating and ingenious or banal, very difficult or trivially easy, significant challenges or dull exercises, novel or trite, narrowly specialized or broadly appealing, memorable or ephemeral, singular or type-cast. They can be solved by systematic techniques or through the right perspective and strokes of inspiration. Of course, the book does not offer guidance and the reader just has to dive in and try the problems. However, a teacher on the lookout for material suitable for various situations will find her efforts rewarded by problems of the right level of difficulty.

The long term viability of Rabinowitz's project is questionable. Problems are pouring into the literature regularly, many of which are neither novel nor particularly inventive. It is clear that the influx of new material is bound to outrun the capacity of the most energetic publisher to collect and classify it. Only the Internet can begin to keep on top of this task, and it may be that future books will have to focus on following the evolution of the most popular, interesting or elegant problems. However, journals with problem sections can help out with regular indexes of their own collections.

Rabinowitz has provided many resources on the Internet. A visit to `http://www.problemcorner.org` will provide you with 20000 problems that can be searched by keyword or author, and for which references to solutions in the literature are given. His main website (`http://www.mathpropress.com`) contains lists of problem books published during the last two centuries, all problem books published recently, web sites with problems from local, regional, and international competitions, 32 books with solutions to IMO problems, journals with problems sections, and mathematical dictionaries.

An Efficient Construction of the Golden Section

Kurt Hofstetter

We wish to divide efficiently, by ruler and compass, a given segment according to the golden section. We denote by $X(XY)$ the circle with X as centre and XY as radius. Let AB be the given segment.

1. Construct circle $c_1 = A(AB)$.

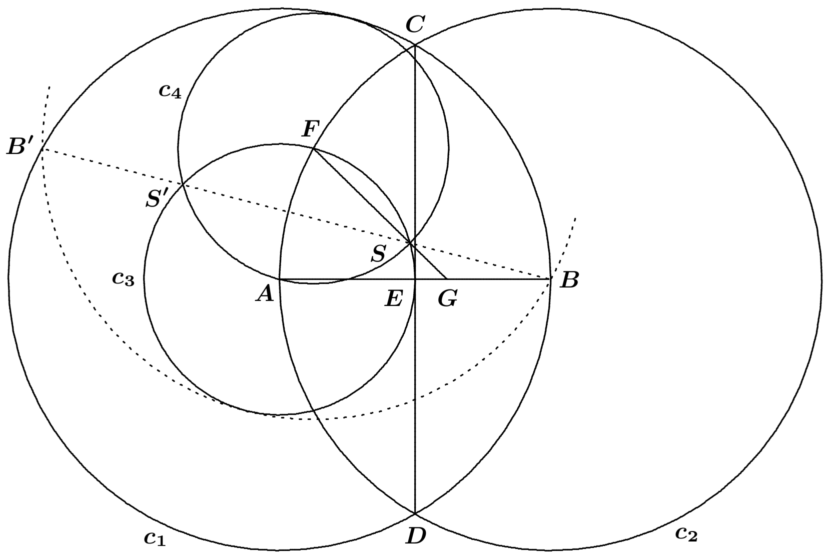
2. Construct circle $c_2 = B(AB)$. Let C and D be the points of intersection of c_1 and c_2.

3. Draw line CD. Let CD intersect AB at E.

4. Construct circle $c_3 = A(AE)$. Let F be one of the points of intersection of c_3 and c_2.

5. Construct circle $c_4 = F(AE)$. Let S and S' be the points of intersection of c_4 and c_3 such that FS extended will intersect the segment AB.

6. Extend FS to intersect AB at G.



Proposition. G divides AB according to the golden section.

Proof: Extend $S'B$ to intersect c_1 at B'. Because

$$AS = FS = AE = \tfrac{1}{2}AB = \tfrac{1}{2}FB\,,$$

we see that B and S are on the perpendicular bisector of FA, as are S' and B'. That is, S, S', B, B' lie on a line, and

$$AB = AB' = FB = FB' = 2AF\,.$$

Following the result in [1], we conclude that S divides $S'B$ according to the golden section. Finally, since $AS' = S'F = FS = SA$, we have $S'A$ parallel to FS $(= FG)$; hence, $S'S : SB = AG : GB$. Therefore, G divides AB according to the golden section.

Remarks.

1. This 6-step construction involves two appearances of the *vesica piscis*, which (according to the *Oxford English Dictionary*) is the name applied in the art world to the pointed oval region between the arcs of two equal circles, each through the centre of the other. Early artists employed the figure as an aureole enclosing the figure of Christ; it is also a common architectural feature. It seems somewhat ironic that this bit of Christian symbolism should be used to construct an object that was sacred to the Pythagoreans.

2. Reference [2] provides a 5-step division of a segment according to the golden section by ruler and rusty compass. The question remains, however, which construction is simpler?

References.

[1] K. Hofstetter, A Simple Construction of the Golden Section, *Forum Geom.* **2** (2002), 65–66.

`http://forumgeom.fau.edu/FG2002volume2/FG200208.pdf`

[2] K. Hofstetter, Division of a Segment in the Golden Section with Ruler and Rusty Compass, *Forum Geom.* **5** (2005), 135–136.

`http://forumgeom.fau.edu/FG2005volume5/FG200518.pdf`

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PROBLEMS

Toutes solutions aux problèmes dans ce numéro doivent nous parvenir au plus tard le **1er avril 2007**. *Une étoile* ($\star$) *après le numéro indique que le problème a été soumis sans solution.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais. Dans la section des solutions, le problème sera publié dans la langue de la principale solution présentée.

La rédaction souhaite remercier Jean-Marc Terrier et Martin Goldstein, de l'Université de Montréal, d'avoir traduit les problèmes.

3163. *Proposé par José Luis Díaz-Barrero, Université Polytechnique de Catalogne, Barcelone, Espagne.*

Calculer

$$\lim_{n\to\infty} \ln\left(\prod_{k=1}^{n}\left(\frac{k^2+n^2}{n^2}\right)^{k}\right)^{\frac{1}{n^2}} .$$

3164. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit P un point quelconque dans le plan du triangle ABC. Soit D, E et F les points milieu respectifs des côtés BC, CA et AB. Si G est le centre de gravité du triangle ABC, montrer que

$$0 \le 3PG+PA+PB+PC-2(PD+PE+PF) \le \tfrac{1}{2}(AB+BC+CA) .$$

3165. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Montrer que, pour tout entier positif n, il existe un polynôme $P(x)$, de degré au moins $8n$, tel que

$$\sum_{k=1}^{(2n+1)^2} |P(k)| < |P(0)| .$$

3166. *Proposé par Mihály Bencze et Marian Dinca, Brasov, Roumanie.*

Soit P un point intérieur du triangle ABC. Désignons respectivement par d_a, d_b et d_c les distances entre P et les côtés BC, CA et AB, et par D_A, D_B et D_C les distances entre P et les côtés A, B et C. Finalement, soit P_A, P_B et P_C les mesures respectives des angles BPC, CPA et APB.

Montrer que

$$d_ad_b\sin\left(\tfrac{1}{2}(P_A+P_B)\right)+d_bd_c\sin\left(\tfrac{1}{2}(P_B+P_C)\right)+d_cd_a\sin\left(\tfrac{1}{2}(P_C+P_A)\right)$$
$$\le \tfrac{1}{4}(D_BD_C\sin P_A+D_CD_A\sin P_B+D_AD_B\sin P_C) .$$

3167. *Proposé par Arkady Alt, San Jose, CA, USA.*

Soit ABC un triangle aux angles non obtus et R le rayon de son cercle circonscrit. Si a, b et c sont les longueurs des côtés opposés aux angles respectifs A, B et C, montrer que

$$a\cos^3 A + b\cos^3 B + c\cos^3 C \;\leq\; \frac{abc}{4R^2}\,.$$

3168. *Proposé par Arkady Alt, San Jose, CA, USA.*

Soit x_1, x_2, $\dots$, x_n des nombres réels positifs satisfaisant $\prod\limits_{i=1}^{n} x_i = 1$. Montrer que

$$\sum_{i=1}^{n} x_i^n(1+x_i) \;\geq\; \frac{n}{2^{n-1}} \prod_{i=1}^{n}(1+x_i)\,.$$

3169. *Proposé par Vesselin Dimitrov, National Highschool of Mathematics and Science, Sofia, Bulgarie.*

Soit A un ensemble fini de nombres réels tel que tout $a \in A$ puisse univoquement s'écrire sous la forme $a = b + c$, où b, $c \in A$ et $b \leq c$.

(a) Montrer qu'il existe des éléments distincts a_1, a_2, $\dots$, $a_k \in A$, tels que $a_1 + a_2 + \cdots + a_k = 0$.

(b)★ Le résultat ci-dessus reste-t-il nécessairement vrai si l'on omet l'unicité de la représentation de a comme $a = b + c$?

