Mayhem Solutions

M182. Proposed by Babis Stergiou, Chalkida, Greece.

If $a$, $b$, $c$ are positive numbers, such that $a + b + c = 1$, prove that

$$(1 + a)(1 + b)(1 + c) \geq 8(1 - a)(1 - b)(1 - c).$$

**Solution by Titu Zvonaru, Comănești, Romania.**

By the AM–GM Inequality we have

$$1 + a = a + b + c + a = a + b + a + c \geq 2\sqrt{(a + b)(a + c)}.$$  

Hence,

$$1 + a \geq 2\sqrt{(1-c)(1-b)}. \quad (1)$$

Similarly,

$$1 + b \geq 2\sqrt{(1-a)(1-c)} \quad (2)$$

and

$$1 + c \geq 2\sqrt{(1-b)(1-a)}. \quad (3)$$

From (1), (2), and (3), the given inequality follows. Equality holds if and only if $a + b = a + c = b + c$; that is, if and only if $a = b = c = \frac{1}{3}$.

Also solved by the Austrian IMO-Team 2005: Yimin Ge, Peter Gila, Bernhard Kinninger, Michael Moshammer, Jakob Preininger, Thomas Takeas.

M183. Proposed by the Mayhem Staff.

In the array at right, two letters are called **neighbouring** letters if they are adjacent to each other horizontally, vertically, or diagonally. Starting from any letter “M” on the outside of the array, find the number of ways of spelling “MATH” by moving only between neighbouring letters.

**Solution by Robert Bilinski, Collège Montmorency, Laval, QC.**

We will count the MATH words using the As, since the H is common to all words. The 4 corners As have 5 adjacent Ms and 1 adjacent T; thus, the 4 corners give us 20 occurrences of MATH. The 8 As adjacent to a corner A have 3 Ms and 2 Ts adjacent; whence, these 8 As give us 48 occurrences of MATH. The 4 middle As have 3 Ms and 3 Ts adjacent, which means that the 4 middle As give us 36 more occurrences of MATH. Therefore, we have $20 + 48 + 36 = 104$ occurrences of MATH in the array.

Also solved by Andrew Fischer and Frank Barlow (Humke’s Raiders), Washington and Lee University, Lexington, VA, USA; and Titu Zvonaru, Comănești, Romania. One incorrect solution was received.
**M184. Proposed by the Mayhem Staff.**

Find all solutions \((a, b)\) for the equation \(ab - 24 = 2a\), where \(a\) and \(b\) are positive integers.

**Solution by Geoffrey Siu, London Central Secondary School, London, ON.**

Rearranging \(ab - 24 = 2a\), we get \(a(b - 2) = 24\). Thus, for positive integer solutions, \(a\) and \(b - 2\) must be factors of 24. Hence, there are 8 solutions for \((a, b)\), namely

\[(1, 26), (2, 14), (3, 10), (4, 8), (6, 6), (8, 5), (12, 4), \text{ and } (24, 3).\]

*Also solved by Robert Bilinski, Collège Montmorency, Laval, QC; and Titu Zvonaru, Comănești, Romania.*

**M185. Proposed by Neven Jurčić, Zagreb, Croatia.**

A lake has the shape of a triangle with sides of length \(a\), \(b\), and \(c\). From a helicopter, which is hovering in a stationary position above the lake, the lines-of-sight to the three vertices of the triangle are pairwise perpendicular. How high is the helicopter above the lake?

**Solution by Titu Zvonaru, Comănești, Romania.**

Let \(ABC\) be the lake and let \(H\) be the helicopter. We denote \(HA = x\), \(HB = y\) and \(HC = z\). Since \(HA \perp HB\), \(HB \perp HC\), and \(HC \perp HA\), we have

\[
x^2 + y^2 = c^2, \\
y^2 + z^2 = a^2, \\
z^2 + x^2 = b^2.
\]

Solving the system, we obtain

\[
x^2 = \frac{-a^2 + b^2 + c^2}{2}, \\
y^2 = \frac{a^2 - b^2 + c^2}{2}, \\
z^2 = \frac{a^2 + b^2 - c^2}{2}.
\]

Hence, \(\triangle ABC\) must be acute-angled. If \(h\) is the distance from \(H\) to the lake, we can write the volume of the tetrahedron \(HABC\) in two ways:

\[
\frac{1}{3}[ABC]h = \frac{1}{3}[HAB] \cdot HC,
\]

which yields

\[
h = \frac{xyz}{2[ABC]} = \sqrt{2(-a^2 + b^2 + c^2)(a^2 - b^2 + c^2)(a^2 + b^2 - c^2)} \\
8\sqrt{s(s-a)(s-b)(s-c)}
\]

where \(s = \frac{1}{2}(a + b + c)\) is the semi-perimeter of \(\triangle ABC\).
M186. Proposed by the Mayhem Staff.

