Pólya's Paragon

It Ain't So Complex (Part 4)

Shawn Godin

In this issue, we wrap up our treatment of complex numbers by looking at some applications to geometry. These problems all appeared in last month's homework; let's see how you did.

First we will show how to use complex numbers to prove the Triangle Inequality. In Part 2 (October issue, p. 371), we saw that arrows representing \( z_1, \ z_2, \) and \( z_1 + z_2 \) can be arranged into a triangle with sides of length \( |z_1|, \ |z_2|, \) and \( |z_1 + z_2| \). We also saw that \( z \bar{z} = |z|^2 \). Thus,

\[
|z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1 + z_2}) = (z_1 + z_2)(\overline{z_1} + \overline{z_2}) = |z_1|^2 + |z_2|^2 + \bar{z}_1z_2 + \bar{z}_2z_1.
\]

If \( z_1 = a_1 + ib_1 \) and \( z_2 = a_2 + ib_2 \), then

\[
z_1\overline{z_2} = a_1a_2 + b_1b_2 + i(a_2b_1 - a_1b_2)
\]

and

\[
\overline{z}_1z_2 = a_1a_2 + b_1b_2 + i(a_1b_2 - a_2b_1).
\]

Thus, \( z_1\overline{z_2} + \overline{z}_1z_2 = 2(a_1a_2 + b_1b_2) = 2\Re(z_1\overline{z}_2) = 2\Re(\overline{z}_1z_2) \). Therefore,

\[
|z_1 + z_2|^2 = |z_1|^2 + 2\Re(z_1\overline{z}_2) + |z_2|^2 \leq |z_1|^2 + 2|z_1||\overline{z}_2| + |z_2|^2 = (|z_1| + |z_2|)^2.
\]

Since \( |z_1 + z_2| \geq 0 \) and \( |z_1| + |z_2| \geq 0 \), when we take square roots of both sides of the above inequality, we must have \( |z_1 + z_2| \leq |z_1| + |z_2| \), which is the Triangle Inequality.

We can go one step further by writing

\[
|z_1| = |(z_1 + z_2) + (-z_2)| \leq |z_1 + z_2| + |-z_2| = |z_1 + z_2| + |z_2|.
\]

That is, \( |z_1| - |z_2| \leq |z_1 + z_2| \). We can use a similar argument for \( z_2 \) to see that \( |z_2| - |z_1| \leq |z_1 + z_2| \). These last two inequalities can be combined to yield

\[
||z_1| - |z_2|| \leq |z_1 + z_2|.
\]

This is another form of the Triangle Inequality.

Next we will prove that the medians of a triangle meet at a common point. But first, we need to convince ourselves of the following fact. For any two unequal complex numbers, \( z_1 \) and \( z_2 \), drawn on the complex plane, the complex number given by \( z = (1 - t)z_1 + tz_2 \), \( 0 < t < 1 \) lies on the line segment joining \( z_1 \) and \( z_2 \) and divides it in the ratio \( t : (1 - t) \).
In the diagram above we see that, if we take a complex number $z$ which satisfies our conditions, then $z = z_1 + t(z_2 - z_1) = (1 - t)z_1 + tz_2$.

Thus, if we consider a triangle whose vertices are the three complex numbers $z_1, z_2, z_3$ in the complex plane, then $\frac{1}{2}(z_2 + z_3)$ corresponds to the mid-point of the side with vertices at $z_2$ and $z_3$. Thus, a point on the median has co-ordinates

$$z = (1 - t)z_1 + t \cdot \frac{z_2 + z_3}{2}.$$

If we choose a point which is $\frac{2}{3}$ of the way from the vertex $z_3$ to the mid-point of $z_1z_2$, it will have co-ordinates

$$z = \frac{z_1 + z_2 + z_3}{3}.$$

From the symmetry of this expression, we see that the same could be said for the point which is $\frac{2}{3}$ of the way from the vertex $z_2$ to the mid-point of $z_1z_3$ and also for the point which is $\frac{2}{3}$ of the way from the vertex $z_1$ to the mid-point of $z_2z_3$. Thus, these three points coincide; that is, the medians of a triangle intersect at a point which divides the medians in the ratio 2 : 1. This point is called the centroid or centre of gravity of the triangle.

We want to draw your attention to one last point. In the solutions to last month’s homework you were pointed in the direction of looking at $n$th roots of complex numbers. If you continue your investigation, you will find that the $n$ such roots, when plotted on the complex plane, are the vertices of a regular $n$-gon centred at the origin.

Well, that wraps up our discussion of complex numbers. Until February, happy problem solving.