3170. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit a et b deux nombres réels satisfaisant $0 \leq a \leq \frac{1}{2} \leq b \leq 1$. Montrer que

(a) $2(b-a) \leq \cos\pi a - \cos\pi b$;

(b) $(1-2a)\cos\pi b \leq (1-2b)\cos\pi a$.

3171. *Proposé par Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.*

Étant donné un point P dans le premier quadrant, on sait que le segment (dit de Philon) de longueur minimale, passant par P et joignant les axes de coordonnées n'est pas constructible avec la règle et le compas. Cependant, le segment définissant (avec les deux axes) un triangle dans le premier quadrant, de périmètre minimal, lui, est constructible. Donner une telle construction.

3172. *Proposé par Vincenţiu Rădulescu, Université de Craiova, Craiova, Roumanie.*

Soit f une fonction continue positive définie sur $(0,\infty)$ et telle que $\liminf_{x\to\infty} f(x) > 0$. Montrer qu'il n'existe pas de fonction positive g, deux fois (continuement) différentiable, définie sur $[0,\infty)$ et satisfaisant $g''+f\circ g = 0$.

3173. *Proposé par Juan-Bosco Romero Márquez, Université de Valladolid, Valladolid, Espagne.*

Soit OAB un triangle rectangle avec l'angle droit en O. Soit OO' la bissectrice l'angle O, avec O' sur AB. Soit D et E les pieds des perpendiculaires respectives abaissées de O' sur les côtés OA et OB. Soit $F = OO'\cap DE$, $G = AE\cap O'D$, et $H = BD\cap O'E$.

Montrer que le triangle FGH est un triangle rectangle isocèle avec l'angle droit en F.

3174. *Proposé par Juan-Bosco Romero Márquez, Université de Valladolid, Valladolid, Espagne.*

Dans un triangle ABC, soit A' le point d'intersection de la bissectrice intérieure de l'angle A avec le côté BC. Soit respectivement B' et C' les pieds des perpendiculaires issues de A' sur les côtés AC et AB. Montrer que BB' et CC' se coupent sur la hauteur issue de A.

3175. *Proposé par D.J. Smeenk, Zaltbommel, Pays-Bas.*

Soit ABC un triangle avec l'angle $B > 90^\circ$ et l'angle $A < 60^\circ$. Soit P un point sur le côté AB de sorte que l'angle $CPB = 60^\circ$. Soit D le point sur CP qui se trouve aussi sur la bissectrice intérieure de l'angle A. Si l'angle $CBD = 30^\circ$, montrer que CP est une droite trisectrice de l'angle ACB.

. . . .

3163. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Calculate

$$\lim_{n\to\infty} \ln\left(\prod_{k=1}^{n}\left(\frac{k^2+n^2}{n^2}\right)^k\right)^{\frac{1}{n^2}}.$$

3164. *Proposed by Mihály Bencze, Brasov, Romania.*

Let P be any point in the plane of $\triangle ABC$. Let D, E, and F denote the mid-points of BC, CA, and AB, respectively. If G is the centroid of $\triangle ABC$, prove that

$$0 \le 3PG+PA+PB+PC-2(PD+PE+PF) \le \tfrac{1}{2}(AB+BC+CA).$$

3165. *Proposed by Mihály Bencze, Brasov, Romania.*

For any positive integer n, prove that there exists a polynomial $P(x)$, of degree at least $8n$, such that

$$\sum_{k=1}^{(2n+1)^2} |P(k)| \ < \ |P(0)| \,.$$

3166. *Proposed by Mihály Bencze and Marian Dinca, Brasov, Romania.*

Let P be an interior point of the triangle ABC. Denote by d_a, d_b, d_c the distances from P to the sides BC, CA, AB, respectively, and denote by D_A, D_B, D_C the distances from P to the vertices A, B, C, respectively. Further let P_A, P_B, and P_C denote the measures of $\angle BPC$, $\angle CPA$, and $\angle APB$, respectively.

Prove that

$$\begin{aligned} d_a d_b \sin\left(\tfrac{1}{2}(P_A+P_B)\right) + d_b d_c \sin\left(\tfrac{1}{2}(P_B+P_C)\right) + d_c d_a \sin\left(\tfrac{1}{2}(P_C+P_A)\right) \\ \leq \ \tfrac{1}{4}(D_B D_C \sin P_A + D_C D_A \sin P_B + D_A D_B \sin P_C)\,. \end{aligned}$$

3167. *Proposed by Arkady Alt, San Jose, CA, USA.*

Let ABC be a non-obtuse triangle with circumradius R. If a, b, c are the lengths of the sides opposite angles A, B, C, respectively, prove that

$$a\cos^3 A + b\cos^3 B + c\cos^3 C \ \leq \ \frac{abc}{4R^2}\,.$$

3168. *Proposed by Arkady Alt, San Jose, CA, USA.*

Let x_1, x_2, $\dots$, x_n be positive real numbers satisfying $\prod\limits_{i=1}^{n} x_i = 1$. Prove that

$$\sum_{i=1}^{n} x_i^n(1+x_i) \ \geq \ \frac{n}{2^{n-1}} \prod_{i=1}^{n}(1+x_i)\,.$$

3169. *Proposed by Vesselin Dimitrov, National Highschool of Mathematics and Science, Sofia, Bulgaria.*

Let A be a finite set of real numbers such that each $a \in A$ is uniquely expressible as $a = b + c$, where b, $c \in A$ and $b \leq c$.

(a) Prove that there exist distinct elements a_1, a_2, $\dots$, $a_k \in A$ such that $a_1 + a_2 + \cdots + a_k = 0$.

(b)★ Does this necessarily hold if it is no longer assumed that each representation $a = b + c$ is unique?

3170. *Proposed by Mihály Bencze, Brasov, Romania.*

Let a and b be real numbers satisfying $0 \leq a \leq \frac{1}{2} \leq b \leq 1$. Prove that

(a) $2(b-a) \leq \cos \pi a - \cos \pi b$;

(b) $(1-2a) \cos \pi b \leq (1-2b) \cos \pi a$.

3171. *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.*

Given a point P in the first quadrant, it is known that the line segment in the first quadrant joining the coordinate axes, passing through P, and having minimum length (Philo's line) is not constructible using straightedge and compass. However, the line which (together with the two axes) defines a triangle in the first quadrant with minimum perimeter is constructible. Give such a construction.

3172. *Proposed by Vincenţiu Rădulescu, University of Craiova, Craiova, Romania.*

Let f be a positive continuous function defined on $(0, \infty)$ such that $\liminf\limits_{x \to \infty} f(x) > 0$. Prove that there is no positive, twice differentiable function g defined on $[0, \infty)$ which satisfies $g'' + f \circ g = 0$.

3173. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let OAB be a right triangle with right angle at O. Let OO' be the bisector of angle O, with O' on AB. Let D and E be the feet of the perpendiculars from O' to the legs OA and OB, respectively. Let $F = OO' \cap DE$, $G = AE \cap O'D$, and $H = BD \cap O'E$.

Prove that $\triangle FGH$ is an isosceles right triangle with right angle at F.

3174. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Given $\triangle ABC$, we define A' to be the point where the internal angle bisector of angle A meets the side BC. Let B' and C' be the feet of the perpendiculars from A' to the sides AC and AB, respectively. Prove that BB' and CC' intersect on the altitude from A.

3175. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Let $\triangle ABC$ be a triangle with $\angle B > 90^\circ$ and $\angle A < 60^\circ$. Let P be a point on the side AB such that $\angle CPB = 60^\circ$. Let D be the point on CP which also lies on the interior angle bisector of $\angle A$. If $\angle CBD = 30^\circ$, prove that CP is a trisector of angle ACB.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

3059. [2005 : 334, 337] *Proposed by Gabriel Dospinescu, Paris, France.*

Let a, b, c, d be real numbers such that $a^2+b^2+c^2+d^2 \le 1$. Prove that

$$ab+bc+cd+da+ac+bd \ \le\ 4abcd+\frac{5}{4}\,.$$

I. *Solution by Dionne Bailey, Elsie Campbell, and Charles R. Diminnie, Angelo State University, San Angelo, TX, USA, modified by the editor.*

Note first that the AM–GM Inequality implies that

$$4\sqrt[4]{(abcd)^2} \ \le\ a^2+b^2+c^2+d^2 \ \le\ 1\,,$$

and hence, $\sqrt{|abcd|} \le \frac{1}{4}$, or

$$|abcd| \ \le\ \tfrac{1}{16}\,. \qquad (1)$$

Further, equality is attained in (1) if and only if $a^2=b^2=c^2=d^2$ and $a^2+b^2+c^2+d^2=1$; that is, if and only if $a^2=b^2=c^2=d^2=\frac{1}{4}$.

