SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

We apologize for omitting the name of WALther Janous, Ursulinen-Gymnasium, Innsbruck, Austria from the list of solvers of 2972 and from the list of those resolving a conjecture of Bencze in 1984 [2006: 51-52].


Given a quadrilateral \(ABCD\), let \(P, Q, R, S\) be points on the sides \(AB, BC, CD, DA\), respectively, such that
\[
\frac{AP}{PB} \cdot \frac{BQ}{QC} \cdot \frac{CR}{RD} \cdot \frac{DS}{SA} = 1.
\]

Let \(O\) be the intersection of \(PR\) and \(QS\). Prove that
\[
\frac{DS \cdot AP}{PB} + \frac{AS \cdot DR}{RC} = \frac{AD \cdot SO}{OQ}.
\]

Solution by Kin Fung Chung, student. University of Toronto, Toronto, ON.

We rearrange the given condition, setting
\[
\frac{x}{y} = \frac{DR \cdot QC}{AP \cdot BQ} = \frac{DS \cdot CR}{BP \cdot AS}.
\]

Attach masses \(xBP, xAP, yDR, yCR\) at points \(A, B, C, D\), respectively. We locate the centre of mass \(G\) of the system in two ways:

1. The centre of mass of \(A\) and \(B\) is at \(P\), and the centre of mass of \(C\) and \(D\) is at \(R\); hence, \(G\) lies on \(PR\).

2. Note that \(xAP \cdot BQ = yDR \cdot CQ\) by the definition of \(x\) and \(y\). This implies that the centre of mass of \(B\) and \(C\) is at \(Q\). Similarly, the centre of mass of \(A\) and \(D\) is at \(S\). Hence, \(G\) lies on \(SQ\).

By the second step, we have
\[
\frac{SO}{OQ} = \frac{\text{mass at } Q}{\text{mass at } S} = \frac{xAP + yDR}{xBP + yCR} = \frac{x}{y} \frac{AP + DR}{BP + CR}
\]
\[
= \frac{DR \cdot QC}{BQ} + \frac{DR}{CR \left(\frac{DS}{AS} + 1\right)} = \frac{DR \cdot QC}{BQ} \frac{CR \left(\frac{AD}{AS}\right)}{CR \left(\frac{AD}{AS}\right)} + \frac{DR}{CR \left(\frac{AD}{AS}\right)}
\]
\[
= \frac{DR \cdot QC}{CR \cdot BQ} \cdot \frac{AS}{AD} + \frac{DR \cdot AS}{CR \cdot AD} = \frac{DS \cdot AP}{AD \cdot PB} + \frac{AS \cdot DR}{AD \cdot RC},
\]

which immediately yields the desired result.
Also solved by MICHEL BATAILLE, Rouen, France; WALther JANous, Ursulinen-
gymnasium, Innsbruck, Austria; JoEL SCHLOSSBERG, Bayside, NY, USA; PeteR Y. WOO, Biola
University, La Mirada, CA, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto,
ON; Li ZHUO, Polk Community College, Winter Haven, FL, USA; Titu ZVONARu, Comanaști,
Romania; and the proposer.

Semiyama comments that we get a known theorem in the special case of his problem where
AP : PB = DR : RC and BQ : QC = AS : SD; in this case, the conclusion becomes
SO : OQ = AP : PB. Compare this with the version found in Coxter's Introduction to
Geometry, exercise 2 on page 76: When all the points P on AB are related by a similarity to
all the points P' on A'B', the points dividing the segments PP' in the ratio AB : A'B'
(internally or externally) are distinct and collinear or else they all coincide. As a consequence
of Semiyama's problem, we can divide the segments PP' in any fixed ratio, not just AB : A'B',
and the division points will still be collinear.

3021. [2005 : 107, 109] Proposed by Pierre Bornsztein, Maisons-Laffitte,
France.

Let E be a finite set of points in the plane, no three of which are
collinear and no four of which are concyclic. If A and B are two distinct
points of E, we say that the pair \{A, B\} is good if there exists a closed disc
in the plane which contains both A and B and which contains no other point
of E. We denote by \( f(E) \) the number of good pairs formed by the points
of E.

Prove that if the cardinality of E is 1003, then 2003 \( \leq f(E) \leq 3003 \).

Solution by Li ZHUO, Polk Community College, Winter Haven, FL, USA.

More generally, we show that \( f(E) = 3n - k - 3 \) when \( n = |E| \geq 3 \)
and \( k \) is the number of sides of the convex hull \( H \) of \( E \).

Let \( G \) denote the graph with \( E \) as the set of vertices and all line
segments connecting each good pair of points of \( E \) as edges. We first claim
that no two edges of \( G \) can cross each other. Suppose edges \( AB \) and \( CD \)
cross each other. Consider the quadrilateral \( ACBD \). Since \( \{A, B\} \) is a good
pair, there is a circle passing through \( A \) and \( B \) and containing neither \( C \) nor
\( D \) in its interior. Hence, \( \angle C + \angle D < \pi \). Similarly, \( \angle A + \angle B < \pi \). Hence,
\( \angle A + \angle B + \angle C + \angle D < 2\pi \), a contradiction. Hence, \( G \) is a planar graph.

Next, we claim that every side of \( H \) is an edge of \( G \). Let \( AB \) be a side
of the polygon \( H \). Then one side of \( AB \) contains no points of \( E \). Thus, we
can draw a circle centred on that side of \( AB \) with radius sufficiently large so
that it passes through \( A \) and \( B \) but contains no other points of \( E \). Hence,
\( \{A, B\} \) is a good pair. That is, \( AB \) is an edge of \( G \).

Note that, for each point \( A \) in \( E \), if \( B \) is a point closest to \( A \), then \( AB \)
is an edge of \( G \).