Let \( \lfloor x \rfloor \) denote the greatest integer less than or equal to \( x \). For example, \( \lfloor 2.5 \rfloor = 2 \) and \( \lfloor -7.4 \rfloor = -8 \). Given that \( \sum_{i=1}^{n} \lfloor \sqrt{i} \rfloor = 217 \), determine the value of \( n \).


For any positive integers \( i \) and \( j \), if \( j^2 \leq i < (j+1)^2 \), then \( \lfloor \sqrt{i} \rfloor = j \). Thus,

\[
\sum_{i=j^2}^{(j+1)^2 - 1} \lfloor \sqrt{i} \rfloor = \sum_{i=j^2}^{(j+1)^2 - 1} j = j((j+1)^2 - j^2) = j(2j+1).
\]

Hence,

\[
\sum_{i=1}^{48} \lfloor \sqrt{i} \rfloor = \sum_{i=1}^{3} \lfloor \sqrt{i} \rfloor + \sum_{i=4}^{8} \lfloor \sqrt{i} \rfloor + \sum_{i=9}^{15} \lfloor \sqrt{i} \rfloor + \sum_{i=16}^{24} \lfloor \sqrt{i} \rfloor + \sum_{i=25}^{35} \lfloor \sqrt{i} \rfloor + \sum_{i=36}^{48} \lfloor \sqrt{i} \rfloor
\]

\[
= 1 \cdot 3 + 2 \cdot 5 + 3 \cdot 7 + 4 \cdot 9 + 5 \cdot 11 + 6 \cdot 13 = 203,
\]

which is 14 short. Then

\[
\sum_{i=1}^{50} \lfloor \sqrt{i} \rfloor = \sum_{i=1}^{48} \lfloor \sqrt{i} \rfloor + \sum_{i=49}^{50} \lfloor \sqrt{i} \rfloor = 203 + 2 \cdot 7 = 217.
\]

Thus, the required value of \( n \) is 50.

Also solved by Robert Bilinski, Collège Montmorency, Laval, QC; Alper Cay, Uzman Private School, Kayseri, Turkey; Chris Dadak, Washington and Lee University, Lexington, VA, USA; and Titu Zvonaru, Comănești, Romania.

M187. Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.

A circle \( \Gamma \) of radius \( 2r \) is inscribed in a square \( KLMN \). Line segment \( AB \) is a diameter of this circle, where \( A \) and \( B \) are mid-points of opposite sides of the square. Two circles \( \Gamma_1 \) and \( \Gamma_2 \) of radii \( r \) have centres on \( AB \) and are externally tangent to one another, and each is internally tangent to \( \Gamma \). Two circles \( \Gamma_3 \) and \( \Gamma_4 \) are externally tangent to \( \Gamma_1 \) and \( \Gamma_2 \) and internally tangent to \( \Gamma \).

Construct the common tangent to \( \Gamma_1 \) and \( \Gamma_3 \) using straight edge and compass with a minimum use of the compass.

What is the minimum number of times that the compass has to be used?

We will prove that the common tangent to $\Gamma_1$ and $\Gamma_3$ passes through $M$. This implies that the tangent can be drawn without a compass by joining $M$ to the intersection of $\Gamma_1$ and $\Gamma_3$.

Let $C_i$ be the centre of $\Gamma_i$ for $i = 1, 2, 3, 4$, and let $D$ be the centre of $\Gamma$. Let the radius of $\Gamma_3$ be $a$. By symmetry, $C_3D \perp AB$. By applying the Pythagorean Theorem to $\triangle C_3 DC_1$, we get

$$r^2 + (2r - a)^2 = (a + r)^2,$$

which gives $a = \frac{2}{3}r$.

Let $E$ be the point of tangency of $\Gamma_1$ and $\Gamma_3$. Let $F$ be the point of intersection of the common tangent to $\Gamma_1$ and $\Gamma_3$ with $AB$, and let $P$ be the point of intersection of this tangent with the line through $M$ and $N$ (see figure). By ASA congruence, we have $\triangle C_3 DC_1 \cong \triangle FEC_1$. Thus,

$$FC_1 = C_3 C_1 = \frac{5}{3}r,$$

$$FB = \frac{5}{3}r + r = \frac{8}{3}r,$$

and

$$FE = C_3 D = 2r - \frac{2}{3}r = \frac{4}{3}r.$$

By AAA similarity, $\triangle FEC_1$ is similar to $\triangle FBP$. Hence,

$$\frac{PB}{EC_1} = \frac{FB}{FE},$$

$$PB = FB \cdot EC_1 \cdot \frac{FB}{FE} = \left(\frac{5}{3}r\right) \left(\frac{8}{3}r\right) = 2r.$$

Therefore, $P$ is located at $M$.

We conclude that the common tangent to $\Gamma_1$ and $\Gamma_3$ passes through $M$.