For real numbers x and y, the Cauchy–Schwarz Inequality implies that

$$x+y \ \le\ \sqrt{2}\sqrt{x^2+y^2}\,,$$

with equality if and only if $x=y\ge 0$. Using this and the AM–GM Inequality, we obtain

$$\begin{aligned}(ab+cd)+(bc+da) &\le \sqrt{2}\sqrt{(ab+cd)^2+(bc+da)^2}\\ &= \sqrt{2}\sqrt{(a^2+c^2)\,(b^2+d^2)+4abcd}\\ &\le \sqrt{2}\sqrt{\left(\tfrac{1}{2}(a^2+c^2+b^2+d^2)\right)^2+4abcd}\\ &\le \sqrt{2}\sqrt{\tfrac{1}{4}+4abcd} \ =\ \tfrac{\sqrt{2}}{2}\sqrt{1+16abcd}\,,\end{aligned}$$

with equality if and only if $ab+cd=bc+da\ge 0$ and $a^2+c^2=b^2+d^2=\frac{1}{2}$.

Similarly, we get

$$(bc+da)+(ac+bd) \ \le\ \tfrac{\sqrt{2}}{2}\sqrt{1+16abcd}\,,$$

with equality if and only if $bc+da=ac+bd\ge 0$ and $a^2+b^2=c^2+d^2=\frac{1}{2}$, and

$$(ab+cd)+(ac+bd) \ \le\ \tfrac{\sqrt{2}}{2}\sqrt{1+16abcd}\,,$$

with equality if and only if $ab+cd=ac+bd\ge 0$ and $a^2+d^2=b^2+c^2=\frac{1}{2}$.

Therefore,

$$\begin{aligned} ab + bc + cd + da + ac + bd \\ &= \tfrac{1}{2}[(ab+cd) + (bc+da) + (bc+da) + (ac+bd) \\ &\qquad + (ab+cd) + (ac+bd)] \\ &\le \tfrac{3\sqrt{2}}{4}\sqrt{1+16abcd}\,, \end{aligned} \tag{2}$$

with equality if and only if $ab + cd = bc + da = ac + bd \ge 0$ and $a^2 = b^2 = c^2 = d^2 = \frac{1}{4}$.

Let $x = 1 + 16abcd$. From (1), we have $0 \le x \le 2$, and hence

$$(x+4)^2 - \left(3\sqrt{2}\sqrt{x}\right)^2 \;=\; x^2 - 10x + 16 \;=\; (x-8)(x-2) \;\ge\; 0\,,$$

with equality if and only if $x = 2$. Thus,

$$3\sqrt{2}\sqrt{1+16abcd} \;\le\; 16abcd + 5\,, \tag{3}$$

with equality if and only if $abcd = \frac{1}{16}$.

Finally, combining (1), (2), and (3), we obtain

$$ab + bc + cd + da + ac + bd \;\le\; \tfrac{3\sqrt{2}}{4}\sqrt{1+16abcd} \;\le\; 4abcd + \tfrac{5}{4}\,,$$

with equality if and only if $a^2 = b^2 = c^2 = d^2 = \frac{1}{4}$, $abcd = \frac{1}{16}$, and $ab+cd = bc+da = ac+bd \ge 0$; that is, if and only if $a = b = c = d = \pm\frac{1}{2}$.

II. *Solution by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain; and the proposer.*

Let t be a real number, and let $S = ab + bc + cd + da + ac + bd$. Consider the polynomial

$$\begin{aligned} A(x) &= (x-a)(x-b)(x-c)(x-d) \\ &= x^4 - (a+b+c+d)x^3 + Sx^2 \\ &\qquad - (abc + bcd + abd + acd)x + abcd\,. \end{aligned}$$

Since $|p + iq| \ge |p|$, we obtain

$$\begin{aligned} |A(it)|^2 &= \left| t^4 + it^3 \sum_{\text{cyclic}} a - St^2 - it \sum_{\text{cyclic}} abc + abcd \right|^2 \\ &\ge \left| t^4 - St^2 + abcd \right|^2\,. \end{aligned}$$

On the other hand,

$$|A(it)|^2 \;=\; A(it)\cdot\overline{A(it)} \;=\; \prod_{\text{cyclic}}(a - it)\cdot\prod_{\text{cyclic}}(a + it) \;=\; \prod_{\text{cyclic}}(a^2 + t^2)\,.$$

Thus,

$$\left|t^4 - St^2 + abcd\right|^2 \;\le\; \prod_{\text{cyclic}} (a^2 + t^2)\,.$$

Set $t = 1/2$ in this inequality, and then use the AM–GM Inequality and the given condition to obtain

$$\begin{aligned}\left|\tfrac{1}{16} - \tfrac{1}{4}S + abcd\right|^2 \;&\le\; \prod_{\text{cyclic}} \left(a^2 + \tfrac{1}{4}\right) \;\le\; \Bigl(\tfrac{1}{4}\sum_{\text{cyclic}} \left(a^2 + \tfrac{1}{4}\right)\Bigr)^4 \\ &=\; \left(\tfrac{1}{4}(a^2 + b^2 + c^2 + d^2 + 1)\right)^4 \;\le\; \tfrac{1}{16}\,.\end{aligned}$$

Therefore, $\left|\frac{1}{16} - \frac{1}{4}S + abcd\right| \le \frac{1}{4}$, which yields $S \le 4abcd + \frac{5}{4}$, completing the proof.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria. There were two incorrect (incomplete) solutions submitted.

3060. [2005 : 334, 337] *Proposed by Gabriel Dospinescu, Paris, France.*

Let a and b be positive real numbers such that $a < 2$. For each integer $n \ge 1$, let $x_n = \lfloor an + b \rfloor$. Prove that the sequence $\{x_n\}_{n\ge 1}$ has an infinite number of terms whose sum of digits is even. (Note: $\lfloor z \rfloor$ is the greatest integer not exceeding z.)

Solution by Tom Leong, Brooklyn, NY, USA, with some detail added by the editor.

If $0 < a \le 1$, then $\{x_n\}$ assumes every integer value greater than or equal to $\lfloor a + b \rfloor$, and the result is clearly true. Hence, we may assume that $1 < a < 2$. Then all the terms of the sequence $\{x_n\}$ are distinct. It is easy to verify (by induction or otherwise) that for $k \ge 2$, exactly half of the $9(10^{k-1})$ positive integers with k digits have an even sum of digits (and half have an odd sum).

The number of terms x_n with k digits is equal to the number of positive integers n such that $10^{k-1} \le an + b < 10^k$. We rewrite these inequalities as

$$\frac{10^{k-1} - b}{a} \;\le\; n \;<\; \frac{10^k - b}{a}\,.$$

If k is large enough so that $10^{k-1} > b$, then all integers n satisfying the above inequalities are positive. The number of such integers is at least

$$\frac{10^k - b}{a} - \frac{10^{k-1} - b}{a} - 1 \;=\; \frac{9\bigl(10^{k-1}\bigr)}{a} - 1\,.$$

Since $a < 2$, this number will be greater than $9\bigl(10^{k-1}\bigr)/2$ for all k sufficiently large (large enough so that $9\bigl(10^{k-1}\bigr) > \left(\frac{1}{a} - \frac{1}{2}\right)^{-1}$).

Thus, for all k sufficiently large, the number of terms x_n with k digits is greater than the number of k–digit positive integers with an odd sum of digits, and hence there exists a term x_n with k digits and an even sum of digits. The desired result follows.

Also solved by the proposer.

3061. [2005 : 335, 337] *Proposed by Gabriel Dospinescu, Paris, France.*

Find the smallest non-negative integer n for which there exists a non-constant function $f : \mathbb{Z} \to [0, \infty)$ such that for all integers x and y,

(a) $f(xy) = f(x)f(y)$, and

(b) $2f(x^2 + y^2) - f(x) - f(y) \in \{0, 1, \ldots, n\}$.

For this value of n, find all the functions f which satisfy (a) and (b).

Solution by Michel Bataille, Rouen, France, modified by the editor.

The solution makes use of the following known result.

Proposition 1. If p is prime, $p \equiv 3 \pmod 4$, and a and b are integers such that $p \mid (a^2 + b^2)$, then $p \mid a$ and $p \mid b$.

We show that the smallest n is $n = 1$.

Let $f : \mathbb{Z} \to [0, \infty)$ be a non-constant function satisfying (a). Since $f(1) = f(1 \cdot 1) = \big(f(1)\big)^2$, we have $f(1) = 1$ or $f(1) = 0$. But the latter yields $f(x) = f(x \cdot 1) = f(x)f(1) = 0$ for all $x \in \mathbb{Z}$, contradicting the fact that f is not constant. It follows that $f(1) = 1$. Similarly, $f(-1) = 1$ and $f(0) = 0$.