Next, suppose \( AB \) is any edge of \( G \). Then there is a point \( C \) such that
the closed disc \( \Gamma \) formed by the circumcircle of \( \triangle ABC \) intersects no other
points of \( E \) (since no four points are concyclic). By perturbing \( \Gamma \) very slightly,
we see that all three sides of \( \triangle ABC \) are edges of \( G \). If \( AB \) is not an edge of
\( H \), then we similarly have another point \( D \) on a different side of \( AB \) from
C such that all three sides of \( \triangle ABD \) are edges of \( G \). This implies that the interior of \( H \) is triangulated by the edges of \( G \).

Let \( m \) denote the number of faces of \( G \), including the exterior one. Since each interior face is enclosed by 3 edges and the exterior face is enclosed by \( k \) edges, we have \( 3(m - 1) + k = 2f(E) \). Thus, \( m = \frac{2}{3}f(E) - \frac{1}{3}k + 1 \), which together with Euler's Formula, \( m - f(E) + n = 2 \), yields \( f(E) = 3n - k - 3 \).

Since \( 3 \leq k \leq n \), we have \( 2n - 3 \leq f(E) \leq 3n - 6 \). The proposed problem is the special case when \( n = 1003 \).

Also solved by JOEL SCHLOSBERG, Bayside, NY, USA (who also derived the same formula obtained by Zhou featured above); PETER Y. WOO, Biola University, La Mirada, CA, USA, and the proposer.

**3022.** [2005 : 107, 110] Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.

Given \( \triangle ABC \), let \( C' \) be any point on the side \( AB \), and let \( M \) and \( N \) be points on the sides \( BC \) and \( AC \), respectively, such that \( C'M \parallel AC \) and \( C'N \parallel BC \).

Prove that the area of \( \triangle C'CN \) is the geometric mean of the areas of \( \triangle AC'N \) and \( \triangle C'BM \).

Essentially the same solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Francisco Bellot Rosado, i.B. Emilio Ferrari, Valladolid, Spain; and Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.

Since \( \triangle NCM \) is a parallelogram, we have \( [\triangle C'CN] = [\triangle C'CM] \) (where \([XYZ]\) denotes the area of \( XYZ \)). Hence,

\[
\frac{[\triangle C'CN]^2}{[\triangle AC'N][\triangle C'BM]} = \frac{[\triangle C'CN]}{[\triangle AC'N]} \cdot \frac{[\triangle C'CM]}{[\triangle C'BM]} = \frac{CN}{NA} \cdot \frac{CM}{MB} = \frac{MC'}{NA} \cdot \frac{NC'}{MB}.
\]

Since triangles \( \triangle ANC' \) and \( \triangle C'MB \) have parallel sides, they are similar, which implies that \( \frac{MC'}{NA} = \frac{MB}{NC'} \). Consequently,

\[
\frac{[\triangle C'CN]^2}{[\triangle AC'N][\triangle C'BM]} = 1,
\]

which is exactly what we wanted to prove.

Also solved by HOUZA ANOUN, Bordeaux, France; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; PAUL DEIERMANN, Southeast Missouri State University, Cape Girardeau, MO, USA; JOHN G. HEUVER, Grande Prairie, AB; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GEOFFREY A. KANDALL, Hamden, CT, USA; R. LAUMEN, Deurne, Belgium; RAFAEL MARTINEZ CALAFAT, I.E.S. La Plana, Castellon, Spain; XIAO LIANG QI, student, Memorial University of Newfoundland; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, NL; JOEL SCHLOSBERG, Bayside, NY, USA; BOB SERKEY, Leonia, NJ, USA; D.J. SMEENK, Zaltbommel, the Netherlands; Mª JESÚS VILLAR RUBIO, Santander, Spain; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comănești, Romania; and the proposer.
3023. [2005 : 107, 110] Proposed by Bogdan Nica. McGill University, Montreal, QC.

Find all integer solutions of the system:

\[ a^c + b^c - 2 = c^3 - c, \]
\[ b^n + c^n - 2 = a^3 - a, \]
\[ c^b + a^b - 2 = b^3 - b. \]

Solution by Li Zhou. Polk Community College, Winter Haven, FL, USA; and Kin Fung Chung, student. University of Toronto, Toronto, ON, modified by the editor.

First, we claim that \( 3^n \geq n^3 - n + 3 \) for all \( n \geq 1 \). The claim is true for \( n = 1 \). Assume that it is true for some \( n > 1 \). Then

\[
3^{n+1} - (n + 1)^3 + (n + 1) - 3 \\
\geq 3(n^3 - n + 3) - (n + 1)^3 + (n + 1) - 3 \\
= (n - 1)(n - 2)(2n + 3) \geq 0.
\]

By induction, the claim is proved. We conclude from this claim that

\[ m^n - 2 > n^3 - n \]  

(\*)

for all \( m \geq 3 \) and \( n \geq 1 \).

Now let \((a, b, c)\) be a solution of the given system, where \( a, b, c \) are integers. It is clear that no two of \( a, b, c \) can be zero. By symmetry, we may suppose \( a \leq b \leq c \).

If \( a \geq 3 \), then \( 3 \leq a \leq b \leq c \). From (\*), we see that such \( a, b, c \) cannot be a solution of the system.

If \( a \leq -2 \), then \( b^a + c^a - 2 \geq -4 \) and \( a^3 - a \leq -6 \). For such values of \( a \), the second equation of the original system has no solution.

If \( a = -1 \), then the second equation becomes \( \frac{1}{b} + \frac{1}{c} - 2 = 0 \). Thus, \( b = c = 1 \). But \( a = -1, b = c = 1 \) is not a solution of the first equation.

If \( a = 0 \), then \( 0 < b \leq c \). By (\*) and the first and third equations of the system, we conclude that \( (b, c) \in \{(1, 1), (1, 2), (2, 1), (2, 2)\} \), but none of these possible values of \( a, b, c \) are a solution of the system.

