Variations on an IMO Inequality

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In July 2004, the 45th International Mathematical Olympiad for high-
school students (IMO 2004) was held in Athens, Greece. It brought together
486 participants from 85 countries. As in previous years, the problem set
consisted of six problems. The fourth IMO problem read as follows:

[IMO problem 2004/4]:
Let $n \geq 3$ be an integer, and let $x_1, \ldots, x_n$ be positive real numbers that satisfy

$$
(x_1 + x_2 + \cdots + x_n) \left( \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} \right) < n^2 + 1.
$$

(1)

Prove that for all $i, j, k$ with $1 \leq i < j < k \leq n$, the numbers $x_i, x_j, x_k$
are the side lengths of a non-degenerate triangle.

An equivalent formulation of the conclusion in this problem is that
$x_i < x_j + x_k$ holds for any three pairwise distinct indices $i, j, k$.

The generalized result

In this note, we will prove and discuss the following generalization of the
above IMO problem:

**Generalization.** Let $m \geq 1$ and $n \geq m + 1$ be integers, and let $\alpha > 1/m$ be
a real number. Let $x_1, \ldots, x_n$ be positive real numbers that satisfy

$$
\sum_{i=1}^{n} x_i \sum_{i=1}^{n} \frac{1}{x_i} < \left( n - m - 1 + \sqrt{m^2 + 1 + m^2 \alpha + 1/\alpha} \right)^2.
$$

(2)

Then, for any set $I \subset \{1, 2, \ldots, n\}$ with $|I| = m$ and for any index
\( \ell \in \{1, \ldots, n\} \setminus I \), the inequality $x_\ell < \alpha \sum_{i \in I} x_i$ holds.

Here are three remarks on this generalization.

1. We may deduce from the Cauchy inequality that the left side in (1)
and (2) is at least $n^2$. Therefore, the two bounds on the corresponding right
sides must be greater than or equal to $n^2$. Note that both right sides actually
are very close to $n^2$.

2. The condition $\alpha > 1/m$ imposed on $\alpha$ is a mild and reasonable condition. Otherwise, for $\alpha \leq 1/m$, the desired conclusion $x_\ell < \alpha \sum_{i \in I} x_i$ could
never hold true, because the maximum number $x_k$ will always be greater than or equal to the arithmetic mean of the smallest $m$ numbers.

3. The bound on the right side of (2) is the best possible. We justify this in the following way: Let $x_i = 1$ for $1 \leq i \leq m$, let $x_{m+1} = \alpha m$, and let $x_i = m \sqrt{\alpha (\alpha + 1)/(\alpha m^2 + 1)}$ for $m + 2 \leq i \leq n$. Then the left side of (2) becomes equal to the right side, and we get the smallest possible violation of (2). Furthermore, in this case the conclusion does not hold any more, since $x_{m+1} = \alpha \sum_{i=1}^{m} x_i$.

Setting $m = 2$ and $\alpha = 1$ in the generalization yields the following corollary:

**Corollary.** Let $x_1, \ldots, x_n$ be positive real numbers that satisfy

$$\sum_{i=1}^{n} x_i \sum_{i=1}^{n} \frac{1}{x_i} < (n - 3 + \sqrt{10})^2. \quad (3)$$

Then $x_i < x_j + x_k$ holds for any three pairwise distinct indices $i, j, k$.

The corollary demonstrates that without losing the conclusion in the IMO problem, the bound $B_1(n) := n^2 + 1$ in inequality (1) may be replaced by the weaker bound $B_2(n) := (n - 3 + \sqrt{10})^2$. For $n = 3$ both bounds coincide (since $B_1(3) = B_2(3) = 10$), whereas for all $n \geq 4$ we have $B_2(n) > B_1(n)$. The extremal example introduced above specializes to $x_1 = x_2 = 1$, $x_3 = 2$, and $x_i = \sqrt{8/5}$ for $4 \leq i \leq n$. It shows that any further relaxation of the bound $B_2(n)$ would make the conclusion invalid.

**Proof of the generalized result**

Our main tool will be the Cauchy Inequality in the following form: For all positive real numbers $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$, we have

$$\sum_{i=1}^{k} a_i \sum_{i=1}^{k} b_i \geq \left( \sum_{i=1}^{k} \sqrt{a_i b_i} \right)^2. \quad (4)$$

Now suppose, for the sake of contradiction, that the $m + 1$ numbers $x_1, \ldots, x_{m+1}$ satisfy $x_{m+1} \geq \alpha \sum_{i=1}^{m} x_i$. We will show that under these circumstances inequality (2) does not hold.

To this end, we define $S = \sum_{i=1}^{m} x_i$, $T = \sum_{i=1}^{m} \frac{1}{x_i}$, and $z = x_{m+1}/S$. Note that our assumption yields $z \geq \alpha > 1/m$, and that the Cauchy Inequality (4)
yields $S \cdot T \geq m^2$. This leads to the following useful inequality:

$$\sum_{i=1}^{m+1} x_i \sum_{i=1}^{m+1} \frac{1}{x_i} = \left( \sum_{i=1}^{m} x_i \sum_{i=1}^{m} \frac{1}{x_i} \right) + x_{m+1} \frac{1}{x_{m+1}} + \sum_{i=1}^{m} \left( \frac{x_{m+1}}{x_i} + \frac{x_i}{x_{m+1}} \right)

= S \cdot T + 1 + x_{m+1} T + \frac{S}{x_{m+1}} \geq m^2 + 1 + x_{m+1} \frac{m^2}{S} + \frac{S}{x_{m+1}} = m^2 + 1 + m^2 \alpha + \frac{1}{\alpha} .$$

(5)

Here the final inequality follows from $z \geq \alpha$, and from the fact that the function $f(z) = m^2 z + 1/z$ for $z > 1/m$ is an increasing function.