If $n = 0$, then f cannot satisfy (b), since $2f(x^2+y^2) \neq f(x)+f(y)$ for $x = 1$ and $y = 0$. The function $f_0 : \mathbb{Z} \to [0, \infty)$ defined by $f_0(0) = 0$ and $f_0(x) = 1$ for all non-zero integers x, is non-constant and clearly satisfies (a). Moreover, if $K_0(x, y) = 2f_0(x^2+y^2) - f_0(x) - f_0(y)$, we have $K(x, y) = 0$ if $x = y = 0$; otherwise $K_0(x, y) = 2 - f_0(x) - f_0(y)$ with at least one of $f_0(x)$, $f_0(y)$ equal to 1. In all cases, $K_0(x, y) \in \{0, 1\}$ and f_0 satisfies (b) as well.

For $n = 1$ we will show that in addition to f_0, the solutions for f are the functions $f_p : \mathbb{Z} \to [0, \infty)$ defined by $f_p(x) = 0$ if $p \mid x$ and $f_p(x) = 1$ otherwise, where p is prime and $p \equiv 3 \pmod 4$. Such a function f_p is not constant, and satisfies (a) (since $p \mid xy$ implies $p \mid x$ or $p \mid y$). Let $K_p(x, y) = 2f_p(x^2+y^2) - f_p(x) - f_p(y)$. If $p \mid x$ and $p \mid y$, then $p \mid x^2+y^2$ and $K_p(x, y) = 0$. If $p \mid x$, and $p \nmid y$, then p does not divide $x^2 + y^2$, and $K_p(x, y) = 1$. Finally, if $p \nmid x$ and $p \nmid y$, then $p \nmid x^2 + y^2$ by Proposition 1, and $K_p(x, y) = 0$. In all cases $K_p(x, y) \in \{0, 1\}$ and f_p satisfies (b); whence, f_p is a solution.

Conversely, let f be a solution. We will show that either $f = f_0$ or $f = f_p$ for some prime $p \equiv 3 \pmod 4$. First, we will show that $f(x)$ is either 0 or 1 for all $x \in \mathbb{Z}$. Let $x \in \mathbb{Z}$ be such that $f(x) \neq 0$. Condition (a) implies that $f(x^2) = \big(f(x)\big)^2$ and (b) yields

$$2\big(f(x)\big)^2 - f(x) \;=\; 2f(x^2+0) - f(x) - f(0) \in \{0,1\}\,.$$

It follows that $f(x)$ is equal to 0, 1, or $1/2$ for all integer x. But if $f(x) = 1/2$ and $f(x^2+1) \in \{0,\,1,\,1/2\}$, then (b) does not hold for $y = 1$.

Next, we will show that $f(2) = 1$ and $f(q) \neq 0$ for any prime q, $q \equiv 1 \pmod 4$. Since $f(1) = 1$, from (b) with $x = y = 1$ we conclude that $f(2) = 1$. If q is a prime, $q \equiv 1 \pmod 4$, then there are integers a and k such that $qk = a^2 + 1$ and

$$2f(k)f(q) \;=\; 2f(kq) \;=\; 2f(a^2+1^2) \;=\; f(a) + f(1) + \varepsilon\,,$$

where ε is equal to 0 or 1. Therefore, $2f(k)f(q) \geq f(1) = 1 \neq 0$ and $f(q) \neq 0$.

Suppose $f \neq f_0$. Then $f(x_0) = 0$ for some integer x_0 with $|x_0| > 1$. By condition (a), we must have $f(p) = 0$ for some prime factor p of x_0 and $p \equiv 3 \pmod 4$. No other prime $p' \equiv 3 \pmod 4$ can satisfy $f(p') = 0$. Otherwise, we would have $2f(p^2+p'^2) \in \{0,\,1\}$; whence $f(p^2+p'^2) = 0$. But then there is a prime factor p'' of $p^2+p'^2$ such that $f(p'') = 0$; thus, $p'' \equiv 3 \pmod 4$. By Proposition 1, this is a contradiction since p'' cannot be a factor of both p and p'. It follows that $f(x) = 0$ if and only if p is a factor of x and $f = f_p$.

Also solved by the proposer.

3062. [2005 : 335, 337] *Proposed by Gabriel Dospinescu, Paris, France.*

Let a, b, c be positive real numbers such that $a+b+c = 1$. Prove that

$$(ab+bc+ca)\left(\frac{a}{b^2+b} + \frac{b}{c^2+c} + \frac{c}{a^2+a}\right) \;\geq\; \frac{3}{4}\,.$$

Essentially the same solution by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain; and the proposer.

Let x be any positive real number. In the Cauchy–Schwarz Inequality $|\overrightarrow{u}|^2\,|\overrightarrow{v}|^2 \geq \left(\overrightarrow{u}\cdot\overrightarrow{v}\right)^2$, we set

$$\overrightarrow{u} \;=\; \left(\frac{\sqrt{a}}{x+b}, \frac{\sqrt{b}}{x+c}, \frac{\sqrt{c}}{x+a}\right) \qquad \text{and} \qquad \overrightarrow{v} \;=\; \left(\sqrt{a}, \sqrt{b}, \sqrt{c}\right)\,.$$

Then, since $|\overrightarrow{v}| = a+b+c = 1$, we obtain

$$\frac{a}{(x+b)^2} + \frac{b}{(x+c)^2} + \frac{c}{(x+a)^2} \;\geq\; \left(\frac{a}{x+b} + \frac{b}{x+c} + \frac{c}{x+a}\right)^2\,.$$

Now use the Cauchy–Schwarz Inequality again, this time with

$$\begin{aligned} \overrightarrow{u} &= \left(\sqrt{\frac{a}{x+b}}, \sqrt{\frac{b}{x+c}}, \sqrt{\frac{c}{x+a}}\right) \\ \text{and} \quad \overrightarrow{v} &= \left(\sqrt{a(x+b)}, \sqrt{b(x+c)}, \sqrt{c(x+a)}\right) . \end{aligned}$$

Since $\overrightarrow{u} \cdot \overrightarrow{v} = a+b+c = 1$, we get

$$\begin{aligned} \left(\frac{a}{x+b} + \frac{b}{x+c} + \frac{c}{x+a}\right)^2 &\geq \frac{1}{\left(a(x+b) + b(x+c) + c(x+a)\right)^2} \\ &= \frac{1}{(x+ab+bc+ca)^2} . \end{aligned}$$

Then

$$\int_0^1 \left(\frac{a}{(x+b)^2} + \frac{b}{(x+c)^2} + \frac{c}{(x+a)^2}\right) dx \geq \int_0^1 \frac{dx}{(x+ab+bc+ca)^2} ;$$

that is,

$$\frac{a}{b^2+b} + \frac{b}{c^2+c} + \frac{c}{a^2+a} \geq \frac{1}{(ab+bc+ca)(1+ab+bc+ca)} .$$

Using the well-known inequality $3(ab+bc+ca) \leq (a+b+c)^2$, we obtain $ab+bc+ca \leq \frac{1}{3}$. Therefore,

$$(ab+bc+ca)\left(\frac{a}{b^2+b} + \frac{b}{c^2+c} + \frac{c}{a^2+a}\right) \geq \frac{1}{1+ab+bc+ca} \geq \frac{3}{4} ,$$

as desired. Equality holds if and only if $a = b = c = \frac{1}{3}$.

Also solved by ARKADY ALT, San Jose, CA, USA; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; MICHEL BATAILLE, Rouen, France; MIHÁLY BENCZE, Brasov, Romania; SILOUANOS BRAZITIKOS, student, Trikala, Greece; OVIDIU FURDUI, student, Western Michigan University, Kalamazoo, MI, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria (two solutions); RONGZHENG JIAO, Yangzhou University, Yangzhou, China; VEDULA N. MURTY, Dover, PA, USA; and PETER Y. WOO, Biola University, La Mirada, CA, USA.

Janous has also proven the following similar results. If a, b, and c are positive numbers with $a+b+c=1$, then

$$(ab+bc+ca)\left(\frac{a}{b^2+1} + \frac{b}{c^2+1} + \frac{c}{a^2+1}\right) \leq \frac{3}{4} ,$$

and

$$(ab+bc+ca)\left(\frac{1}{b^2+1} + \frac{1}{c^2+1} + \frac{1}{a^2+1}\right) \leq \frac{9}{10} .$$

3063. [2005 : 335, 337] *Proposed by Mohammed Aassila, Strasbourg, France.*

Determine all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that, for all $x \in \mathbb{R}$,

$$f\big(f(x)\big) + f(x) \;=\; 2x + a\,,$$

where a is a real constant.

Solution by Michel Bataille, Rouen, France, expanded by the editor.