If \( a = 1 \), then \( 1 \leq b \leq c \), and the second equation becomes \( b + c = 2 \). Hence, \( b = c = 1 \), and \( a = b = c = 1 \) is a solution of the system.

If \( a = 2 \), then \( 2 \leq b \leq c \). The second equation becomes \( b^2 + c^2 = 8 \). Hence, \( b = c = 2 \), and \( a = b = c = 2 \) is a solution of the system.

Therefore, \( a = b = c = 1 \) and \( a = b = c = 2 \) are the only integer solutions of the system.

Also solved by Joel Schlosberg, Bayside, NY, USA; Michel Bataille, Rouen, France; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and the proposer. Two solutions were incomplete and one incorrect.
Proposed by the late Murray S. Klamkin, University of Alberta, Edmonton, AB; and K.R.S. Sastry, Bangalore, India.

Generalize the following identity so that it involves an \( n \)th order determinant in place of a 3rdd order determinant, and prove your generalization:

\[
\begin{vmatrix}
-bc & b^2 + bc & c^2 + bc \\
ca & -ca & c^2 + ca \\
\a & ab & b^2 + ab & -ab \\
\end{vmatrix}
= (bc + ca + ab)^3.
\]

[Ed. Two different generalizations were proved in the solutions that were submitted for this problem. One of these is given in the first solution below, and the other appears in the second and third solutions.]

1. Solution by Walther Janous, Ursulinegymnasium, Innsbruck, Austria, modified slightly by the editor.

The 3 \( \times \) 3 matrix in the given determinant may be written in the form \( M = (ab + bc + ca)I \), where \( I \) is the 3 \( \times \) 3 identity matrix and

\[
M = \begin{bmatrix}
b + c \\
a + c \\
a + b \\
\end{bmatrix}
\cdot
\begin{bmatrix}
b \\
a \\
c \\
\end{bmatrix}
\]

This suggests the following generalization involving \( x_1, x_2, \ldots, x_n \) in place of \( a, b, c \), for \( n \geq 2 \).

Let \( S_1 = \sum x_i \) and \( S_2 = \sum_{i<j} x_i x_j \) (the first and second symmetric functions of \( x_1, x_2, \ldots, x_n \)). Let

\[
U = \begin{bmatrix}
S_1 - x_1 \\
S_1 - x_2 \\
\vdots \\
S_1 - x_n \\
\end{bmatrix}
\text{ and } 
V = \begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n \\
\end{bmatrix}
\]

and let \( M = U \cdot V^\top \) (where \( V^\top \) denotes the transpose of \( V \)). Let \( I \) denote the \( n \times n \) identity matrix.

Since the matrix \( M \) has rank one, it has an eigenspace of dimension \( n - 1 \) corresponding to the eigenvalue 0. Any vector in this eigenspace is also an eigenvector of \( M - S_2I \) corresponding to the eigenvalue \( 0 - S_2 = -S_2 \).

We also note that \( U \) is an eigenvector of \( M \) corresponding to the eigenvalue \( V^\top \cdot U = S_1^2 - \sum x_i^2 = 2S_2 \). This implies that \( U \) is an eigenvector of \( M - S_2I \) corresponding to the eigenvalue \( 2S_2 - S_2 = S_2 \). It follows that

\[
\det(M - S_2I) = (-S_2)^{n-1} S_2 = (-1)^{n-1} S_2^n.
\]
II. Solution by Michel Bataille, Rouen, France.

Let \( x_1, x_2, \ldots, x_n \) be \( n \) indeterminates. For each \( i \), let \( p_i = \prod_{\ell \neq i} x_\ell \), and for all distinct \( i, j, k \), let \( p_{i,j,k} = \prod_{\ell \neq i,j,k} x_\ell \). Then let \( s = \sum p_\ell \) and \( s_{i,j} = \sum_{\ell \neq i,j} p_{i,j,\ell} \) for \( i \neq j \). (The index \( \ell \) runs from 1 to \( n \) subject to the indicated restrictions. A product over an empty set of indices is equal to 1.)

Using these notations, we define a determinant \( \Delta = \Delta(x_1, x_2, \ldots, x_n) \) as follows:

\[
\Delta = \begin{vmatrix} -p_1 & x_1^2 s_{1,2} + p_1 & \cdots & x_1^2 s_{1,n} + p_1 \\ x_1^2 s_{2,1} + p_2 & -p_2 & \cdots & x_2^2 s_{2,n} + p_2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^2 s_{n,1} + p_n & x_2^2 s_{n,2} + p_n & \cdots & -p_n \end{vmatrix}
\]

that is, \( \Delta_{i,i} = -p_i \) and \( \Delta_{i,j} = x_i^2 s_{i,j} + p_i \) for \( i \neq j \). We will prove that \( \Delta = (-1)^{n-1} (n-2) s^n \).

Multiply rows 1, 2, \ldots, \( n \) in \( \Delta \) by \( x_1, x_2, \ldots, x_n \), respectively, and then extract the factors \( x_1, x_2, \ldots, x_n \) from columns 1, 2, \ldots, \( n \), respectively. Entries \( \Delta_{i,i} \) remain unchanged, while each entry \( \Delta_{i,j} \) for \( i \neq j \) becomes \( x_i x_j s_{i,j} + p_j = s - p_i \). Now, adding rows 2, 3, \ldots, \( n \) to the first row gives

\[
\Delta = \begin{vmatrix} (n-2)s & (n-2)s & \cdots & (n-2)s \\ \vdots & \vdots & \ddots & \vdots \\ s - p_n & s - p_n & \cdots & -p_n \end{vmatrix}
\]

Now we factor \( (n-2)s \) from the first row (so that the first row becomes a row of 1s) and then subtract the first column from each of the other columns to get

\[
\Delta = (n-2)s \begin{vmatrix} 1 & 0 & \cdots & 0 \\ s - p_2 & -s & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ s - p_n & 0 & \cdots & -s \end{vmatrix}
\]

From here we conclude that \( \Delta = (n-2)s(-s)^{n-1} = (-1)^{n-1}(n-2)s^n \).