Next, in the Cauchy Inequality (4), we set $a_i = x_{m+1+i}$ for $i = 1, \ldots, n - m - 1$, and $a_{n-m} = \sum_{i=1}^{m+1} x_i$. Furthermore, we set $b_i = 1/x_{m+1+i}$ for $i = 1, \ldots, n - m - 1$, and $b_{n-m} = \sum_{i=1}^{m+1} \frac{1}{x_i}$. Note that (5) provides a lower bound on $a_{n-m} b_{n-m}$. Then the Cauchy Inequality yields

$$\sum_{i=1}^{n} x_i \sum_{i=1}^{n} \frac{1}{x_i} = \sum_{i=1}^{n-m} a_i \sum_{i=1}^{n-m} b_i \geq \left( n - m - 1 + \sqrt{a_{n-m} b_{n-m}} \right)^2 \geq \left( n - m - 1 + \sqrt{m^2 + 1 + m^2 \alpha + 1/\alpha} \right)^2 .$$

To summarize, whenever there are $m+1$ numbers that violate the conclusion of the generalized problem, then also the inequality in (2) is violated. This completes the argument.

**Related results**

We close this note by discussing several results that all are related to the special case $m = 1$. By setting $m = 1$ in the generalization we get the following corollary:

**Corollary.** Let $\alpha > 1$ and $x_1, \ldots, x_n$ be positive real numbers such that

$$\sum_{i=1}^{n} x_i \sum_{i=1}^{n} \frac{1}{x_i} < \left( n - 2 + \sqrt{\alpha} + \frac{1}{\sqrt{\alpha}} \right)^2 .$$

(6)

Then $x_i < \alpha x_j$ holds for any two indices $i$ and $j$.

In this corollary, the conclusion holds for any pair of indices. In the following proposition, a similar conclusion holds for some pair of indices.
Proposition. Let $\alpha > 1$ and $x_1, \ldots, x_n$ be positive real numbers such that
\[ \sum_{i=1}^{n} x_i \sum_{i=1}^{n} \frac{1}{x_i} < \frac{1}{\alpha^{n-1}} \left(1 + \alpha + \alpha^2 + \cdots + \alpha^{n-1}\right)^2. \] \hspace{1cm} (7)

Then there exist two indices $i$ and $j$ such that $x_i \leq x_j < \alpha x_i$.

Proof: Without loss of generality, let $x_1 \leq x_2 \leq \cdots \leq x_n$. Now suppose for the sake of contradiction that the conclusion is violated. Then $x_{i+1} \geq \alpha x_i$ holds for $1 \leq i \leq n-1$. An easy induction yields $x_j/x_i \geq \alpha^{j-i}$ for all $j \geq i$.

Since for $z \geq 1$ the function $f(z) = z + \frac{1}{z}$ is increasing, we conclude from this that $x_j/x_i + x_i/x_j \geq \alpha^{j-i} + \alpha^{i-j}$. This implies
\[
\sum_{i=1}^{n} x_i \sum_{i=1}^{n} \frac{1}{x_i} = n + \sum_{1 \leq i < j \leq n} \left(\frac{x_j}{x_i} + \frac{x_i}{x_j}\right) \geq n + \sum_{1 \leq i < j \leq n} \left(\alpha^{j-i} + \alpha^{i-j}\right)
= \sum_{i=1}^{n} \alpha^i \sum_{i=1}^{n} \alpha^{-i} = \frac{1}{\alpha^{n-1}} \left(1 + \alpha + \alpha^2 + \cdots + \alpha^{n-1}\right)^2.
\]

Since the resulting inequality violates (7), we have reached the desired contradiction.

The example with $x_i = \alpha^i$ for $1 \leq i \leq n$ demonstrates that the bound in the right hand side of (7) is the best possible.

The following proposition gives a kind of reverse statement for the preceding two results. It is a straightforward reformulation of the well-known Schweitzer Inequality. And it is also a special case of the famous Kantorovich Inequality, which was stated and proved in 1948 by Leonid Vitaliyevich Kantorovich (the same Kantorovich who, in 1975, received the Nobel Prize for Economics).

Proposition. Let $\alpha > 1$ and $x_1, \ldots, x_n$ be positive real numbers such that $x_j < \alpha x_i$ holds for any two indices $i$ and $j$. Then
\[ \sum_{i=1}^{n} x_i \sum_{i=1}^{n} \frac{1}{x_i} < \frac{(\alpha + 1)^2}{4\alpha} n^2. \] \hspace{1cm} (8)

For even $n$, the bound in the right side of (8) is the best possible, as shown by the example $x_i = 1$ for $1 \leq i \leq n/2$ and $x_i = \alpha$ for $n/2 < i \leq n$.

For odd $n$, the bound can be further decreased to $\frac{(\alpha + 1)^2}{4\alpha} n^2 - \frac{(\alpha - 1)^2}{4\alpha}$. 

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