It is readily verified that the function $f(x) = x + \frac{1}{3}a$ is a solution (for any a) and that, if $a = 0$, any function of the form $f(x) = -2x + c$ is a solution, where c is an arbitrary real constant. We will prove that these are the only solutions.

Let f be any function that satisfies the given conditions. Clearly, f is one-to-one, since $f(x_1) = f(x_2)$ implies $2x_1 + a = 2x_2 + a$, from which we get $x_1 = x_2$. Being also continuous, f must be strictly monotone on $\mathbb{R}$; hence, f must have a limit (finite or infinite) as $x \to \infty$ and as $x \to -\infty$. If $\lim\limits_{x\to\infty} f(x) = L \in \mathbb{R}$, then $\lim\limits_{x\to\infty} f\big(f(x)\big) + f(x) = f(L) + L$ (since f is continuous), which contradicts $\lim\limits_{x\to\infty}(2x+a) = \infty$. Thus, $\lim\limits_{x\to\infty} f(x) = \pm\infty$. Similarly, $\lim\limits_{x\to-\infty} f(x) = \pm\infty$. Since f is monotone, either $\lim\limits_{x\to\infty} f(x) = \infty$ and $\lim\limits_{x\to-\infty} f(x) = -\infty$, or $\lim\limits_{x\to\infty} f(x) = -\infty$ and $\lim\limits_{x\to-\infty} f(x) = \infty$. Hence, f is a continuous bijection from $\mathbb{R}$ onto $\mathbb{R}$. Now we consider two cases separately:

Case (i). f is strictly increasing.

Let $x \in \mathbb{R}$ be arbitrary but fixed. Define the sequence $\{u_n\}$ as follows: $u_0 = x$, $u_1 = f(x)$, and $u_{n+1} = f(u_n)$ for all $n \ge 1$. Then the given functional equation implies that, for all $n = 0, 1, 2, \ldots,$

$$u_{n+2} + u_{n+1} \;=\; 2u_n + a\,. \tag{1}$$

We solve this recurrence relation by the usual method. The characteristic equation of the corresponding homogeneous relation is $t^2 + t - 2 = 0$, and the characteristic roots are 1 and -2. By inspection, a particular solution is $u_n = \frac{1}{3}na$. Hence, the general solution of (1) is $u_n = \alpha + \beta(-2)^n + \frac{1}{3}na$. Using $\alpha + \beta = u_0 = x$ and $\alpha - 2\beta + \frac{1}{3}a = u_1 = f(x)$, we easily find that $\alpha = \frac{1}{3}\left(2x + f(x) - \frac{1}{3}a\right)$ and $\beta = \frac{1}{3}\left(x - f(x) + \frac{1}{3}a\right)$. Hence,

$$u_n \;=\; \tfrac{1}{3}\left(2x + f(x) - \tfrac{1}{3}a\right) + \tfrac{1}{3}(-2)^n\left(x - f(x) + \tfrac{1}{3}a\right) + \tfrac{1}{3}na\,. \tag{2}$$

Since f is increasing, the sequence $\{u_n\}$ must be monotone; thus, the sign of $u_{n+1} - u_n$ must be fixed for $n = 0, 1, 2, \ldots$. The second term on the right side of (2) reveals that this is possible only if $x - f(x) + \frac{1}{3}a = 0$. Since this is true for all $x \in \mathbb{R}$, it follows that $f(x) = x + \frac{1}{3}a$.

Case (ii). f is strictly decreasing.

We first show that $a = 0$. Let $\phi(x) = f(x) - x$. Then ϕ is continuous and strictly decreasing. Since $\lim\limits_{x\to-\infty} \phi(x) = \infty$ and $\lim\limits_{x\to\infty} \phi(x) = -\infty$, we deduce that ϕ has exactly one real root. Thus, there exists a unique real number x_0 such that $f(x_0) = x_0$. From $2x_0 + a = f\big(f(x_0)\big) + f(x_0) = 2x_0$, we then have $a = 0$.

Let $g = f^{-1}$ denote the inverse of f. For an arbitrary but fixed real number x, define the sequence $\{v_n\}$ as follows: $v_0 = f(x)$, $v_1 = x$, and $v_{n+1} = g(v_n)$ for all $n \geq 1$. Since $f \circ f$ is also a bijection, there exists $y \in \mathbb{R}$ such that $f\big(f(y)\big) = x$. Then the functional equation $f\big(f(y)\big) + f(y) = 2y$ becomes $x + g(x) = 2g\big(g(x)\big)$, which implies that, for all $n = 0, 1, 2, \ldots$,

$$2v_{n+2} = 2g(v_{n+1}) = 2g\big(g(v_n)\big) = g(v_n) + v_n = v_{n+1} + v_n .$$

Solving this recurrence relation as in Case (i), we find that

$$v_n = \tfrac{1}{3}\big(f(x) + 2x\big) + \tfrac{2}{3}\left(-\tfrac{1}{2}\right)^n \big(f(x) - x\big) .$$

Letting $m = \frac{1}{3}\big(f(x) + 2x\big)$, we have $\lim\limits_{n\to\infty} v_n = m$. From the recurrence relation $v_{n+1} = g(v_n)$, we obtain $g(m) = m$ and hence, $f(m) = m$. Since x_0 is the only fixed point of f, we infer that $x_0 = m = \frac{1}{3}\big(f(x) + 2x\big)$. Since this is true for all $x \in \mathbb{R}$, we conclude that $f(x) = -2x + 3x_0 = -2x + c$, where $c = 3x_0$. Our proof is complete.

Also solved by ROBERT B. ISRAEL, University of British Columbia, Vancouver, BC; PETER Y. WOO, Biola University, La Mirada, CA, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; and the proposer. There were also an incomplete solution and a solution which gave the correct answers but with faulty argument.

3064. [2005 : 397, 399] *Proposed by J. Chris Fisher, University of Regina, Regina, SK.*

(a) Starting with four points A, B, C, D in the plane, no three of which are collinear, let P, Q, R, S be the mid-points of AB, CD, AC, BD, respectively. Let L be the point of intersection of AQ and DP, and let M be the point of intersection of BR and CS. Prove that the mid-point of BC lies on the line LM if and only if $AD \parallel BC$.

(b) Let A_0, A_1, A_2, A_3, and A_4 be the vertices of a non-degenerate pentagon. Define a *median* to be a line that joins a vertex A_j either to the mid-point of the opposite side $A_{j+2}A_{j-2}$ or to the mid-point of the opposite diagonal $A_{j+1}A_{j-1}$ (where subscripts are taken modulo 5). Prove that the pentagon is affinely regular if and only if the ten medians are concurrent.

The result is based on a theorem of Zvonco Čerin, *Journal of Geometry*, 77 (2003), 22–34.

Note: A pentagon is said to be *affinely regular* if it is the image under a linear transformation of a regular pentagon or a regular pentagram.

(a) *Solution by Michel Bataille, Rouen, France.*

We shall see that the claim in (a) is not correct: although $AD \parallel BC$ does imply that the mid-point of BC lies on LM, the converse is false. We choose affine coordinates with the origin at B such that $A(0,2)$, $C(2,0)$, and $D(2a,2b)$ for real numbers a,b. Then $P(0,1)$, $Q(1+a,b)$, $R(1,1)$, $S(a,b)$, and the equations of the required lines are:

$$\begin{array}{rl} AQ: & (b-2)x-(a+1)y+2(a+1) \;=\; 0\,, \\ DP: & (2b-1)x-2ay+2a \;=\; 0\,, \\ BR: & x-y \;=\; 0\,, \\ CS: & bx-(a-2)y-2b \;=\; 0\,. \end{array}$$

From the equations of BR and CS, we find that $M\left(\dfrac{2b}{b-a+2}, \dfrac{2b}{b-a+2}\right)$. Denoting by I the mid-point of BC, we have $I(1,0)$ and IM has equation $2bx-(b+a-2)y-2b=0$. It follows that I lies on the line LM if and only if L lies on the line IM; that is, the lines AQ, DP, IM pass through a point:

$$\begin{vmatrix} b-2 & -a-1 & 2a+2 \\ 2b-1 & -2a & 2a \\ 2b & 2-a-b & -2b \end{vmatrix} \;=\; 0\,,$$

or finally,

$$(1-b)(a^2-ab-3a-4b+2) \;=\; 0\,. \tag{1}$$

Observe that $AD \parallel BC$ is equivalent to $b=1$. Thus, if $AD \parallel BC$, then the relation (1) is satisfied and I lies on LM, as desired. But not conversely—we may have I on LM but AD *not* parallel to BC. For example, this is the case if $a=-2$ and $b=6$ in the calculation above. It is readily checked that $L\left(\frac{4}{5},-\frac{6}{5}\right)$ and $M\left(\frac{6}{5},\frac{6}{5}\right)$, so that I lies on LM (indeed, I is the mid-point of LM), but AD is not parallel to BC (AD is in the direction of the vector $(-4,10)$, while BC has the direction vector $(2,0)$).