III. Solution by the proposers.

First we give a short proof of the given identity. This will lead to a generalization.

In the given identity, replace \( a, b, c \) by \( 1/a, 1/b, 1/c \), respectively, and multiply both sides by \( (abc)^4 \) to obtain

\[
\begin{vmatrix} -a^2 & a(b+c) & a(b+c) \\ b(c+a) & -b^2 & b(c+a) \\ c(a+b) & c(a+b) & -c^2 \end{vmatrix} = abc(a + b + c)^3.
\]
Dividing the rows by $a$, $b$, and $c$, respectively, we get
\[
\begin{vmatrix}
-a & b+c & b+c \\
c+a & -b & c+a \\
a+b & a+b & -c
\end{vmatrix} = (a+b+c)^3.
\]

(1)

It will suffice to prove this identity.

In the determinant on the left side of (1), if we let $a + b + c = 0$, then we get three identical columns. Hence, the determinant has the factor $(a + b + c)^2$. Since the determinant is symmetric and homogeneous of the third degree, the third factor has the form $k(a + b + c)$ for some constant $k$. Since the coefficient of $a^3$ in the determinant is 1, we must have $k = 1$. This proves (1).

We will now generalize (1) (which is equivalent to generalizing the given identity). In place of $a$, $b$, $c$, we consider $a_1$, $a_2$, ..., $a_n$. Let $S = \sum_i a_i$, and let $D$ be the matrix whose entries are $D_{ii} = -a_i$ and $D_{ij} = S - a_i$ for $i \neq j$. We will show that $|D| = (-1)^{n-1}(n-2)S^n$.

If we set $S = 0$ in $D$, we get $n$ identical columns. Therefore, $S^{n-1}$ is a factor of $|D|$. Since the determinant is a symmetric and homogeneous polynomial of degree $n$, the remaining factor is $kS$ for some constant $k$. Setting $a_1 = 1$ and $a_i = 0$ for $i \neq 1$, we find that $-k$ is the determinant $|B|$, where $B$ is the $(n-1) \times (n-1)$ matrix whose entries are $B_{ii} = 0$ and $B_{ij} = 1$ for $i \neq j$. This determinant is a special case of the known determinant $|C| = \left(1 + \sum_i b_i\right)/\prod_i b_i$, where $C_{ii} = 1 + 1/b_i$ and $C_{ij} = 1$ for $i \neq j$. Setting $b_i = -1$ for all $i$, we get $k = (-1)^{n-1}(n-2)$.

Also solved by JOEL SCHLOSBERG, Bayside, NY, USA; MARIAN TETIVA, Birlad, Romania; and LI ZHOU, Polk Community College, Winter Haven, FL, USA. There was one incomplete solution.

Joe Howard, Portales, NM, USA observed that the adjoint of the 3 \times 3 matrix $A$ in the given determinant—that is, the matrix $\text{adj} A$ which is the transpose of the matrix of cofactors of the entries of $A$—has the property that $\det(\text{adj} A) = (bc + ca + ab)^3 \det(A)$. He then noted that, since $\det(\text{adj} A) = (\det A)^{n-1}$ for any $n \times n$ matrix $A$, it follows immediately that $\det A = (bc + ca + ab)^3$ for the given matrix $A$.

\section*{3025. [2005:107, 110] Proposed by Neven Jurić, Zagreb, Croatia.}

For each chess piece, we assign to each square of a chessboard a number which is the number of moves available to that piece from that square. The power of the piece is then defined to be the sum of all these numbers over all the squares of the chessboard.

Do there exist integers $m \geq 2$ and $b \geq 2$ such that, on an $m \times b$ chessboard, the power of a rook is equal to the sum of the powers of a bishop and a knight?

[Ed: In the solutions below, $P(R)$, $P(B)$, and $P(K)$ denote the power of a rook, a bishop, and a knight, respectively.]
I. Solution by Edward T.H. Wang. Wilfrid Laurier University. Waterloo, ON.

Yes, a $3 \times 3$ board is such an example.

Clearly, $P(R) = 9 \times 4 = 36$, and the arrays displayed below show that $P(B) = 20$ and $P(K) = 16$, where the number in a square is the number of moves available to a bishop (knight, respectively) from that square.

\[
\begin{array}{ccc}
2 & 2 & 2 \\
2 & 4 & 2 \\
2 & 2 & 2
\end{array}
\]

$P(B) = 20$

\[
\begin{array}{ccc}
2 & 2 & 2 \\
2 & 0 & 2 \\
2 & 2 & 2
\end{array}
\]

$P(K) = 16$

II. Solution by Li Zhou. Polk Community College. Winter Haven, FL, USA.

Yes, an ordinary $8 \times 8$ chessboard provides such an example, as shown by the arrays displayed below. [Ed: Clearly, $P(R) = 64 \times 14 = 896$. Thus, $P(R) = P(B) + P(K)$]

\[
\begin{array}{cccccc}
7 & 7 & 7 & 7 & 7 & 7 \\
7 & 9 & 9 & 9 & 9 & 7 \\
7 & 9 & 11 & 11 & 11 & 9 \\
7 & 9 & 11 & 13 & 13 & 11 \\
7 & 9 & 11 & 13 & 11 & 9 \\
7 & 9 & 9 & 9 & 9 & 7 \\
7 & 7 & 7 & 7 & 7 & 7
\end{array}
\]

$P(B) = 560$

\[
\begin{array}{cccccc}
2 & 3 & 4 & 4 & 4 & 4 \\
2 & 3 & 4 & 4 & 4 & 3 \\
4 & 6 & 8 & 8 & 8 & 6 \\
4 & 6 & 8 & 8 & 8 & 6 \\
4 & 6 & 8 & 8 & 8 & 6 \\
3 & 4 & 6 & 6 & 6 & 4 \\
2 & 3 & 4 & 4 & 4 & 3
\end{array}
\]

$P(K) = 336$

Also solved by Richard I. Hess, Rancho Palos Verdes, CA, USA; Walter Janous, Ursulinen-Gymnasium, Innsbruck, Austria; and the proposer.

In general, on an $m \times b$ board with $m \geq b \geq 3$, it is clear that $P(R) = mb(m+b-2)$. The formulas $P(B) = 2b(b-1)(3m-b-1)/3$ and $P(K) = 8bm - 12(m+b) + 16$ were obtained by both Hess and Janous. Janous also arrived at the $8 \times 8$ solution by considering square boards and equating $P(R)$ with $P(B) + P(K)$. In addition to the $3 \times 3$ and $8 \times 8$ solutions, Hess obtained two more solutions, namely $4 \times 3$ and $6 \times 4$ boards. He claimed without proof that these are the only solutions (up to transposing the board).


Let $a > 0$. Prove that

\[
\frac{a^2}{e^a} + \frac{3a^2 - 1}{3e^{3a}} + \frac{5a^2 + 1}{5e^{5a}} + \frac{7a^2 - 1}{7e^{7a}} + \cdots < \frac{\pi}{4}.
\]

Composite of essentially the same solution by Edward Doolittle, University of Regina, Regina, SK; and Li Zhou, Polk Community College, Winter Haven, FL, USA.

Let $f(a)$ denote the expression on the left side of the inequality. Then

\[
f(a) = a^2(e^{-a} + e^{-3a} + e^{-5a} + \cdots) + (e^{-a} - \frac{1}{3} e^{-3a} + \frac{1}{5} e^{-5a} - \cdots).
\]
Since $e^{-a} < 1$, we can sum the series to get

$$f(a) = a^2 \left( \frac{e^{-a}}{1 - e^{-2a}} \right) + \tan^{-1}(e^{-a}) = \frac{1}{2} a^2 \csch a + \tan^{-1}(e^{-a}).$$

Then

$$f'(a) = a \csch a - \frac{1}{2} a^2 \csch a \coth a - \frac{e^{-a}}{1 + e^{-2a}}$$

$$= a \coth a \sech a - \frac{1}{2} a^2 \sech a \coth^2 a - \frac{1}{2} \sech a$$

$$= -\frac{1}{2} \sech a \left( a^2 \coth^2 a - 2a \coth a + 1 \right)$$

$$= -\frac{1}{2} \sech a \left( a \coth a - 1 \right)^2 < 0.$$ 

Hence, $f$ is strictly decreasing on $(0, \infty)$. Using L'Hôpital's Rule, we get

$$\lim_{a \to 0^+} f(a) = \lim_{a \to 0^+} \left( \frac{2a}{e^a + e^{-a}} \right) + \tan^{-1}(1) = \frac{\pi}{4}.$$

It follows that $f(a) < \frac{\pi}{4}$ for all $a > 0$.

Also solved by DIANNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; WALTHER JANOUS, Ursulinen Gymnasium, Innsbruck, Austria; and the proposer.


Let $ABCD$ be any quadrilateral, and let $M$ be the mid-point of $AB$. On the sides $CB$, $DC$, and $AD$, equilateral triangles $CBE$, $DCF$, and $ADG$ are constructed externally. Let $N$ be the mid-point of $EF$ and $P$ be the midpoint of $FG$.

Prove that $\triangle MNP$ is equilateral.

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

Identify a point $X$ with a complex number $x$. Let $\omega = e^{2\pi i/3}$. Then $\omega^3 = 1$ and $1 + \omega + \omega^2 = 0$. Since triangles $CBE$, $DCF$, and $ADG$ are equilateral and oriented counterclockwise, it follows that $c + \omega b + \omega^2 e = 0$, $d + \omega c + \omega^2 f = 0$, and $a + \omega d + \omega^2 g = 0$.

Thus, $e = -\omega c - \omega^2 b$, $f = -\omega d - \omega^2 c$, and $g = -\omega a - \omega^2 d$. Then

$$2n = e + f$$

$$= -\omega (c + d) - \omega^2 (b + c)$$

and

$$2p = f + g$$

$$= -\omega (d + a) - \omega^2 (c + d).$$

We also have $2m = a + b$. Therefore,
\[2(m + \omega n + \omega^2p) = a + b - \omega^2(c + d) - (b + c) - (d + a) - \omega(c + d) = -(1 + \omega + \omega^2)(c + d) = 0,\]

which shows that \(\triangle MNP\) is equilateral.

Also solved by ŠEFKET ARSLANAGIC, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; KIN FUNG CHUNG, student, University of Toronto, Toronto, ON; EDWARD DOOLITTLE, University of Regina, Regina, SK; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie, AB; WALTER JANOUS, Ursulinen-gymnasium, Innsbruck, Austria; JOEL SCHLOSBURG, Bayside, NY, USA; YUEFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; and the proposer.

All solutions made use of the same idea. It would be nice to see a purely geometric proof.

The editor was reminded of Napoleon’s Theorem here, and wondered if there was a generalization where the apexes of the equilateral triangles are replaced by points on the perpendicular bisectors of the sides \(BC, CD\) and \(DA\), at the same proportional distance with respect to the lengths of the respective sides. But this is not true, as shown by the simple example where the apexes are replaced by the centres of the squares on the sides and the mid-points of the sides. What is it about the equilateral triangles that makes this result true?


Let \(a_1, a_2, \ldots, a_n\) be positive real numbers, and let \(S_k = 1 + 2 + \cdots + k\). Prove the following

\[
1 + \frac{(a_1 a_2)^{\frac{1}{n}}}{a_1 + 2a_2} + \frac{(a_1 a_2 a_3)^{\frac{1}{n}}}{a_1 + 2a_2 + 3a_3} + \cdots + \frac{(a_1 a_2 \cdots a_n)^{\frac{1}{n}}}{a_1 + 2a_2 + \cdots + na_n} \leq \frac{2n}{n + 1}.
\]

Solution by Edward Doollittle, University of Regina, Regina, SK.