(b) *Incomplete solutions by Bataille and the proposer, completed by the editor.*

Any median of a regular pentagon or pentagram is a line of symmetry: the line from a vertex to the midpoint of the opposite side also passes through the midpoint of the opposite diagonal, so a regular pentagon has just five medians, and all five pass through the centre of the circumcircle. Affine transformations preserve mid-points and concurrency, so the five medians of affinely regular pentagons and pentagrams are concurrent.

For the converse, we let any four of the vertices of our pentagon, labeled consecutively, play the role of $ABCD$ from part (a). The concurrence of the medians says that $L=M$. We shall see that this is sufficient to prove that $AD \parallel BC$ and either $AD : BC = \tau : 1$ or $BC : AD = \tau : 1$, where $\tau = \frac{1+\sqrt{5}}{2}$ is the golden section, which is the ratio of the diagonal of a regular pentagon to a side. The condition holds for each set of four points;

thus, we can uniquely construct vertices A_3 and A_4 of an affinely regular pentagon from any given triangle A_0, A_1, A_2. It therefore remains to show, in the notation of part (a), that $b = 1$ and a is either τ or its reciprocal. To satisfy the condition that $L = M$, we must have both AQ and DP meeting $BR : x - y = 0$ at M. Setting $y = x$ in the equation for DP (in part (a)), we find that

$$x = \frac{2a}{-2b + 1 + 2a} = \frac{2b}{b - a + 2},$$

where the last entry is the x–coordinate of M. The same process with AQ yields

$$x = \frac{2(1 + a)}{3 - b + a} = \frac{2b}{b - a + 2}.$$

These equations represent a pair of conics in the variables a and b,

$$\begin{aligned} a^2 - 2b^2 + ab - 2a + b &= 0, \\ \text{and} \qquad a^2 - b^2 - a + 2b - 2 &= 0. \end{aligned}$$

Setting $b = 1$, both these equations reduce to $a^2 - a - 1 = 0$, whose solution is $a = \tau$ or $a = -\frac{1}{\tau}$.

The common points of these two conics are, therefore, $(a, b) = (\tau, 1)$, $\left(-\frac{1}{\tau}, 1\right)$, $(2, 0)$, and the point at infinity of the line $a = b$. Of these, only $a = 2$ and $b = 0$ produces a point of the affine plane that is a zero of the second factor $a^2 - ab - 3a - 4b + 2$ in the determinant of part (a); but these values force the points $B(0, 0)$, $C(2, 0)$, and $D(2a, 2b)$ to be collinear, contrary to the assumption that we start with five non-collinear points. The other two common points have $b = 1$ as desired; the choice $(a, b) = (\tau, 1)$ produces an affinely regular pentagon while $(a, b) = \left(-\frac{1}{\tau}, 1\right)$ produces an affinely regular pentagram.

Part (a) *also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria* (*including a counterexample*)*; part* (b) *solved by JOEL SCHLOSBERG, Bayside, NY, USA.*

3065. [2005 : 397, 399] *Proposed by Gabriel Dospinescu, Paris, France.*

Let ABC be an acute-angled triangle, and let M be an interior point of the triangle. Prove that

$$\frac{1}{MA} + \frac{1}{MB} + \frac{1}{MC} \ge 2\left(\frac{\sin\angle AMB}{AB} + \frac{\sin\angle BMC}{BC} + \frac{\sin\angle CMA}{CA}\right).$$

Solution by Michel Bataille, Rouen, France.

Let d_a, d_b, and d_c denote the distances from point M to the sides BC, CA, and AB, respectively. Using the Law of Sines for triangle BMC, we obtain

$$\begin{aligned} 2 \cdot \frac{\sin \angle BMC}{BC} &= \frac{\sin \angle MCB}{MB} + \frac{\sin \angle MBC}{MC} \\ &= \frac{d_a}{MB \cdot MC} + \frac{d_a}{MB \cdot MC} = 2 \cdot \frac{d_a}{MB \cdot MC} \end{aligned}$$

Thus,

$$\frac{\sin \angle BMC}{BC} = \frac{d_a}{MB \cdot MC} .$$

Similarly,

$$\frac{\sin \angle CMA}{CA} = \frac{d_b}{MC \cdot MA} \quad \text{and} \quad \frac{\sin \angle AMB}{AB} = \frac{d_c}{MA \cdot MB} .$$

It follows that the proposed inequality is equivalent to the inequality

$$MB \cdot MC + MC \cdot MA + MA \cdot MB \geq 2(MA \cdot d_a + MB \cdot d_b + MC \cdot d_c) ,$$

which is known to be true (see [1] or [2]). This completes the proof.

References

[1] M. Bataille, Solution to problem 85G, *Mathematical Gazette*, 86(2002), pp. 149–151.

[2] A. Oppenheim, The Erdős Inequality and Other Inequalities for a Triangle, *American Mathematical Monthly*, 68(1961), pp. 226–230.

Also solved by ARKADY ALT, San Jose, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

3066. [2005 : 397, 400] *Proposed by Gabriel Dospinescu, Paris, France.*

Given an integer $n > 2$, let $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$ be subsets of $S = \{1, 2, \ldots, n\}$ with the property that for all $i, j \in S$, the subsets A_i and B_j have exactly one element in common. Prove that, if there are at least two distinct subsets among $B_1, B_2, \ldots, B_n$, then there exists a non-empty subset $T \subseteq S$ that has an even number of elements in common with each of the subsets $A_1, A_2, \ldots, A_n$.

Solution by Tom Leong, Brooklyn, NY, USA.

Let x_{ij} denote the element common to A_i and B_j, and let $B_1 \neq B_2$, say. Then $T = (B_1 \cup B_2) \setminus (B_1 \cap B_2)$ is non-empty and meets each of $A_1, A_2, \ldots, A_n$ in zero or two elements. Indeed, if $x_{i1} \in B_1 \cap B_2$, then $x_{i1} = x_{i2}$ and $T \cap A_i$ is empty; while if $x_{i1} \in B_1 \setminus B_2$, then $x_{i2} \in B_2 \setminus B_1$ and $T \cap A_i = \{x_{i1}, x_{i2}\}$.

Also solved by KATHLEEN E. LEWIS, SUNY Oswego, Oswego, NY, USA; JOEL SCHLOSBERG, Bayside, NY, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

As in the featured solution, most solvers used the symmetric difference $B_1 \Delta B_2$ (which is the set of all elements contained in exactly one of the two sets B_1 and B_2) for the desired subset T. Only Schlosberg observed that the symmetric difference has an even intersection with *each A_i more generally when each A_i has an odd intersection with each B_j (rather than just a single common element). Specifically,*

> Let A_1, A_2, $\ldots$, A_m, B, and C be subsets of a finite set such if $|A_i \cap B|$ and $|A_i \cap C|$ are both odd for all i, then $|A_i \cap (B \Delta C)|$ is even for all i.

3067. [2005 : 398, 400] *Proposed by Gabriel Dospinescu, Paris, France.*

Find all functions $f : (0,\infty) \to (0,\infty)$ such that

1. $f(f(f(x))) + 2x = f(3x)$ for all $x > 0$, and
2. $\lim\limits_{x\to\infty} \big(f(x) - x\big) = 0$.

Composite of very similar solutions by Joel Schlosberg, Bayside, NY, USA; and the proposer.

The function $f(x) = x$ clearly has the required properties. We will prove that it is the only function with these properties.

Suppose that a function $f : (0,\infty) \to (0,\infty)$ has properties 1 and 2. For any $x > 0$, property 1 implies that

$$f(x) \;=\; \tfrac{2}{3}x \;+\; f\Big(f\big(f(\tfrac{1}{3}x)\big)\Big) \;>\; \tfrac{2}{3}x\,.$$

Define a sequence $\{a_n\}_{n=1}^{\infty}$ by setting $a_1 = \frac{2}{3}$ and $a_{n+1} = \frac{1}{3}a_n^3 + \frac{2}{3}$ for all $n \in \mathbb{N}$. We will prove by induction that $f(x) > a_n x$ for all x and n.

The case $n = 1$ was proven above. Assume that $f(x) > a_n x$ for some $n \in \mathbb{N}$ and all $x > 0$. Then, for all $x > 0$,

$$f\Big(f\big(f(\tfrac{1}{3}x)\big)\Big) \;>\; a_n f\big(f(\tfrac{1}{3}x)\big) \;>\; a_n^2 f\big(\tfrac{1}{3}x\big) \;>\; a_n^3 \cdot \tfrac{1}{3}x\,,$$

and hence,

$$f(x) \;=\; \tfrac{2}{3}x \;+\; f\Big(f\big(f(\tfrac{1}{3}x)\big)\Big) \;>\; \tfrac{2}{3}x + a_n^3 \cdot \tfrac{1}{3}x \;=\; a_{n+1}x\,.$$

This completes the induction.