By the Weighted AM–GM Inequality, we have

\[
(a_1 a_2 \cdots a_k)^{\frac{1}{n}} \leq \frac{a_1 + 2a_2 + \cdots + ka_k}{S_k},
\]

with equality if and only if \(a_1 = a_2 = \cdots = a_k\). Now, since

\[
\frac{1}{S_k} = \frac{2}{k(k + 1)} = \frac{2}{k} - \frac{2}{k + 1},
\]

we have

\[
1 + \frac{(a_1 a_2)^{\frac{1}{n}}}{a_1 + 2a_2} + \frac{(a_1 a_2 a_3)^{\frac{1}{n}}}{a_1 + 2a_2 + 3a_3} + \cdots + \frac{(a_1 a_2 \cdots a_n)^{\frac{1}{n}}}{a_1 + 2a_2 + \cdots + na_n} \leq \frac{1}{S_1} + \frac{1}{S_2} + \cdots + \frac{1}{S_n} = \frac{2}{i} - \frac{2}{i + 1} = \frac{2n}{n + 1},
\]

with equality if and only if \(a_1 = a_2 = \cdots = a_n\).
Also solved by ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MIHLÁKY BENCZE, Brasov, Romania; WALTHER JANOUS, Ursulengymnasium, Innsbruck, Austria; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; KIN FUNG CHUNG, student, University of Toronto, Toronto, ON; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; OVIDIU FURDUI, student, Western Michigan University, Kalamazoo, MI, USA; JOHN G. HEUVER, Grande Prairie, AB; JOE HOWARD, Portales, NM, USA; HENRY RICARDO, Medgar Evers College (CUNY), Brooklyn, NY, USA; JOEL SCHLOSBERG, Bayside, NY, USA; PANOS E. TSAOUSOGLOU, Athens, Greece; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comănești, Romania; and the proposer. All submitted solutions were essentially the same.

Mihály Bencze, Brasov, Romania, actually submitted the following generalization of the result. If \( a_k \) and \( \alpha_k \) are positive for \( k = 1, 2, \ldots, n \), then

\[
\sum_{k=1}^{n} \frac{a_1a_2\cdots a_k}{\alpha_1\alpha_2\cdots \alpha_k} \leq \sum_{k=1}^{n} \frac{1}{\alpha_1 + \alpha_2 + \cdots + \alpha_k}.
\]


Let \( a_1, a_2, \ldots, a_n \) be real numbers greater than \(-1\), and let \( \alpha \) be any positive real number. Prove that if \( a_1 + a_2 + \cdots + a_n \leq \alpha n \), then

\[
\frac{1}{a_1 + 1} + \frac{1}{a_2 + 1} + \cdots + \frac{1}{a_n + 1} \geq \frac{n}{\alpha + 1}.
\]

I. Composite of nearly identical solutions by the Austrian IMO-Team, 2005; Mihály Bencze, Brasov, Romania; Kin Fung Chung, student, University of Toronto, Toronto, ON; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON; Li Zhou, Polk Community College, Winter Haven, FL, USA; and Titu Zvonaru, Comănești, Romania.

Since \( a_i + 1 > 0 \) for all \( i \), we have, by the AM–HM Inequality,

\[
\frac{n}{\sum_{i=1}^{n} a_i + 1} \geq \frac{n^2}{\sum_{i=1}^{n} (a_i + 1)} = \frac{n^2}{n + \sum_{i=1}^{n} a_i} \geq \frac{n^2}{n + \alpha n} = \frac{n}{\alpha + 1}.
\]

Equality holds if and only if all the \( a_i \)’s are equal.

II. Composite of very similar solutions by Michel Bataille, Rouen, France; Charles R. Diminnie, Angelo State University, San Angelo, TX, USA; Edward Doolittle, University of Regina, Regina, SK; Joe Howard, Portales, NM, USA; Walther Janous, Ursulengymnasium, Innsbruck, Austria; and Henry Ricardo, Medgar Evers College (CUNY), Brooklyn, NY, USA.

Since the function \( f(x) = \frac{1}{x + 1} \) is decreasing and strictly convex on the interval \((-1, \infty)\), we have, by Jensen’s Inequality,

\[
\frac{1}{n} \sum_{i=1}^{n} \frac{1}{a_i + 1} \geq \frac{1}{n} \sum_{i=1}^{n} f(a_i) \geq f \left( \frac{1}{n} \sum_{i=1}^{n} a_i \right) \geq f(\alpha) = \frac{1}{\alpha + 1}.
\]
with equality if and only if $a_1 = a_2 = \cdots = a_n = \alpha$. [Ed: Note that $-1 < \frac{1}{n} \sum_{i=1}^{n} a_i < \alpha$.]

Also solved by ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; OVIDIU FURDUI, student, Western Michigan University, Kalamazoo, MI, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie, AB; JOEL SCHLOSBERG, Bayside, NY, USA; PANOS E. TSAOUSSOGLIOU, Athens, Greece; and the proposer.


Show that, if $a_1$, $a_2$, \ldots, $a_n$ are positive real numbers, then

\[
\frac{1}{a_1} + \frac{2}{(a_2)^{\frac{1}{2}}} + \frac{3}{(a_3)^{\frac{1}{3}}} + \cdots + \frac{n}{(a_n)^{\frac{1}{n}}} \geq \frac{S_n}{(a_1a_2 \cdots a_n)^{\frac{1}{n}}}
\]

where $S_n = 1 + 2 + \cdots + n$.

Composite solution extracted from essentially the same solutions by those solvers marked with an asterisk (*) below.

By the AM–GM Inequality, we have

\[
\frac{1}{a_1} + \frac{2}{(a_2)^{\frac{1}{2}}} + \frac{3}{(a_3)^{\frac{1}{3}}} + \cdots + \frac{n}{(a_n)^{\frac{1}{n}}} \geq (1 + 2 + \cdots + n) \left( \frac{1}{a_1 \left( \frac{1}{a_2} \right)^{\frac{1}{2}} \cdots \left( \frac{1}{a_n} \right)^{\frac{1}{n}}} \right) = \frac{S_n}{(a_1a_2 \cdots a_n)^{\frac{1}{n}}},
\]

with equality if and only if all the $a_i$s are equal.

Solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; *the AUSTRIAN IMO TEAM, 2005; *MICHEL BATAILLE, Rouen, France; *MIHÅLY BENCZE, Brasso, Romania; *KIN FUNG CHUNG, student, University of Toronto, Toronto, ON; *CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; CHARLES R. DIMMINIE, Angelo State University, San Angelo, TX, USA; EDWARD DOOLITTLE, University of Regina, Regina, SK; *OVIDIU FURDUI, student, Western Michigan University, Kalamazoo, MI, USA; *JOHN G. HEUVER, Grande Prairie, AB; *WALTHER JANOUŠ, Ursulinen-gymnasium, Innsbruck, Austria; *JOHN LEONARD, University of Arizona, Tucson, AZ, USA; HENRY RICARDO, Medgar Evers College (CUNY), Brooklyn, NY, USA; *JOEL SCHLOSBERG, Bayside, NY, USA; PANOS E. TSAOUSSOGLIOU, Athens, Greece; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; LI ZHOU U, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comanaști, Romania; and the proposer.

Several solvers used the Weighted AM–GM Inequality with weights $k/S_n$ on the numbers $1/(a_k)^{1/k}$, for $k = 1, 2, \ldots, n$. This is actually equivalent to the argument given above, in this case.

A quadruple \((a, b, c, d)\) of positive integers is said to have the Diophantine property if each of the six integers \(ab + 1, ac + 1, ad + 1, bc + 1, bd + 1, cd + 1\) is a perfect square. For example, each of the following nine quadruples has the Diophantine property:

\[
(3, 5, 16, 1008), \quad (3, 8, 21, 2080), \quad (3, 16, 33, 6440), \\
(3, 21, 40, 10208), \quad (3, 33, 56, 22360), \quad (3, 40, 65, 31416), \\
(3, 56, 85, 57408), \quad (3, 65, 96, 75208), \quad (3, 85, 120, 122816).
\]

Find a general expression for the sequence of quadruples \((a_n, b_n, c_n, d_n)\) which have the Diophantine property and for which the above examples represent the first terms.

Solution by Mercedes Sánchez Benito, Universidad Complutense, Madrid, Manuel Benito Muñoz and Emilio Fernández Moral, IES P. M. Sagasta, Logroño, Spain.

In problem 20 of his book \(\Delta\) (the fourth book of Arithmetica), Diophantus of Alexandria proposes: “To find four numbers such that the product of any two increased by unity is a square.” He gives a solution in rational numbers: \(\frac{1}{16}, \frac{33}{16}, \frac{68}{16}, \frac{105}{16}\).

Euler ([1], [2]) gives the following solution to Diophantus’ problem in integers \((a, b, c, d)\):

Let \(a\) and \(b\) be such that \(ab + 1 = p^2\) (with \(p\) an integer), \(c = a + b + 2p\), and \(d = 4p(p + a)(p + b)\). Then we also have

\[
ac + 1 = (p + a)^2, \quad bc + 1 = (p + b)^2, \quad ad + 1 = (2p(p + a) - 1)^2, \\
bd + 1 = (2p(p + b) - 1)^2, \quad cd + 1 = (2pc - 1)^2.
\]

The proposed sequence of quadruples \((a_n, b_n, c_n, d_n)\) can be obtained from Euler’s solution for \(a = 3\) and \(p\) having integer values greater than 3, in increasing order, and such that \(p^2 - 1 \equiv 0 \pmod{3}\). Therefore, the solution to the problem at hand is:

- \(a_n = 3\) for \(n = 1, 2, \ldots\);
- \(p_n = \begin{cases} 3m + 1 & \text{for } n = 2m - 1, \\ 3m + 2 & \text{for } n = 2m; \end{cases}\)
- \(b_n = \frac{p_n^2 - 1}{3} = \begin{cases} m(3m + 2) & \text{for } n = 2m - 1, \\ (m + 1)(3m + 1) & \text{for } n = 2m; \end{cases}\)
- \(c_n = a_n + b_n + 2p_n = \begin{cases} (m + 1)(3m + 5) & \text{for } n = 2m - 1, \\ (m + 2)(3m + 4) & \text{for } n = 2m; \end{cases}\)
- \(d_n = 4p_n(p_n + a_n)(p_n + b_n) = \begin{cases} 4(3m + 1)(3m + 4)(3m^2 + 5m + 1) & \text{for } n = 2m - 1, \\ 4(3m + 2)(3m + 5)(3m^2 + 7m + 3) & \text{for } n = 2m. \end{cases}\)
If $p = 2$, we obtain the quadruple $(3, 1, 8, 120)$, which is not among the proposed ones. Note also that there are quadruples of positive integers with $a_n = 3$ and the Diophantine property which are not among the quadruples of the above solution: For example, the quadruple $(3, 5, 1008, 62496)$ has the Diophantine property, but cannot be obtained from Euler's solution.

Finally, we note that there are quadruples containing the number of this problem, 3031, and having the Diophantine property; we list three of them:

$$(248, 1545, 3031, 4645441488),$$
$$(1545, 3031, 8904, 166786015280),$$
$$(3031, 5013, 15840, 962717421848).$$

References


Also solved by MICHEL BATAILLE, Rouen, France; EDWARD DOOLITTLE, University of Regina, Regina, SK; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulengymnasium, Innsbruck, Austria; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.