Applying the AM–GM Inequality, we get $a_{n+1} = \dfrac{a_n^3 + 1^3 + 1^3}{3} \geq a_n$ for all n. Thus, the sequence $\{a_n\}$ is increasing. Furthermore, $0 < a_n < 1$ for all n, as can be shown by an easy induction. Therefore, the sequence converges. Letting u denote its limit, we have

$$u \;=\; \lim_{n\to\infty} a_{n+1} \;=\; \lim_{n\to\infty} \big(\tfrac{1}{3}a_n^3 + \tfrac{2}{3}\big) \;=\; \tfrac{1}{3}u^3 + \tfrac{2}{3}\,.$$

Hence, $u^3 - 3u + 2 = 0$, from which we get $u = 1$ or $u = -2$. But $u = -2$ is impossible, since $a_n > 0$ for all n. Therefore $u = 1$. Since $f(x) > a_n x$ for all n and x, we obtain, for all $x > 0$,

$$f(x) \;\geq\; \lim_{n\to\infty} a_n x \;=\; x\,.$$

For all $x > 0$, since $f(x) \geq x$, we have

$$f\Big(f\big(f(x)\big)\Big) \;\geq\; f\big(f(x)\big) \;\geq\; f(x)\,,$$

and hence, using property (a) again,

$$f(3x) - 3x \;=\; f\Big(f\big(f(x)\big)\Big) - x \;\geq\; f(x) - x\,.$$

By induction, we then obtain $f(x) - x \leq f(3^n x) - 3^n x$ for all $n \in \mathbb{N}$. Since $\lim\limits_{n\to\infty}\big(f(3^n x) - 3^n x\big) = 0$ (by property 2), we have $f(x) - x \leq 0$; that is, $f(x) \leq x$.

We have shown that $x \leq f(x) \leq x$ for all $x > 0$. We conclude that $f(x) = x$ for all $x > 0$.

There was one incomplete solution submitted.

3068. [2005 : 398, 400] *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

Let a, b, c be non-negative real numbers, no two of which are zero. Prove that

$$\sqrt{1+\frac{48a}{b+c}}+\sqrt{1+\frac{48b}{c+a}}+\sqrt{1+\frac{48c}{a+b}} \;\geq\; 15\,,$$

and determine when there is equality.

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

Without loss of generality, we may assume that $a \geq b \geq c \geq 0$ and that $a+b+c=1$. Let $f(x) = \sqrt{1+\dfrac{48x}{1-x}}$ for $0 \leq x < 1$. Then

$$\begin{aligned} f'(x) &= \frac{24}{\sqrt{(1-x)^3(1+47x)}} > 0 \\ \text{and}\quad f''(x) &= \frac{48(47x-11)}{\sqrt{(1-x)^5(1+47x)^3}}\,. \end{aligned}$$

The tangent line to the graph of $f(x)$ at the point $\left(\frac{1}{3},5\right)$ has equation $T(x) = \dfrac{54x+7}{5}$. Setting $f(x) = T(x)$, we obtain $12(3x-1)^2(27x-2) = 0$, from which we see that the graphs of the functions f and T intersect again at $x = \frac{2}{27}$. Define

$$g(x) \;=\; \begin{cases} T(x) & \text{if } \frac{2}{27} \leq x < \frac{1}{3}, \\ f(x) & \text{if } \frac{1}{3} \leq x < 1. \end{cases}$$

Clearly, the function g is convex and $g(x) \leq f(x)$ for $\frac{2}{27} \leq x < 1$.

If $b \leq \frac{2}{27}$, then $a = 1 - b - c \geq \frac{23}{27}$, and therefore,

$$f(a) \;\geq\; f\left(\tfrac{23}{27}\right) \;=\; \sqrt{277} \;>\; 15\,,$$

which implies that the original inequality holds. Hence, we can further assume that $b > \frac{2}{27}$.

If $c > \frac{2}{27}$, then, applying Jensen's Inequality, we obtain

$$\begin{aligned} f(a) + f(b) + f(c) &\geq g(a) + g(b) + g(c) \\ &\geq 3g\left(\tfrac{1}{3}(a+b+c)\right) \;=\; 3g\left(\tfrac{1}{3}\right) \;=\; 15\,, \end{aligned}$$

with equality if and only if $a = b = c = \frac{1}{3}$.

If $\frac{1}{17} < c \leq \frac{2}{27}$, then $f(c) > f(\frac{1}{17}) = 2$ and, applying Jensen's Inequality, we obtain

$$f(a) + f(b) \;\geq\; g(a) + g(b) \;\geq\; 2g\left(\tfrac{1}{2}(a+b)\right) \;\geq\; 2g\left(\tfrac{25}{54}\right) \;>\; 13\,.$$

Thus, $f(a) + f(b) + f(c) > 15$, and the original inequality holds again.

Finally, consider $c \leq \frac{1}{17}$. Then, applying Jensen's Inequality, we have

$$\begin{aligned} f(a) + f(b) + f(c) &\geq g(a) + g(b) + f(c) \;\geq\; 2g\left(\tfrac{1}{2}(a+b)\right) + f(c) \\ &= 2f\left(\tfrac{1}{2}(a+b)\right) + f(c) \;=\; 2f\left(\tfrac{1}{2}(1-c)\right) + f(c)\,. \end{aligned}$$

Define

$$h(x) \;=\; 2f\left(\frac{1-x}{2}\right) + f(x) \;=\; 2\sqrt{\frac{49-47x}{1+x}} + \sqrt{\frac{1+47x}{1-x}}$$

for $0 \leq x \leq \frac{1}{17}$. Then $h(0) = 15$ and

$$h'(x) \;=\; 24\left(\frac{1}{\sqrt{(1-x)^3(1+47x)}} - \frac{4}{\sqrt{(1+x)^3(49-47x)}}\right)\,.$$

Now, $(1+x)^3(49-47x) - 16(1-x)^3(1+47x) = (3x-1)k(x)$, where $k(x) = 235x^3 - 699x^2 + 505x - 33$. It is easy to verify that $k(\frac{1}{17}) < 0$, $k(1) > 0$, $k(\frac{3}{2}) < 0$, and $k(2) > 0$. Hence, $k(x) < 0$ for $0 \leq x \leq \frac{1}{17}$. Thus, $h'(x) > 0$, and therefore, $h(x) \geq h(0) = 15$ for $0 \leq x \leq \frac{1}{17}$.

This completes the proof. In summary, equality holds if and only if $a = b = c$ or two of a, b, c are equal while the third is 0.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposer.

The proposer has proven the following, slightly more general result: If $a \geq 0$, $b \geq 0$, $c \geq 0$, and $0 < m \leq 24$, then

$$\sqrt{1 + \frac{2ma}{b+c}} + \sqrt{1 + \frac{2mb}{c+a}} + \sqrt{1 + \frac{2mc}{a+b}} \;\geq\; 3\sqrt{1+m}\,.$$

3069. [2005 : 398, 400] *Proposed by Cristinel Mortici, Valahia University of Targoviste, Romania.*

Let A, $B \in M_2(\mathbb{C})$ be such that $(AB)^2 = A^2B^2$. Prove that

$$\det(I + AB - BA) \;=\; 1\,.$$

Solution by Michel Bataille, Rouen, France.

We will denote by $\mathrm{tr}(X)$ the trace of $X \in M_2(\mathbb{C})$. Let $S = AB - BA$. Since $\mathrm{tr}(MN) = \mathrm{tr}(NM)$ for any pair of $n \times n$ matrices, $\mathrm{tr}(S) = 0$ and, moreover, $\mathrm{tr}(S^2) = 0$, since

$$\begin{aligned}\mathrm{tr}(S^2) &= \mathrm{tr}\big((AB)^2 - AB^2A - BA^2B + (BA)^2\big)\\ &= \mathrm{tr}\big((AB)^2\big) - \mathrm{tr}\big((AB^2)A\big) - \mathrm{tr}\big(B(A^2B)\big) + \mathrm{tr}\big(B(ABA)\big)\\ &= \mathrm{tr}\big((AB)^2\big) - \mathrm{tr}\big(A^2B^2\big) - \mathrm{tr}\big(A^2B^2\big) + \mathrm{tr}\big((AB)^2\big) \;=\; 0\,,\end{aligned}$$

where the last equality results from the hypothesis $(AB)^2 = A^2B^2$.