Let $a, b, c$ be non-negative real numbers such that $a^2 + b^2 + c^2 = 1$. Prove that

$$\frac{1}{1 - ab} + \frac{1}{1 - bc} + \frac{1}{1 - ca} \leq \frac{9}{2}.$$  

Solution by the proposer, modified slightly by the editor.

Note first that the given inequality is successively equivalent to each of the following:

$$2(1 - ab)(1 - bc) + 2(1 - bc)(1 - ca) + 2(1 - ca)(1 - ab) \leq 9(1 - ab)(1 - bc)(1 - ca),$$

$$6 - 4(ab + bc + ca) + 2abc(a + b + c) \leq 9(ab + bc + ca) + 7abc(a + b + c) - 9a^2b^2c^2,$$

$$0 \leq 3 - 5(ab + bc + ca) + 10abc(a + b + c) + abc(a + b + c - 9abc). \quad (1)$$

By the AM-GM Inequality, we have

$$a + b + c - 9abc = (a + b + c)(a^2 + b^2 + c^2) - 9abc \geq 3\sqrt[3]{abc} \cdot 3\sqrt[3]{a^2b^2c^2} - 9abc = 0. \quad (2)$$
On the other hand,

\[
3 - 5(ab + bc + ca) + 6abc(a + b + c) = 3(a^2 + b^2 + c^2)^2 - 5(ab + bc + ca)(a^2 + b^2 + c^2) + 6abc(a + b + c) = 3(a^4 + b^4 + c^4) + 6(a^2b^2 + b^2c^2 + c^2a^2) + abc(a + b + c) - 5(ab(a^2 + b^2) + bc(b^2 + c^2) + ca(c^2 + a^2)) = [2(a^4 + b^4 + c^4) + 6(a^2b^2 + b^2c^2 + c^2a^2) - 4ab(a^2 + b^2) - 4bc(b^2 + c^2) - 4ca(c^2 + a^2)] + (a^4 + b^4 + c^4 + abc(a + b + c) - ab(a^2 + b^2) - bc(b^2 + c^2) - ca(c^2 + a^2) = [(a - b)^2 + (b - c)^4 + (c - a)^4] + a^2(a - b)(a - c) + b^2(b - c)(b - a) + c^2(c - a)(c - b) \geq 0 ,
\]

(3)

since \((a - b)^4 + (b - c)^4 + (c - a)^4 \geq 0\) and

\[a^2(a - b)(a - c) + b^2(b - c)(b - a) + c^2(c - a)(c - b) \geq 0\]

is the well-known Schur's Inequality. Now (1) follows from (2) and (3).

We see that equality holds if and only if \(a = b = c = \sqrt{3}/3\).

Also Solved by MICHEL BATAILLE, Rouen, France; WALther Janous, Ursulinen-gymnasium, Innsbruck, Austria; Panos E. Tsaoussoglou, Athens, Greece; and Li Zhou, Polk Community College, Winter Haven, FL, USA. There was one incorrect solution.

Using the same proof presented above, the proposer actually proved the more general result that if \(a, b,\) and \(c\) are non-negative real numbers such that \(a^2 + b^2 + c^2 = 1\), then, for all constants \(k \geq 1\), we have

\[
\frac{1}{k - ab} + \frac{1}{k - bc} + \frac{1}{k - ca} \leq \frac{9}{3k - 1} .
\]

3033. [2005 : 175, 177] Proposed by Eckard Specht, Otto-von-Quericke University, Magdeburg, Germany.

Let \(I\) be the incentre of \(\triangle ABC\), and let \(R\) and \(r\) be its circumradius and inradius, respectively. Prove that

\[6r \leq AI + BI + CI \leq \sqrt{12(R^2 - Rr + r^2)} .
\]

1. Solution by Arkady Alt, San Jose, CA, USA.

[Ed: We give Alt's argument for the left inequality only.]

Let \(K\) and \(s\) be the area and the semiperimeter of the triangle. Using the well-known (or easy to prove) formulas

\[AI = \sqrt{\frac{bc(s - a)}{s}} , \quad BI = \sqrt{\frac{ca(s - b)}{s}} , \quad CI = \sqrt{\frac{ab(s - c)}{s}} ,\]
abc = 4KR, K = sr, K = \sqrt{s(s-a)(s-b)(s-c)}$, and the AM-GM Inequality, we obtain

\[
\frac{AI + BI + CI}{3} \geq \sqrt[3]{AI \cdot BI \cdot CI} = \sqrt[3]{\frac{abc}{s^2} \sqrt{s(s-a)(s-b)(s-c)}}
\]

\[
= \sqrt[3]{\frac{abcK}{s^2}} = \sqrt[3]{\frac{4RK^2}{s^2}} = \sqrt[3]{4Rr^2}.
\]

Thus, \( AI + BI + CI \geq 3\sqrt[3]{4Rr^2} \). This inequality is stronger than the one proposed, because Euler's Inequality implies that \( 3\sqrt[3]{4Rr^2} \geq 6r \).

II. Solution by Walther Janous. Ursulengymnasium, Innsbruck, Austria.

We give a solution "from the books". The inequality \( AI + BI + CI \geq 6r \) is item 12.1 in [1]. On the other hand, item 12.2 in [1] is the inequality \( AI + BI + CI \leq 2(R+r) \), which is stronger than the proposed one, because the well-known Euler's inequality \( R \geq 2r \) implies that \( (R-2r)(2R-r) \geq 0 \), and this is equivalent to \( 2(R+r) \leq \sqrt{12(R^2 - r^2)} \).

References

[1] O. Bottema et al., Geometric Inequalities, Groningen, 1969

Also solved by ŠEFKET ARSLANAGIC, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; SCOTT BROWN, Auburn University, Montgomery, AL, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; JOHN G. HEUVER, Grande Prairie, AB; JOE HOWARD, Portales, NM, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.