By the Cayley-Hamilton Theorem, $S^2 - \big(\mathrm{tr}(S)\big)S + \det(S)I = 0$; that is, $S^2 = -\det(S)I$. Taking traces gives $0 = \mathrm{tr}(S^2) = -2\det(S)$. Thus, $\mathrm{rank}(S) < 2$ and $S^2 = 0$. [*Editor's comment:* Those who prefer to avoid the Cayley-Hamilton Theorem can observe that $\mathrm{tr}(S) = 0$ implies that the eigenvalues of S are $\pm\lambda$, while $\mathrm{tr}(S^2) = 0$ implies that $2\lambda^2 = 0$; therefore, $\lambda = 0$ and S is nilpotent.]

Should $S = 0$, then $\det(I+S) = \det(I) = 1$. Otherwise, $\mathrm{rank}(S) = 1$ and S is similar to $T = \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}$ for some non-zero complex number α. (If u is a non-zero vector in $\mathrm{range}(S)$ and v is such that $\{\mathrm{u}, \mathrm{v}\}$ is a basis of $\mathbb{C}^2$, we have $S\mathrm{u} = 0$ (since $S^2 = 0$) and $S\mathrm{v} = \alpha\mathrm{u}$ for some complex α.) Therefore, $I + S$ is similar to $I + T = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$, and $\det(I + S) = \det(I + T) = 1$.

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; RONGZHENG JIAO, Yangzhou University, Yangzhou, China; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

3070. [2005 : 398, 400] *Proposed by Zhang Yun, High School attached to Xi′ An Jiao Tong University, Xi′ An City, Shan Xi, China.*

Let x_1, x_2, ..., x_n be positive real numbers such that

$$x_1 + x_2 + \cdots + x_n \;\geq\; x_1x_2\cdots x_n\,.$$

Prove that

$$(x_1x_2\cdots x_n)^{-1}\left(x_1^{n-1} + x_2^{n-1} + \cdots + x_n^{n-1}\right) \;\geq\; \sqrt[n-1]{n^{n-2}}\,,$$

and determine when there is equality.

Solution by Joel Schlosberg, Bayside, NY, USA.

For $n = 2$, we are required to prove that $(x_1x_2)^{-1}(x_1 + x_2) \geq 1$ if $x_1 + x_2 \geq x_1x_2$. This is trivially true. Equality holds in this case if and only if $x_1 + x_2 = x_1x_2$.

Suppose now that $n \geq 3$. By the AM–GM Inequality,

$$\begin{aligned} \frac{x_1^{n-1} + x_2^{n-1} + \cdots + x_n^{n-1}}{n} &\geq \left(x_1^{n-1}x_2^{n-1}\cdots x_n^{n-1}\right)^{\frac{1}{n}} \\ &= (x_1x_2\cdots x_n)^{\frac{n-1}{n}}\,. \end{aligned}$$

Thus,

$$(x_1^{n-1} + x_2^{n-1} + \cdots + x_n^{n-1})^{\frac{n(n-2)}{n-1}} \geq n^{\frac{n(n-2)}{n-1}}(x_1\cdots x_n)^{n-2}\,, \tag{1}$$

with equality if and only if $x_1^{n-1} = x_2^{n-1} = \cdots = x_n^{n-1}$, which is equivalent to $x_1 = x_2 = \cdots = x_n$.

Since $n - 1 > 1$, the Power Mean Inequality gives us

$$\left(\frac{x_1^{n-1} + x_2^{n-1} + \cdots + x_n^{n-1}}{n}\right)^{\frac{1}{n-1}} \geq \frac{x_1 + x_2 + \cdots + x_n}{n}\,;$$

that is,

$$(x_1^{n-1} + x_2^{n-1} + \cdots + x_n^{n-1})^{\frac{1}{n-1}} \geq n^{-\frac{n-2}{n-1}}(x_1 + x_2 + \cdots + x_n)\,. \tag{2}$$

Equality holds here if and only if $x_1 = x_2 = \cdots = x_n$.

Multiplying (1) and (2), we get

$$(x_1^{n-1} + \cdots + x_n^{n-1})^{n-1} \geq n^{n-2}(x_1\cdots x_n)^{n-2}(x_1 + \cdots + x_n)\,.$$

If $x_1 + x_2 + \cdots + x_n \geq x_1\cdots x_n$, then we conclude that

$$(x_1^{n-1} + \cdots + x_n^{n-1})^{n-1} \geq n^{n-2}(x_1\cdots x_n)^{n-1}\,.$$

and the required inequality follows. Furthermore, we have equality if and only if $x_1 = x_2 = \cdots = x_n$ and $x_1 + x_2 + \cdots + x_n = x_1x_2\cdots x_n$, which occurs if and only if the common value x satisfies $nx = x^n$. Therefore, we get equality if and only if $x_1 = \cdots = x_n = n^{\frac{1}{n-1}}$.

Also solved by MICHEL BATAILLE, Rouen, France; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. The inequality only was proved by MOHAMMED AASSILA, Strasbourg, France; ARKADY ALT, San Jose, CA, USA; WALTHER JANOUS, Ursulinen-gymnasium, Innsbruck, Austria; VEDULA N. MURTY, Dover, PA, USA; and YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON.

3071. [2005 : 398, 400] *Proposed by Arkady Alt, San Jose, CA, USA.*

Let $k > -1$ be a fixed real number. Let a, b, and c be non-negative real numbers such that $a+b+c=1$ and $ab+bc+ca>0$. Find

$$\min\left\{\frac{(1+ka)(1+kb)(1+kc)}{(1-a)(1-b)(1-c)}\right\}\,.$$

Solution by Michel Bataille, Rouen, France, modified by the editor.

The required minimum is $\min\left\{\frac{1}{8}(k+3)^3,(k+2)^2\right\}$.

First, we establish the following inequality:

$$4(ab+bc+ca)\;\le\;1+9abc\,. \qquad (1)$$

Using the fact that $a+b+c=1$, we get

$$\begin{aligned}
&1-4(ab+bc+ca)+9abc\\
&\qquad=\;(a+b+c)^3-4(a+b+c)(ab+bc+ca)+9abc\\
&\qquad=\;a(a-b)(a-c)+b(b-c)(b-a)+c(c-a)(c-b)\;\ge 0\,,
\end{aligned}$$

where the last line is Schur's Inequality. This proves (1).

We also claim that

$$ab+bc+ca\;\ge\;9abc\,. \qquad (2)$$

Indeed,

$$\begin{aligned}
ab+bc+ca &= (ab+bc+ca)(a+b+c)\\
&= a^2b+ab^2+b^2c+bc^2+c^2a+ca^2+3abc\\
&\ge 6abc+3abc\;=\;9abc\,,
\end{aligned}$$

where the inequality follows by an application of the AM–GM Inequality.

Turning back to the problem, we note that it is not possible for a, b, or c to equal 1. If $a=1$, for example, then $b=c=0$, which means that $ab+bc+ca=0$, a contradiction. Thus, a, b, $c\in[0,1)$. Let

$$\begin{aligned}
Q(a,b,c) &= \frac{(1+ka)(1+kb)(1+kc)}{(1-a)(1-b)(1-c)}\\
&= \frac{k^3abc+k^2(ab+bc+ca)+k+1}{ab+bc+ca-abc}\\
&= k^2+(k+1)\frac{k^2abc+1}{ab+bc+ca-abc}\,.
\end{aligned}$$

Note that $Q\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)=\frac{1}{8}(k+3)^3$ and $Q\left(0,\frac{1}{2},\frac{1}{2}\right)=(k+2)^2$.

Case 1. $k^2\le 5$.

We prove that $Q(a,b,c)\ge Q\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$. Since $k+1>0$, a straightforward calculation shows that this inequality is equivalent to

$$k^2(ab+bc+ca-9abc)+27(ab+bc+ca-abc)\;\le\;8\,. \qquad (3)$$

The term involving k^2 is non-negative, in view of (2). Since $k^2 \le 5$, the left side of (3) is at most $8\big(4(ab+bc+ca) - 9abc\big)$ and (3) follows from (1). Thus, $Q\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \frac{1}{8}(k+3)^3$ is the minimum value of Q.

Case 2. $k^2 \ge 5$.

We prove that $Q(a,b,c) \ge Q\left(0, \frac{1}{2}, \frac{1}{2}\right)$. Since $k+1 > 0$, we find that this inequality is equivalent to

$$1 + 4\big(abc - (ab+bc+ca)\big) + k^2 abc \;\ge\; 0\,.$$

This holds by (1) because $k^2 \ge 5$. Thus, $Q\left(0, \frac{1}{2}, \frac{1}{2}\right) = (k+2)^2$ is the minimum value of Q.

Noticing that

$$\tfrac{1}{8}(k+3)^3 - (k+2)^2 \;=\; \tfrac{1}{8}(k+1)(k^2-5)\,,$$

we see that $\frac{1}{8}(k+3)^3 \ge (k+2)^2$ if $k^2 \ge 5$ and $(k+2)^2 \ge \frac{1}{8}(k+3)^3$ if $k^2 \le 5$. The announced result follows.

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